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BLOCK-PULSE FUNCTIONS AND OPERATIONAL MATRIX FOR THE NUMERICAL SOLUTION OF SOME CLASSES OF LINEAR AND NONLINEAR STOCHASTIC INTEGRAL EQUATIONS

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Abstract In this paper, we obtain a new iterative method for solving stochastic nonlinear Volterra-Fredholm integral equations. Using Block-pulse functions and operational matrix of integration, a nonlinear Volterra-Fredholm integral equation can be reduced to a nonlinear system of algebraic equations, which can be solved by iterative method. Error estimate and convergence analysis of the proposed method have been proved. Finally, illustrative examples are included to demonstrate the validity and applicability of the technique.

1 Introduction

Approximation by orthogonal families of basis functions such as Block-pulse functions, Bernoulli polynomials, Fourier series, Taylor collocation, Legendre polynomials,..etc were

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used to estimate solutions of some equations and systems such as integral equations [2, 3, 5, 6, 7]. This technique is an accurate and efficient computational method for solving stochastic integral equations. We know that stochastic Volterra-Fredholm integral equations arise in many problems in mechanics, finance, biology, science medical, social sciences.etc. So it is necessary to study such problems where there is an increasing demand for studying the behavior of a number of sophisticated dynamical systems. These systems are often dependent on a noise source, like e.g. a Gaussian white noise, governed by certain probability laws, so that modeling such phenomena naturally requires the use of various stochastic differential equations [1, 4, 9]. Recenlty, approximate solutions of integral equations have been attracted the attention of many authors, and they obtained solutions using various techniques, for examples with wavelets techniques with Chebychev polynomials of block-pulse functions. Nonlinear integral equations oppear in many problems of physical phenomena and engineering [11]. In this paper, we use Block-pulse functions and their stochastic operational matrix of integration to the following nonlinear Volterra-Fredholm stochastic integral equation

$$
\begin{aligned}
X(t) &= f(t) + \int_0^T k_1(s,t)\mathcal{N}_1(s,X(s))ds + \int_0^t k_2(s,t)\mathcal{N}_2(s,X(s))ds \qquad (1)\\
&+ \int_0^t k_3(s,t)\mathcal{N}_3(s,X(s))dB(s),
\end{aligned}
$$

where $X(t)$, $f(t)$, $k_1(s,t)$, $k_2(s,t)$ and $k_3(s,t)$, for $s,t \in [0,T], T < 1$ are the stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, P)$, and $X(t)$ is unknown process. $B(t)$ is Brownian motion process and $\int_0^t k_3(s,t)\mathcal{N}_3(s,X(s))dB(s)$ is the Itô integral. This work is inspired by [8], where the authors introduced a method for numerical solution of linear Itô-Volterra integral equation. The paper is organized as follow, In section 2, we give the basic properties of the block-pulse functions, functions approximation of smooth functions of one and two variables by block-pulse functions and we introduce the deterministic operational matrix of integration. In section 3, we derive block-pulse stochastic operational matrix of integration. In section 4, we give the proposed technique for solving nonlinear stochastic Fredholm-Volterra integral equations. Section 5 is devoted to solve stochastic Itô Volterra-Fredholm integral equations with several independent white noise sources. In section 6, the error in block-pulse approximation is obtained. Section 7 is devoted to some numerical examples.

2 Block-pulse functions (BPFs)

We define the m set of BPFs as

$$
\phi_i(t) = \begin{cases} 1 & (i-1)h \le t \le ih, \\ 0 & \text{Otherwise}, \end{cases} \qquad (2)
$$

with $t \in [0,T)$, $i = 1, 2, \ldots, m$ et $h = \dfrac{T}{m}$. The elementary properties of BPFs are as follows

1. Disjointness:
$$\phi_i \phi_j = \delta_{ij} \phi_i(t), \tag{3}$$
where $i, j = 1, 2, \ldots, m$ and δ_{ij} is Kronecker delta.

2. Orthogonality: The BPFs are orthogonal with each other in the interval $t \in [0, T]$
$$\int_0^T \phi_i(t)\phi_j(t)dt = h\delta_{ij}, \quad i, j = 1, 2, \ldots, m. \tag{4}$$

3. Completeness: If $m \longrightarrow \infty$, then BPFs set is complete, i.e for every $f \in L^2([0, T])$, Parseval's identity holds,
$$\int_0^T f^2(t)d(t) = \sum_{i=1}^{\infty} f_i^2 \|\phi_i(t)\|^2, \tag{5}$$
where
$$f_i = \frac{1}{h} \int_0^T f(t)\phi_i(t)dt. \tag{6}$$

Consider the first m terms of BPFs and write them as m vector
$$\Phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_m(t))^T, \quad t \in [0, T). \tag{7}$$

From the above properties, we get
$$\Phi(t)\Phi^T(t) = \begin{pmatrix} \phi_1(t) & 0 & \ldots & 0 \\ 0 & \phi_2(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \phi_m(t) \end{pmatrix}_{m \times m}, \tag{8}$$

furthermore, we have
$$\Phi^T \Phi(t) = 1 \quad \text{and} \quad \Phi(t)\Phi^T(t)F^T = A_F \Phi(t), \tag{9}$$

where A_F denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = (f_1, f_2, \ldots, f_m)^T$.

2.1 Function approximation

An arbitrary real bounded function $f(t)$ which is square integrable in the interval $[0, T]$, can be written into a block-pulse series in the sense of minimizing the mean square error between $f(t)$ and its approximation
$$f(t) \simeq \hat{f}_m(t) = F^T \Phi(t) = \Phi^T(t)F, \tag{10}$$

where $F = (f_1, \ldots, f_m)^T$.
Let $K(s, t) \in L^2([0, T_1] \times [0, T_2])$, it can be wirtten with respect of BPFs as follow
$$K(s, t) = \Psi(s)^T K \Phi(t) = \Phi(t)^T K^T \Psi(s), \tag{11}$$

where $\Psi(s)$ et $\Phi(t)$ are m_1, m_2 dimensional BPFs vectors and $K = (K_{ij})$, $i = 1, \dots, m_1$, $j = 1, \dots, m_2$ is the $m_1 \times m_2$ block pulse coefficients with

$$K_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} K(s,t)\Psi_i(s)\Phi_j(t)dtds,$$

où $h_1 = \dfrac{T_1}{m_1}$, $h_2 = \dfrac{T_2}{m_2}$.

