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ON DISCRETE ONE-HARMONIC MAP FLOWS WITH VALUES INTO AN EMBEDDED MANIFOLD ON A MULTI-DIMENSIONAL DOMAIN

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Abstract. The very singular diffusion equation

$$\frac{\partial u}{\partial t} = -\pi_u \left(-\mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) \right)$$

is called one-harmonic map flow equation. By effect of the very singular diffusivity term $\mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right)$, one-harmonic map flow equation admits piecewise constant solutions. Hence, piecewise constant discretizations are one of natural discretizations which have consideration of characteristics of this equation. In this paper, we study the piecewise constant discretization, proposed by Y. Giga, H. Kuroda and N. Yamazaki (2005), for the Neumann problem of the one-harmonic map flow equation. Since they formulated a discretized version of one-harmonic map flow equations as differential inclusions, uniqueness and existence of its solutions are not clear. We establish existence and uniqueness of solutions to this discretized problem.

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1. Introduction

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ and let M be a manifold embedded into $\mathbf{R}^n$. We consider the system of partial differential equations

$$\frac{\partial u}{\partial t} = -\pi_u \left(-\mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) \right) \tag{HF}$$

for vector fields $u : \Omega \times (0, T) \to M \subset \mathbf{R}^n$, where ∇ is the gradient on $\mathbf{R}^d$ and π_p is the orthogonal projection from $\mathbf{R}^n$ to the tangent space $T_pM \subset \mathbf{R}^n$ at $p \in M$. The equation (HF) is called the one-harmonic map flow equation from Ω to M. This equation and related ones are commonly known as very singular diffusion equations ([15], [16]) since the evolution speed is determined by a nonlocal quantity at $\nabla u = 0$.

One-harmonic map flow equations are proposed as methods of image processing and as a continuum model of grain boundary. In the case of $M = \mathbf{S}^2$, unite two-dimensional sphere, such equations are proposed as a method of de-noising of color images with preserving sharp edge and the chromaticity ([31],[32]). In the case of $M = \mathbf{SO}(3)$, the space of three-dimensional rotations, such equations are suggested for continuum model of grain boundary ([26], [24], [25]). In the case of $M = \mathbf{SPD}(3)$, the space of symmetric positive definite three-dimensional matrices, such equations are proposed for denosing DT-MRI ([5], [9], [30], [33]).

1.1. Mathematical analysis

Mathematical analysis of one-harmonic map flows has been done by several authors. Notions of solutions to one-harmonic map flow equations are not clear because of a singularity at $\nabla u = 0$. In [20], Y. Giga and H. Kuroda proved that rotational symmetric solutions in the classical sense to one-harmonic map flows equation from unit disk $\mathbf{D}$ to unit sphere $\mathbf{S}^2$ may break down in finite time. Moreover, in [12], L. Giacomelli and S. Moll established the optimal blowup criterion for initial datum given in [20] and proved that so-called reverse bubbling blowup may happen. On the other hand, we need to consider notions of weak solution to treat solutions which admit singularities at $\nabla u = 0$. In the case of unconstraint equation version of (HF), that is to say the total variation flow equation of the form

$$\frac{\partial u}{\partial t} = \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right), \tag{TVF}$$

we can formulate (TVF) as an evolution equation with a monotone law associated with a convex energy, so-called the total variation,

$$\int_\Omega |\nabla u|.$$

For this model, we can apply a nonlinear semigroup theory initiated by Y. Kōmura ([27]) and developed by H. Brezis ([8]) and others to show well-posedness. However, in the case of constraint equation, such approach is not available since the convexity of the energy is lost because of the constraint of map with values into M. For (HF), two notions of weak solution are proposed.