2.2 Integration operational matrix

In this subsection, we give deterministic operational matrix, we have

$$\int_0^t \phi_i(s)ds = \begin{cases} 0 & 0 \leq t < (i-1)h, \\ t-(i-1)h & (i-1)h \leq t < ih, \\ h & ih \leq t < T. \end{cases} \tag{12}$$

Since $(i-1)h \leq t < ih$ then $t = (1-\lambda)((i-1)h) + \lambda(ih)$, $\lambda \in [0,1)$, we can approximate $t-(i-1)h$, for $(i-1)h \leq t < ih$ by λh then

$$\int_0^t \phi_i(s)ds \simeq (0, \dots, 0, \lambda h,\, h, \dots, h)\, \Phi(t),$$

where λh is the ith component. Therefore,

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{13}$$

where P is the deterministic operational matrix given by

$$P = h \begin{pmatrix} \lambda & 1 & 1 & \dots & 1 \\ 0 & \lambda & 1 & \dots & 1 \\ 0 & 0 & \lambda & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}_{m\times m}. \tag{14}$$

Then $\displaystyle\int_0^t f(s)ds \simeq \int_0^t F^T\Phi(s)ds \simeq F^T P\Phi(t)$.

3 Stochastic integration operational matrix

The integration of the components of $\Phi(t)$ can be given as follows,

$$\int_0^t \phi_i(s)dB(s) = \begin{cases} 0 & 0 \leq t < (i-1)h, \\ B(t) - B((i-1)h) & (i-1)h \leq t < ih, \\ B(ih) - B((i-1)h) & ih \leq t < T. \end{cases} \tag{15}$$

We approximate $B(t) - B((i-1)h)$ for $(i-1)h \leq t < ih$ by $B((i+(\lambda-1))h) - B((i-1)h)$, then

$$\int_0^t \phi_i(s)dB(s) \simeq \Big(B((i+(\lambda-1))h) - B((i-1)h)\Big)\phi_i(t) + \Big(B(ih) - B((i-1)h)\Big)\sum_{j=i+1}^{m} \phi_j(t),$$

which can be written as

$$\int_0^t \phi_i(s)dB(s) \simeq \Big(0,\ldots,0,B((i+(\lambda-1))h)-B((i-1)h),B(ih)-B((i-1)h),\ldots,B(ih)-B((i-1)h)\Big)\Phi(t),$$

with $B((i+(\lambda-1))h)-B((i-1)h)$ is the ith component. Therefore

$$\int_0^t \Phi(s)dB(s) \simeq P_s\Phi(t), \tag{16}$$

where P_s is given by

$$P_s = \begin{pmatrix} B(\lambda h) & B(h) & B(h) & \ldots & B(h) \\ 0 & B((\lambda+1)h)-B(h) & B(2h)-B(h) & \ldots & B(2h)-B(h) \\ 0 & 0 & B((\lambda+2)h)-B(2h) & \ldots & B(3h)-B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & B\Big((\lambda+m-1)h\Big)-B((m-1)h)\Big) \end{pmatrix}_{m\times m}. \tag{17}$$

4 Solving stochastic Itô Volterra-Fredholm integral equation

We consider the following nonlinear stochastic Itô Volterra-Fredholm integral equation

$$\begin{aligned} X(t) &= f(t)+\int_0^T k_1(s,t)\mathcal{N}_1(s,X(s))ds+\int_0^t k_2(s,t)\mathcal{N}_2(s,X(s))ds \\ &+ \int_0^t k_3(s,t)\mathcal{N}_3(s,X(s))dB(s),\ t\in[0,T]. \end{aligned} \tag{18}$$

Our problem is to solve equation (18). First, we define

$$\left\{ \begin{array}{l} p_1(t)=\mathcal{N}_1\big(t,f(t)+\int_0^T k_1(s,t)p_1(s)ds+\int_0^t k_2(s,t)p_2(s)ds+\int_0^t k_3(s,t)p_3(s)dB(s)\big) \\ p_2(t)=\mathcal{N}_2\big(t,f(t)+\int_0^T k_1(s,t)p_1(s)ds+\int_0^t k_2(s,t)p_2(s)ds+\int_0^t k_3(s,t)p_3(s)dB(s)\big) \\ p_3(t)=\mathcal{N}_3\big(t,f(t)+\int_0^T k_1(s,t)p_1(s)ds+\int_0^t k_2(s,t)p_2(s)ds+\int_0^t k_3(s,t)p_3(s)dB(s)\big). \end{array} \right. \tag{19}$$

If we approximate the functions $f(t),k_1(s,t),k_2(s,t),k_3(s,t)$ and $p_1(t),p_2(t),p_3(t)$ in terms of block-pulse functions, then we have

$$\left\{ \begin{array}{l} X(t)\simeq\Phi^T(t)X\simeq X^T\Phi(t), \quad f(t)\simeq\Phi^T(t)F\simeq F^T\Phi(t), \\ p_1(t)\simeq\Phi^T(t)P_1\simeq P_1^T\Phi(t), \quad p_2(t)\simeq\Phi^T(t)P_2\simeq P_2^T\Phi(t), \\ p_3(t)\simeq\Phi^T(t)P_3\simeq P_3^T\Phi(t), \quad k_1(s,t)\simeq\Psi^T(s)K_1\Phi(t)=\Phi^T(t)K_1^T\Psi(s), \\ k_2(s,t)\simeq\Psi^T(s)K_2\Phi(t)=\Phi^T(t)K_2^T\Psi(s), \quad k_3(s,t)\simeq\Psi^T(s)K_3\Phi(t)=\Phi^T(t)K_3^T\Psi(s), \end{array} \right.$$

where the vectors X,F,P_1,P_2,P_3 and matrix K_1,K_2 and K_3 are stochastic block pulse coefficients of $X(t),f(t),p_1(t),p_2(t),p_3(t)$ and $K_1(s,t),K_2(s,t),K_3(s,t)$ respectively.

First, by taking $m_1 = m_2$ and with substituting above approximation in (18), we get

$$\begin{aligned}\Phi^T(t)X &\simeq \Phi^T(t)F + \int_0^T \Phi^T(t)K_1^T\Phi(s)\Phi^T(s)P_1 ds \\ &+ \int_0^t \Phi^T(t)K_2^T\Phi(s)\Phi^T(s)P_2 ds + \int_0^t \Phi^T(t)K_3^T\Phi(s)\Phi^T(s)P_3 dB(s)\end{aligned}$$

and

$$\begin{aligned}\Phi^T(t)X &\simeq \Phi^T(t)F + P_1^T\Big(\int_0^T \Phi(s)\Phi^T(s)ds\Big)K_1\Phi(t) + P_2^T\Big(\int_0^t \Phi(s)\Phi^T(s)ds\Big)K_2\Phi(t) \\ &+ P_3^T\Big(\int_0^t \Phi(s)\Phi^T(s)dB(s)\Big)K_3\Phi(t)\end{aligned}$$

we have $\int_0^T \Phi(s)\Phi^T(s)ds = hI$. Then

$$P_1^T\Big(\int_0^T \Phi(s)\Phi^T(s)ds\Big)K_1\Phi(t) = P_1^T hIK_1\Phi(t) = hP_1^T K_1\Phi(t), \tag{20}$$

and

$$P_2^T\Big(\int_0^t \Phi(s)\Phi^T(s)ds\Big)K_2\Phi(t) = P_2^T A_1\Phi(t), \tag{21}$$

where

$$A_1 = \begin{pmatrix} \lambda hK_2^{11} & hK_2^{12} & hK_2^{13} & \dots & hK_2^{1m} \\ 0 & \lambda hK_2^{22} & hK_2^{23} & \dots & hK_2^{2m} \\ 0 & 0 & \lambda hK_2^{33} & \dots & hK_2^{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda hK_2^{mm} \end{pmatrix}_{m\times m}, \tag{22}$$

and K_2^{ij}, $i, j = 1, \dots, m$ are the coefficients of the matrix K_2. Also, we get

$$P_3^T\Big(\int_0^t \Phi(s)\Phi^T(s)dB(s)\Big)K_3\Phi(t) = P_3^T A_3\Phi(t), \tag{23}$$

where

$$A_3 = \begin{pmatrix} K_3^{11}B(\lambda h) & K_3^{12}B(h) & K_3^{13}B(h) & \dots & K_3^{1m}B(h) \\ 0 & K_3^{22}\Big(B((\lambda+1)h) - B(h)\Big) & K_3^{23}\Big(B(2h) - B(h)\Big) & \dots & K_3^{2m}\Big(B(2h) - B(h)\Big) \\ 0 & 0 & K_3^{33}\Big(B((\lambda+2)h) - B(2h)\Big) & \dots & K_3^{3m}\Big(B(3h) - B(2h)\Big) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & K_3^{mm}\Big(B\Big((\lambda+m-1)h\Big) - B((m-1)h)\Big) \end{pmatrix}_{m\times m}. \tag{24}$$

With substituting relations (20), (21) and (23) in equation (20), we obtain

$$X(t) \simeq f(t) + hP_1^T K_1\Phi(t) + P_2^T A_1\Phi(t) + P_3^T A_3\Phi(t). \tag{25}$$

Now, with substituting relations (20), (21) and (23) in (19), we get

$$\begin{cases} P_1^T\Phi(t) = \mathcal{N}_1\Big(t, \Phi^T(t)F + hP_1^T K_1\Phi(t) + P_2^T A_1\Phi(t) + P_3^T A_3\Phi(t)\Big) \\ P_2^T\Phi(t) = \mathcal{N}_2\Big(t, \Phi^T(t)F + hP_1^T K_1\Phi(t) + P_2^T A_1\Phi(t) + P_3^T A_3\Phi(t)\Big) \\ P_3^T\Phi(t) = \mathcal{N}_3\Big(t, \Phi^T(t)F + hP_1^T K_1\Phi(t) + P_2^T A_1\Phi(t) + P_3^T A_3\Phi(t)\Big). \end{cases} \tag{26}$$

Now, we collocate equation (26) at m Newton -Cotes nodes defined as $t_i = \dfrac{2i-1}{2m}$, $i = 1, 2, \dots, m$. Then, we obtain nonlinear system of algebraic equations, given by

$$\begin{cases} P_1^T \Phi(t_i) = \mathcal{N}_1\Big(t_i, \Phi^T(t_i)F + hP_1^T K_1 \Phi(t_i) + P_2^T A_1 \Phi(t_i) + P_3^T A_3 \Phi(t_i)\Big) \\ P_2^T \Phi(t_i) = \mathcal{N}_2\Big(t_i, \Phi^T(t_i)F + hP_1^T K_1 \Phi(t_i) + P_2^T A_1 \Phi(t_i) + P_3^T A_3 \Phi(t_i)\Big) \\ P_3^T \Phi(t_i) = \mathcal{N}_3\Big(t_i, \Phi^T(t_i)F + hP_1^T K_1 \Phi(t_i) + P_2^T A_1 \Phi(t_i) + P_3^T A_3 \Phi(t_i)\Big), \end{cases} \tag{27}$$

where the unknown vectors are P_1, P_2 and P_3.
The approximate solution of equation (1) can be obtained as

$$X(t) \simeq \Phi^T(t)F + hP_1^T K_1 \Phi(t) + P_2^T A_1 \Phi(t) + P_3^T A_3 \Phi(t). \tag{28}$$

5 Solving multidimensional stochastic Itô Volterra-Fredholm integral equation

In this section, Block-pulse functions will be applied to find an approximate solution for the following stochastic Volterra-Fredholm integral equation with multi-stochastic terms.