1.1.1. Giga–Kobayashi solutions

In [19], Y. Giga and R. Kobayashi proposed a notion of solution as a solution of an evolution inclusion with the subdifferential of the convex energy in some real Hilbert space. We shall refer to such a notion of solution as Giga–Kobayashi solution (for short, GK solution). In this formulation, they established existence and uniqueness of piecewise constant solution to general manifold valued problems in bounded open interval when initial data is piecewise constant by computing the sub-differential of the total variation in the space of piecewise constant functions. Moreover, they proved a finite time stopping phenomenon for $\mathbf{S}^1$-valued problem. However, in [21], Y. Giga and H. Kuroda constructed a counter example of this phenomenon for a flow values into $\mathbf{S}^2$. In [20], Y. Giga, Y. Kashima and N. Yamazaki established a local-in-time solution with small initial datum which is unnecessary piecewise constant. Unfortunately, uniqueness for local-in-time solutions is not clear. On the other hand, in [22] and [23], Y. Giga, H. Kuroda and N. Yamazaki considered a discretization of one-harmonic map flow equations with values into a unit sphere in multi-dimensional domain and proved global solvability of this discretized problem by reducing the problem to ordinary differential equations. Unfortunately, uniqueness was not clear. In this paper, we establish uniqueness. We shall call such solutions to the discretized problem discrete Giga–Kobayashi solutions (for short, discrete GK solution). Discrete GK solutions may not correspond to GK solutions except one-dimensional problem. This is also observed in unconstrained problem of crystalline flow ([7], [16]), for instance.

1.1.2. Giacomelli–Mazón–Moll solutions

In [4] and [10], X. Feng and his collaborators proposed a notion of a BV-solution to a sphere-valued problem with Neumann boundary condition and presented an existence result. However, in [13], L. Giacomelli, M. Mazón and S. Moll pointed out that jump of values of solution is not considered in their argument. They introduced an appropriate notion of a BV-solution to $\mathbf{S}^1$-valued problem with the Neumann boundary condition in [13] and [14]. We shall refer to their notion of solution as Giacomelli–Mazón–Moll solution (simply, GMM solution). In [13], they established the time-global existence and uniqueness of solution of a semicircle valued problem in a multi-dimensional bounded domain and time-global existence of solution to a circle valued problem in multi-dimensional bounded domain with initial datum whose angle is bounded essentially and does not jump larger than π. In [14], they generalized the notion of solutions in [13] to a hyper-octant valued problem with the Neumann boundary condition and established an existence of time-global solutions. Uniqueness for GMM solutions to a hyper-octant valued problem is still open.

1.1.3. Giga–Kobayashi solutions vs Giacomelli–Mazón–Moll solutions

GK solutions and GMM solutions may not coincide. The difference of two notations is how to measure the jump of values of solutions. GK solutions measure the jump of values of solutions by an extrinsic metric. On other hand, GMM solutions measure it by the intrinsic metric. In Section 5, we will construct a GK solution and two GMM solutions to a Neumann problem with certain piecewise constant initial datum to check their difference.

1.2. Contribution of this paper

The main object of this paper is to show a existence and uniqueness of discrete Giga–Kobayashi solution to an embedded manifold-valued problem with the Neumann boundary condition in a multi-dimensional domain. Difficulty of this problem is that discrete Giga–Kobayashi solutions are solutions to an evolution inclusion. We assume that manifolds are compact, path-connected and embedded into $\mathbf{R}^n$. This assumption is reasonable because we are interested in $\mathbf{S}^2$ and $\mathbf{SO}(3)$-valued problems for image processing and grain boundary problem, respectively. We emphasize that only existence of $\mathbf{S}^2$-valued problem was established in [22] and [23], so an existence and uniqueness for general manifold valued problems were not considered. In particular, uniqueness was not clear even if we considered $\mathbf{S}^2$-valued problem. On the other hand, we construct two examples of piecewise constant GMM solutions. These examples show that difference of GK solutions and GMM solutions, and GMM solutions do not hold similar uniqueness result in [19] unlike GK solutions.