$$\begin{aligned} X(t) &= f(t) + \int_0^T k_1(s,t)\mathcal{N}_1(s, X(s))ds + \int_0^t k_2(s,t)\mathcal{N}_2(s, X(s))ds \\ &+ \sum_{j=1}^{l} \int_0^t k_{3j}(s,t)\mathcal{N}_{3j}(s, X(s))dB_j(s), \end{aligned} \tag{29}$$

where $B(t) = (B_1(t), B_2(t), \dots, B_n(t))$ is an l-dimensional Brownian motion process. Let

$$\begin{cases} p_1(t) = \mathcal{N}_1(t, f(t) + \int_0^T k_1(s,t)p_1(s)ds + \int_0^t k_2(s,t)p_2(s)ds + \sum_{j=1}^{l}\int_0^t k_{3j}(s,t)p_{3j}(s)dB_j(s)) \\ p_2(t) = \mathcal{N}_2(t, f(t) + \int_0^T k_1(s,t)p_1(s)ds + \int_0^t k_2(s,t)p_2(s)ds + \sum_{j=1}^{l}\int_0^t k_{3j}(s,t)p_{3j}(s)dB_j(s)) \\ p_3(t) = \mathcal{N}_3(t, f(t) + \int_0^T k_1(s,t)p_1(s)ds + \int_0^t k_2(s,t)p_2(s)ds + \sum_{j=1}^{l}\int_0^t k_{3j}(s,t)p_{3j}(s)dB_j(s)). \end{cases} \tag{30}$$

As in previous section, we approximate $f(t), k_1(s,t), k_2(s,t), k_{3j}(s,t)$ and $p_1(t), p_2(t), p_{3j}(t)$ in terms of block-pulse functions, then we have, for $j = 1, \dots, l$,

$$\begin{cases} X(t) \simeq \Phi^T(t)X \simeq X^T\Phi(t), \quad f(t) \simeq \Phi^T(t)F \simeq F^T\Phi(t), \\ p_1(t) \simeq \Phi^T(t)P_1 \simeq P_1^T\Phi(t), \quad p_2(t) \simeq \Phi^T(t)P_2 \simeq P_2^T\Phi(t), \\ p_{3j}(t) \simeq \Phi^T(t)P_{3j} \simeq P_{3j}^T\Phi(t), \quad k_1(s,t) \simeq \Psi^T(s)K_1\Phi(t) = \Phi^T(t)K_1^T\Psi(s), \\ k_2(s,t) \simeq \Psi^T(s)K_2\Phi(t) = \Phi^T(t)K_2^T\Psi(s), \quad k_{3j}(s,t) \simeq \Psi^T(s)K_{3j}\Phi(t) = \Phi^T(t)K_{3j}^T\Psi(s), \end{cases}$$

where the vectors X, F, P_1, P_2, P_{3j} and matrix K_1, K_2 and K_{3j} are stochastic block pulse coefficients of $X(t), f(t), p_1(t), p_2(t), p_{3j}(t)$ and $K_1(s,t), K_2(s,t), K_{3j}(s,t)$ respectively.

By taking $m_1 = m_2$ and with substituting above approximation in (29), we get

$$\begin{aligned} \Phi^T(t)X &\simeq \Phi^T(t)F + \int_0^T \Phi^T(t)K_1^T\Phi(s)\Phi^T(s)P_1 ds + \int_0^t \Phi^T(t)K_2^T\Phi(s)\Phi^T(s)P_2 ds \\ &+ \sum_{j=1}^{l}\int_0^t \Phi^T(t)K_{3j}^T\Phi(s)\Phi^T(s)P_{3j}dB_j(s), \end{aligned} \tag{31}$$

hence

$$\begin{aligned}\Phi^T(t)X &\simeq \Phi^T(t)F + P_1^T\Big(\int_0^T \Phi(s)\Phi^T(s)ds\Big)K_1\Phi(t) + P_2^T\Big(\int_0^t \Phi(s)\Phi^T(s)ds\Big)K_2\Phi(t)\\ &+ \sum_{j=1}^{l} P_{3j}^T\Big(\int_0^t \Phi(s)\Phi^T(s)dB(s)\Big)K_{3j}\Phi(t).\end{aligned}$$

We have $\int_0^T \Phi(s)\Phi^T(s)ds = hI$. Then, we obtain

$$P_1^T\Big(\int_0^T \Phi(s)\Phi^T(s)ds\Big)K_1\Phi(t) = P_1^T hIK_1\Phi(t) = hP_1^T K_1\Phi(t). \tag{32}$$

$$P_2^T\Big(\int_0^t \Phi(s)\Phi^T(s)ds\Big)K_2\Phi(t) = P_2^T A_1\Phi(t), \tag{33}$$

where the matrix A_1 is given by (22) and K_2^{ij}, $i, j = 1, \ldots, m$ are the coefficients of the matrix K_2. Also

$$\sum_{j=1}^{l} P_{3j}^T\Big(\int_0^t \Phi(s)\Phi^T(s)dB_j(s)\Big)K_{3j}\Phi(t) = \sum_{j=1}^{m} P_{3j}^T A_{3j}\Phi(t), \tag{34}$$

where

$$A_{3j} = \begin{pmatrix} K_{3j}^{11}B_j(\lambda h) & K_{3j}^{12}B_j(h) & K_{3j}^{13}B_j(h) & \ldots & K_{3j}^{1m}B_j(h)\\ 0 & K_{3j}^{22}\Big(B_j((\lambda+1)h) - B_j(h)\Big) & K_{3j}^{23}\Big(B_j(2h) - B_j(h)\Big) & \ldots & K_{3j}^{2m}\Big(B_j(2h) - B_j(h)\Big)\\ 0 & 0 & K_{3j}^{33}\Big(B_j((\lambda+2)h) - B_j(2h)\Big) & \ldots & K_{3j}^{3m}\Big(B_j(3h) - B_j(2h)\Big)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & K_{3j}^{mm}\Big(B_j\Big((\lambda+m-1)h\Big) - B_j((m-1)h)\Big)\end{pmatrix}. \tag{35}$$

With substituting relations (32), (33) and (34) in equation (29), we obtain

$$X(t) \simeq \Phi^T(t)F + hP_1^T K_1\Phi(t) + P_2^T A_1\Phi(t) + \sum_{j=1}^{l} P_{3j}^T A_{3j}\Phi(t). \tag{36}$$

and we get

$$\begin{cases} P_1^T\Phi(t) = \mathcal{N}_1\Big(t, \Phi^T(t)F + hP_1^T K_1\Phi(t) + P_2^T A_1\Phi(t) + \sum_{j=1}^{l} P_{3j}^T A_{3j}\Phi(t)\Big)\\ P_2^T\Phi(t) = \mathcal{N}_2\Big(t, \Phi^T(t)F + hP_1^T K_1\Phi(t) + P_2^T A_1\Phi(t) + \sum_{j=1}^{l} P_{3j}^T A_{3j}\Phi(t)\Big)\\ P_{3j}^T\Phi(t) = \mathcal{N}_{3j}\Big(t, \Phi^T(t)F + hP_1^T K_1\Phi(t) + P_2^T A_1\Phi(t) + \sum_{j=1}^{l} P_{3j}^T A_{3j}\Phi(t)\Big). \end{cases} \tag{37}$$

Now, we collocate equation (37) at m Newton -Cotes nodes defined as $t_i = \dfrac{2i-1}{2m}$, $i = 1, 2, \ldots, m$. Then, we get the following nonlinear system of algebraic equations

$$\begin{cases} P_1^T\Phi(t_i) = \mathcal{N}_1\Big(t_i, \Phi^T(t_i)F + hP_1^T K_1\Phi(t_i) + P_2^T A_1\Phi(t_i) + \sum_{j=1}^{l} P_{3j}^T A_{3j}\Phi(t)\Big)\\ P_2^T\Phi(t_i) = \mathcal{N}_2\Big(t_i, \Phi^T(t_i)F + hP_1^T K_1\Phi(t_i) + P_2^T A_1\Phi(t_i) + \sum_{j=1}^{l} P_{3j}^T A_{3j}\Phi(t)\Big)\\ P_{3j}^T\Phi(t_i) = \mathcal{N}_{3j}\Big(t_i, \Phi^T(t_i)F + hP_1^T K_1\Phi(t_i) + P_2^T A_1\Phi(t_i) + \sum_{j=1}^{l} P_{3j}^T A_{3j}\Phi(t)\Big) \end{cases} \tag{38}$$

where the unknown vectors are P_1, P_2 and P_{3j}. The approximate solution of equation (1) is as follow

$$X(t) \simeq \Phi^T(t)F + hP_1^T K_1\Phi(t) + P_2^T A_1\Phi(t) + \sum_{j=1}^{l} P_{3j}^T A_{3j}\Phi(t).$$

6 Error Analysis

In this section, we will show that the rate of convergence of the proposed method for solving Itô-Volterra-Fredholm integral equations with multi independant stochastic terms is $O(h)$.

Theorem 6.1. *Suppose that $g(t)$ is an arbitrary real bounded function, which is square integrable in the interval $[0,1]$, and $e(t) = g(t) - \hat{g}_m(t)$, $t \in I = [0,1]$, where $\hat{g}_m(t) = \sum_{i=1}^{m} g_i\Phi_i(t)$ is the block pulse series of $g(t)$. Then, $\|e(t)\| \le \frac{h}{2\sqrt{3}} \sup_{t\in I} |g'(t)|$.*

Proof. See [8].

Theorem 6.2. *Suppose that $g(t,s) \in L^2([0,1]\times[0,1])$ and $e(s,t) = g(s,t) - \hat{g}_m(s,t)$, $(s,t) \in D = [0,1]\times[0,1]$, where $\hat{g}_m(s,t) = \sum_{i=1}^{m}\sum_{j=1}^{m} g_{ij}\Psi_i(t)\Phi_j(t)$ is the bolck-pulse series of $g(s,t)$. Then, $\|e(s,t)\| \le \frac{h}{2\sqrt{3}}\left(\sup_{(x,y)\in D} |g_s'(x,y)|^2 + \sup_{(x,y)\in D} |g_t'(x,y)|^2\right)^{\frac{1}{2}}$.*

Proof. See [8].
Suppose that $g(t)$ is any arbitrary real bounded function which is square integrable in the interval $[0,1]$ and $\hat{g}_m(t)$ be the approximation of $g(t)$ by using block-Pulse functions. Using theorem 6.1, we get

$$||e(t)|| = ||g(t) - \hat{g}_m(t)|| \le ch. \tag{39}$$

Let $g(s,t) \in L^2([0,1]\times[0,1])$ and $e(s,t) = g(s,t) - \hat{g}(s,t)$, $s,t \in [0,1]\times[0,1]$ and $\hat{g}_m(s,t)$ is the approximation of $g(s,t)$ by using block-Pulse functions. Then, by theorem 6.2, we have

$$||e(s,t)|| \le c'h. \tag{40}$$

In addition nonlinear terms $\mathcal{N}_1(t,X(t))$, $\mathcal{N}_2(t,X(t))$ and $\mathcal{N}_3(t,X(t))$ satisfy Lipschitz and linear growth conditions such that

$$|\mathcal{N}_i(t,x_1) - \mathcal{N}_i(t,x_2)| \le L_i|x_1 - x_2|, \quad i = 1,2,3, \tag{41}$$

and

$$\sum_{i=1}^{3} |\mathcal{N}_i(t,x)| < d_i\big(1 + |x|\big), \tag{42}$$

these assumptions ensure the existence of a unique solution of the sde (1).