1.3. Organization

The plan of this paper is as follows. In Section 2, we explain the notations and the mathematical tools which are used in this paper. In Section 3, we recall the notions of solutions to the Neumann problem of (HF) proposed in [19] and its discretized problem proposed in [22]. Moreover, we state our main result in this paper. In Section 4, we prove the main result in this paper. We split this section into existence part and uniqueness part. In existence part, we use the Moreau–Yosida approximation in order to construct solutions. In uniqueness part, evolution variational inequalities have an important role. These inequalities are used for formulating gradient flows in a metric space, and the definition implies that uniqueness of gradient flows holds. We focus on this strong property, and we prove that discrete one-harmonic map flows satisfy some evolution variational inequality in order to obtain uniqueness. In Section 5, we give non-uniqueness example by constructing explicit GMM solutions to the Neumann problem of one-harmonic map flow equations with values into a circle.

2. Notation and Preliminaries

We explain and recall several notion, notation and tools in mathematics which are used in this paper. We consider the space $\mathbf{R}^N$ as the normed space associated with the standard norm $\|\cdot\|_{\mathbf{R}^N}$ induced by the standard inner product $\langle\cdot,\cdot\rangle_{\mathbf{R}^N}$:

$$\|x\|_{\mathbf{R}^N} := (\langle x, x\rangle_{\mathbf{R}^N})^{1/2},$$

$$\langle x, y\rangle_{\mathbf{R}^N} := \sum_{j=1}^{N} x^j y^j,$$

where $x := (x^1, \ldots, x^N), y := (y^1, \ldots, y^N) \in \mathbf{R}^N$. A choice of norms of the vector space $\mathbf{R}^N$ plays important role since total variations of $\mathbf{R}^N$-valued functions depends on norms of $\mathbf{R}^N$. On the other hand, we denote by

- $\mathcal{L}^N$: the N-dimensional Lebesgue measure for an integer $N \geq 0$,
- $\mathcal{H}^D$: the D-dimensional Hausdorff measure for $D \geq 0$.

2.1. Quantities in manifolds

Let M be a manifold embedded into $\mathbf{R}^n$. For $p \in M$, we denote by

- π_p : the orthogonal projection from $\mathbf{R}^n$ to the tangent space T_pM of M at p,
- $\pi_p^\perp$: the orthogonal projection from $\mathbf{R}^n$ to the normal space N_pM of M at p.

We denote by $\mathrm{diam}(M)$ the diameter of M, i.e.,

$$\mathrm{diam}(M) := \sup_{p,q \in M} \|p - q\|_{\mathbf{R}^n}.$$

On the other hand, we denote by dist_M the intrinsic distance on M, i.e., dist_M is defined by the formula:

$$\mathrm{dist}_M(p, q) := \inf_\gamma \int_0^1 \|\gamma'(t)\|_{\mathbf{R}^n} dt. \tag{L}$$

Here $\gamma : [0, 1] \to M$ is a smooth curve in M with $\gamma(0) = p$ and $\gamma(1) = q$. It is well known that minimizers $\gamma_* : [0, 1] \to M$ of the minimizing problem (L) satisfy that $\gamma_*(0) = p$, $\gamma_*(1) = q$ and $\pi_{\gamma(t)}\gamma_*''(t) = 0$ for all $t \in (0, 1)$. In general, a smooth curve $\gamma : [t_0, t_1] \to M$ is called a geodesic if γ satisfies that $\pi_{\gamma(t)}\gamma''(t) = 0$ for all $t \in (t_0, t_1)$. In addition, for each two distinct points p and q in M, there exists an arc-length parameterized geodesic $\gamma : [0, \mathrm{dist}_M(p, q)] \to M$ such that $\gamma(0) = p$ and $\gamma(\mathrm{dist}_M(p, q)) = q$ when M is a compact and path-connected manifold.