Theorem 6.3. *Let $X(t)$ be the exact solution and $\hat{X}_m(t)$ be the approximation solution of equation (1) which is the solution of (28), we suppose that, the conditions (41) and (42) are satisfied and*

1. *$P\{\omega \in \Omega : ||X(t,\omega)|| < k\} = 1$,*
2. *$||\mathcal{N}_i(t,X(t))|| \le \rho_i$, $i = 1,2,3$, $t \in [0,1]$,*
3. *$||k_i(s,t)|| \le M_i$, $\quad (s,t) \in [0,1]\times[0,1]$, $i = 1,2,3$,*

4. $\sum_{i=1}^{2} L_i\Big(M_i+\Gamma_i(h)\Big)+L_3\Big(M_3+\Gamma_3(h)\Big)\sup_{t\in[0,1]}|B(t)|<1.$

then

$$\begin{aligned}||X(t)-\hat{X}_m(t)|| &\leq \frac{\gamma(h)+\sum_{i=1}^{2}\Big(\Gamma_i(h)\rho_i+\Big(M_i+\Gamma_i(h)\Big)\gamma_i(h)\Big)}{1-\sum_{i=1}^{2}L_i\Big(M_i+\Gamma_i(h)\Big)-L_3\Big(M_3+\Gamma_3(h)\Big)\sup_{t\in[0,1]}|B(t)|}\\ &+ \frac{\Big[\Big(M_3+\Gamma_3(h)\Big)\gamma_3(h)+\Gamma_3(h)\rho_3\Big]\sup_{t\in[0,1]}|B(t)|}{1-\sum_{i=1}^{2}L_i\Big(M_i+\Gamma_i(h)\Big)-L_3\Big(M_3+\Gamma_3(h)\Big)\sup_{t\in[0,1]}|B(t)|}\end{aligned}$$

where

- $||X||^2=E[|X(t)|^2]$,
- $||f(t)-\hat{f}_m(t)||\leq\gamma(h),\ ||p_i^m(s)-\hat{p}_i(s)||\leq\gamma_i(h),\ ||k_i(s,t)-\hat{k}_i^m(s,t)||\leq\Gamma_i(h).$

and $\gamma(h)$, $\gamma_i(h)$ *and* $\Gamma_i(h)$ *are given by theorem* 6.1 *and theorem* 6.2.

Proof. Let $\hat{p}_i(s)=\hat{\mathcal{N}}_i((s,\hat{X}_m(s))$ and $\hat{k}_i^m(s,t)$, $i=1,2,3$ are the approximation solution of $p_i(s)$ and $k_i(s,t)$ by block-pulse functions. Also, let $p_i^m(s)=\mathcal{N}_i(s,\hat{X}_m(s))$. From equation (1), we get

$$\begin{aligned}X(t)-\hat{X}_m(t) &= f(t)-\hat{f}_m(t)+\int_0^T\Big(k_1(s,t)p_1(s)-\hat{k}_1^m(s,t)\hat{p}_1(s)\Big)ds\\ &+ \int_0^t\Big(k_2(s,t)p_2(s)-\hat{k}_2^m(s,t)\hat{p}_2(s)\Big)ds\\ &+ \int_0^t\Big(k_3(s,t)p_3(s)-\hat{k}_3^m(s,t)\hat{p}_3(s)\Big)dB(s)\end{aligned}$$

then by mean value theorem, we can write

$$\begin{aligned}||X(t)-\hat{X}_m(t)|| &\leq ||f(t)-\hat{f}_m(t)||+T||k_1(s,t)p_1(s)-\hat{k}_1^m(s,t)\hat{p}_1(s)||+t||k_2(s,t)p_2(s)-\hat{k}_2^m\hat{p}_2(s)||\\ &+ |B(t)|||k_3(s,t)p_3(s)-\hat{k}_3^m(s,t)\hat{p}_3(s)||\end{aligned}$$

by using conditions (39) and (40) and conditions of the theorem 6.3, we get

$$\begin{aligned}||p_i(s)-\hat{p}_i(s)|| &= ||p_i(s)-p_i^m(s)+p_i^m(s)-\hat{p}_i(s)||\\ &\leq ||p_i(s)-p_i^m(s)||+||p_i^m(s)-\hat{p}_i(s)||\leq L_i||X(s)-\hat{X}_m(s)||+\gamma_i(h),\quad i=1,2,3.\end{aligned}$$

then

$$\begin{aligned}||k_1(s,t)p_1(s)-\hat{k}_1^m(s,t)\hat{p}_1(s)|| &\leq ||k_1(s,t)||\,||p_1(s)-\hat{p}_1(s)||\\ &+ ||k_1(s,t)-\hat{k}_1^m(s,t)||\Big(||p_1(s)-\hat{p}_1(s)||+||p_1(s)||\Big).\end{aligned}$$

Consequently

$$||k_1(s,t)p_1(s)-\hat{k}_1^m(s,t)\hat{p}_1(s)|| \leq \Big(M_1+\Gamma_1(h)\Big)\Big(L_1||X(s)-\hat{X}_m(s)||+\gamma_1(h)\Big)+\Gamma_1(h)\rho_1,$$

and

$$||k_2(s,t)p_2(s) - \hat{k}_2^m(s,t)\hat{p}_2(s)|| \quad \leq \Big(M_2 + \Gamma_2(h)\Big)\Big(L_2||X(s) - \hat{X}_m(s)|| + \gamma_2(h)\Big) + \Gamma_2(h)\rho_2,$$

and

$$||k_3(s,t)p_3(s) - \hat{k}_3^m(s,t)\hat{p}_3(s)|| \quad \leq \Big(M_3 + \Gamma_3(h)\Big)\Big(L_3||X(s) - \hat{X}_m(s)|| + \gamma_3(h)\Big) + \Gamma_3(h)\rho_3.$$

Hence

$$\begin{aligned} ||X(t) - \hat{X}_m(t)|| &\leq \gamma(h) + T\Big[\Big(M_1 + \Gamma_1(h)\Big)\Big(L_1||X(s) - \hat{X}_m(s)|| + \gamma_1(h)\Big) + \Gamma_1(h)\rho_1\Big] \\ &+ t\Big[\Big(M_2 + \Gamma_2(h)\Big)\Big(L_2||X(s) - \hat{X}_m(s)|| + \gamma_2(h)\Big) + \Gamma_2(h)\rho_2\Big] \\ &+ |B(t)|\Big[\Big(M_3 + \Gamma_3(h)\Big)\Big(L_3||X(s) - \hat{X}_m(s)|| + \gamma_3(h)\Big) + \Gamma_3(h)\rho_3\Big]. \end{aligned}$$

Then it follows from the last inequality that the conclusion of theorem is true.

7 Illustrative examples

To illustrate the effictiveness of the proposed method, some examples are carried out in this section. In this regard, we have presented tables 1 to 6. All results are computed by using a program written in Matlab. We compare the values of approximate solution and exact solution at some selected points via definition of absolute error which defined as $e(t) = |X(t) - \hat{X}(t)|$, $t \in [0, 1]$, where $X(t)$ and $\hat{X}(t)$ denote exact and approximate solutions, respectively.

Example 7.1 Consider the linear Volterra integral equation

$$X(t) = \frac{1}{12} + \int_0^t \cos(s)X(s)ds + \int_0^t \sin(s)X(s)dB(s),\ s, t \in [0, 0.5).$$

The exact solution is $X(t) = \frac{1}{12}\exp(-\frac{t}{4} + \sin(t) + \frac{\sin(2t)}{8} + \int_0^t \sin(s)dB(s))$. In this example, we take $X_0 = \frac{1}{12}$, $m = 5$ and $\lambda = 1/2$. The results are summarized in Table 1.

Table 1: Computed errors for Example 7.1 for $M = 10, 20, 40$ simulations.

t	$M = 10$	$M = 20$	$M = 40$
0	0.0087237	0.0089509	0.0090448
0.1	0.0168600	0.0153020	0.0160146
0.2	0.0284360	0.0267211	0.0271932
0.3	0.0389988	0.0343539	0.0306550
0.4	0.0471885	0.0506237	0.0446878

Example 7.2 Let us consider the problem

$$X(t) = X_0 + \int_0^t a^2\cos(X(s))\sin^3(X(s))ds - a\int_0^t \sin^2(X(s))dB(s),\ t \in [0, 1].$$

The exact solution is $X(t) = arccot(aB(s) + \cot(X0))$. By taking $m = 8$, $a = 1/8$, $\lambda = 1/2$, $X_0 = 1, 0.1, 0.01$ and $M = 30$ simulations, the numerical results are given in table 2.

Table 2: Errors obtained from block-pulse functions approximation of Example 7.2.

t	$X_0 = 1$	$X_0 = 0.1$	$X_0 = 0.01$
0	0.0000000000	0.00000000000	0.00000000000
0.1	$2.0740618\,E-2$	$3.0013044E-4$	$2.9267713E-6$
0.2	$3.144773\,E-2$	$3.6286150E-4$	$5.4044779E-6$
0.3	$3.8726924\,E-2$	$5.0387071E-4$	$7.0485413E-6$
0.4	$4.4614610\,E-2$	$5.6967794E-4$	$7.5217966E-6$
0.5	$5.6207276\,E-2$	$7.2342694E-4$	$7.8643065E-6$
0.6	$5.7219980\,E-2$	$8.7314034E-4$	$8.6918163E-6$
0.7	$6.0158915\,E-2$	$9.8637195E-4$	$8.9194592E-6$
0.8	$6.8378217\,E-2$	$1.1110927E-4$	$9.8411892E-6$
0.9	$6.9958887\,E-2$	$1.2330252E-4$	$9.6181820E-6$

Example 7.3 Let given the logistic differential equation $u'(t) = \rho u(t)(1-u(t))$, $t > 0$, $\rho > 0$, $u(0) = u_0$, $u_0 > 0$. The exact solution to this problem is given by $u(t) = \dfrac{u_0}{(1-u_0)e^{-\rho t} + u_0}$. By taking $m = 7$, $\lambda = 1/2$, and with different values of ρ and $u_0 = 0.85$. The numerical results are given in Tables 3-4.

Table 3: Errors obtained from block-pulse functions approximation of Example 7.3.

t	$\rho = 1/20$	$\rho = 1/8$	$\rho = 1/2$
0	9.06177 E-4	2.24858 E-3	8.66835 E-3
0.01	8.42438 E-4	2.08928 E-3	8.03197 E-3
0.02	7.78722 E-4	1.93011 E-3	7.39781 E-3
0.03	7.15027 E-4	1.77109 E-3	6.76588 E-3
0.04	6.51355 E-4	1.61220 E-3	6.13616 E-3
0.05	5.87706 E-4	1.45345 E-3	5.50866 E-3
0.06	5.24079 E-4	1.29484 E-3	4.83388 E-3
0.07	4.60473 E-4	1.13637 E-3	4.26030 E-3
0.08	3.9689 E-4	9.78045 E-4	3.63943 E-3
0.09	3.33330 E-4	8.19853 E-4	3.020 76 E-3

Example 7.4 (The basic Black-Scholes model) Let given the following linear stochastic equation $dX(t) = \gamma X(t)dt + \mu X(t)dW(t)$, $X(0) = X_0$, $t \in [0,1]$. Where the exact solution is given by $X(t) = exp((\gamma - \frac{1}{2}\mu^2)t + \mu B(t))$. The results obtained for $m = 7$, $X_0 = 1$, and $\lambda = 1/2$ of this example are given in Table 5.

Example 7.5 Consider the following Volterra integral equation

$$dX(t) = \frac{1}{1000}t^3X(t)dt - \frac{1}{20}t^3X(t)dB(t),\ X(0) = -\frac{1}{50}. \tag{43}$$

The exact solution is $X(t) = -\frac{1}{50}\exp(\frac{1}{4000}t^4 - \frac{1}{2800}t^7 - \frac{1}{20}\int_0^t s^3dB(s))$, $s \in [0,T]$, $T < 1$. The numerical results of this example by taking $m = 7$ and $m = 5$ with different values of λ are given in Tables 6-7.

Table 4: Errors obtained from block-pulse functions approximation of Example 7.3.

t	$\rho = 1/20$	$\rho = 1/8$	$\rho = 2$
0	9.06177 E-4	2.24858 E-3	3.01408 E-2
0.1	2.69792 E-4	6.61800 E-4	6.38301 E-3
0.2	8.02490 E-4	1.94310 E-3	1.29611 E-2
0.4	1.72776 E-4	3.97798 E-4	1.86854 E-3
0.5	4.37912 E-4	1.03124 E-3	3.80215 E-3
0.6	7.00768 E-4	1.46466 E-4	4.96487 E-3
0.8	6.00998 E-4	7.65879 E-4	2.19125 E-3
0.9	6.26407 E-4	1.37164 E-3	3.00982 E-4

Table 5: Errors obtained from block-pulse functions approximation of Example 7.4.

t	$\gamma = -100$, $\mu = 1$	$\gamma = -100$, $\mu = 10$	$\gamma = -100$, $\mu = 0$
0	9.33523 E-1	9.22224 E-1	9.34579 E-1
0.1	6.64558 E-2	7.77751 E-2	6.53751 E-2
0.2	3.73369 E-5	3.26583 E-2	4.27984 E-3
0.3	4.60091 E-6	2.03320 E-2	2.79990 E-4
0.4	4.60091 E-6	2.03320 E-2	2.79990 E-4
0.4	4.04514 E-7	5.95492 E-3	1.83172 E-5
0.5	3.43206 E-8	1.27450 E-3	1.19831 E-6
0.6	3.43206 E-8	1.27450 E-3	1.19831 E-6
0.8	5.06274 E-9	1.07329 E-3	7.83944 E-8
0.9	2.69752 E -10	5.84276 E-5	5.12861 E-9

Table 6: Errors obtained from block-pulse functions approximation of Example 7.5.

t	$\lambda = 1/4$, $m = 7$	$\lambda = 1/2$, $m = 7$	$\lambda = 3/4$, $m = 7$
0	1.23058 E-7	1.74893 E-7	2.14667 E-7
0.1	5.13511 E-7	5.65345 E-7	6.05119 E-7
0.2	1.27859 E-5	1.46883 E-5	1.61483 E-5
0.3	2.72661 E-5	2.81524 E-5	2.88325 E-5
0.4	3.97351 E-5	4.06214 E-5	4.13016 E-5
0.4	5.58231 E-6	6.27466 E-6	6.80589 E-6
0.5	6.82685 E-5	1.11223 E-4	1.44309 E-4
0.6	4.98187 E-5	6.86356 E-5	2.62221 E-5
0.8	6.75617 E-4	6.62710 E-4	6.52796 E-4
0.9	4.68001 E -4	4.64133 E-4	4.61167 E-4

Table 7: Errors obtained from block-pulse functions approximation of Example 7.5.

t	$\lambda = 1/2,\, m = 5$	$\lambda = 3/4,\, m = 5$
0	5.66585 E-7	6.95715 E-7
0.1	4.52901 E-7	5.82017 E-7
0.2	2.88211 E-5	3.35683 E-5
0.3	2.20147 E-5	2.67619 E-5
0.4	2.34378 E-5	2.56531 E-5
0.4	7.22459 E-5	7.00306 E-5
0.5	3.49573 E-5	3.71726 E-5
0.6	4.43868 E-4	4.42164 E-4
0.8	1.22716 E-3	1.34211 E-3
0.9	8.35702 E -4	9.50645 E-4

8 Conclusion

In this paper, an integral collocation approach based on block-pulse functions is introduced for solving numerically stochastic Itô-Fredholm-Volterra integral equations. The properties of block-pulse functions are used to reduce the proposed problems to system of algebraic equations which is solved by suitable numerical method. Some advantages of the proposed method are: The implementation of the method is easier and the effort required is very low, while the accuracy is high. When the solution is sufficiently smooth, a small number of basis functions is enough to obtain a high accuracy solution. Using block-pulse functions as basis functions to solve nonlinear stochastic Itô-Volterra-Fredholm integral equations with white noise source is very simple and effective in comparaison with other methods. Its applicability and accuracy is checked on some examples.

References

[1] L. Arnold, *Stochastic Differential Equations*, Wiley, New York, 1974.

[2] S. Bazm, *Bernoulli polynomials for the numerical solution of some classes of linear and nonlinear integral equations*, Journal of Computational and Applied Mathematics. 275(2015) 44-60.

[3] A.H. Bhrawy, E. Tohidi, F. Soleymani, *A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals*, Appl. Math. Comput. 219 (2) (2012) 482-497.

[4] Kloeden, P.E, E. Platen, *Numerical solution of stochastic differential equations*, Springer, Berlin, 1992.

[5] K. Maleknejad, B. Rahimi, *Modification of block pulse functions and their application to solve numerically Volterra integral equation of the first kind*, Commun. Nonlinear Sci. Numer. Simul. 16 (6) (2011) 2469-2477.

[6] K. Maleknejad, E. Hashemizadeh, B. Basirat, *Computational method based on Bernstein operational matrices for nonlinear Volterra-Fredholm- Hammerstein integral equations*, Commun. Nonlinear Sci. Numer. Simul. 17 (1) (2012) 52-61.

[7] K. Maleknejad, R. Mollapourasl, M. Alizadeh, *Numerical solution of the Volterra type integral equation of the first kind with wavelet basis*, Appl. Math.Comput. 194 (2) (2007) 400-405.

[8] K. Maleknejad, M. Khodabin, M. Rostami, *Numerical method for solving $m-$dimensional stochastic Itô Volterra integral equations by stochastic operational matrix based on block-pulse functions*, Comput.Math. Appl, 63(2012) 133-143.

[9] G.N. Milstein, *Numerical Integration of stochastic differential equations, Mathematics and Its application*, Kluwer, Dordrecht, 1995.

[10] G. N. Milstein and M. V. Tretyakov, *Stochastic Numerics for Mathematical Physics, Scientific Computing*, Springer-Verlag, New York,2004.

[11] Wazwaz.A.M, *Linear and nonlinear integral equations, Methods and applications*, Higher education, Springer 2011.