2.1.1. Curvature

Let p be a point in M and v be a vector in T_pM with $\|v\|_{\mathbf{R}^n} = 1$. Then we denote by $\kappa(p, v)$ the normal curvature of M at p with the direction $v \in T_pM$ is given by

$$\kappa(p, v) := \|\pi_p^\perp \gamma''(c)\|_{\mathbf{R}^n},$$

where γ is a curve such that $\gamma(c) = p$ and $\gamma'(c) = v$ with $\|v\|_{\mathbf{R}^n} = 1$ for some $c \in \mathbf{R}$. This quantity is independent of a choice of curves. In addition, we denote by $\mathrm{curv}(M)$ the normal curvature of M, i.e.,

$$\mathrm{curv}(M) := \sup_{p \in M} \sup_{v \in T_pM, \|v\|_{\mathbf{R}^n} = 1} \kappa(p, v).$$

2.1.2. Local feature size

We recall several notion and notation developed in the computation geometry. A point $x \in \mathbf{R}^n$ is said to have the unique nearest point if there exists a unique point $p(x) \in M$ such that $p(x) \in \mathrm{argmin}_{p\in M}\|x-p\|_{\mathbf{R}^n}$. Let $S_0(M)$ denote the set of all points in $\mathbf{R}^n$ which **do not** have the unique nearest point. The closure of $S_0(M)$ is called the medial axis of M and is denoted by $S(M)$. Then the local feature size of M is the quantity defined by

$$\mathrm{lfs}(M) := \inf_{p\in M} \inf_{q\in S_0(M)} \|p-q\|_{\mathbf{R}^n}.$$

If M is a compact C^k manifold embedded into $\mathbf{R}^n$ with $k \geq 2$, then lfs(M) is positive ([11]).

2.2. Function spaces

Here we explain the notations which are used in this paper.

2.2.1. Spaces of $\mathbf{R}^n$-valued functions

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$. Then we denote by

- $C_c^\infty(\Omega;\mathbf{R}^n)$: the space of smooth $\mathbf{R}^n$-valued functions with compact support,

- $L^p(\Omega;\mathbf{R}^n)$: the space of p-integrable $\mathbf{R}^n$-valued functions on Ω endowed with the norm $\|u\|_{L^p(\Omega;\mathbf{R}^n)} := (\int_\Omega \|u\|_{\mathbf{R}^n}^p)^{1/p}$ for $1 \leq p < \infty$ and the space of essential bounded $\mathbf{R}^n$-valued functions on Ω endowed with the norm $\|u\|_{L^\infty(\Omega;\mathbf{R}^n)} := \mathrm{esssup}_{x\in\Omega}\|u(x)\|_{\mathbf{R}^n}$ for $p=\infty$.

- $BV(\Omega;\mathbf{R}^n)$: the space of integrable $\mathbf{R}^n$-valued functions on Ω with finite total variation endowed with the norm $\|u\|_{BV(\Omega;\mathbf{R}^n)} := \|u\|_{L^1(\Omega;\mathbf{R}^n)} + \int_\Omega |Du|$, where $\int_\Omega |Du|$ is the isotropic total variation of $u := (u^1,\ldots,u^n)$ given by the formula

$$\int_\Omega |Du| := \sup\left\{ \sum_{j=1}^d \sum_{k=1}^n \int_\Omega u^j \frac{\partial \varphi_{j,k}}{\partial x_j} d\mathcal{L}^d \;\middle|\; \begin{array}{c} \varphi_{j,k} \in C_c^\infty(\Omega), 1 \leq j \leq d, 1 \leq k \leq n, \\ (\sum_{j=1}^d \sum_{k=1}^n |\varphi_{j,k}(x)|^2)^{1/2} \leq 1,\; x \in \Omega. \end{array} \right\}.$$

- $M(\Omega;\mathbf{R}^n)$: the space of $\mathbf{R}^n$-valued finite Radon measures $\mu := (\mu^1,\ldots,\mu^n)$ on Ω endowed with the norm $\|\mu\|_{M(\Omega;\mathbf{R}^n)} := |\mu|(\Omega)$, where $|\mu|$ is the total variation measure of μ given by the formula:

$$|\mu|(E) := \sup\left\{ \sum_{j=1}^n \int_E f^j \mu^j \;\middle|\; \begin{array}{c} f := (f^1,\ldots,f^n) \in C_c^\infty(\Omega;\mathbf{R}^n), \\ \|f(x)\|_{\mathbf{R}^n} \leq 1 \text{ for all } x \in \Omega. \end{array} \right\}.$$

  for all Borel sets E in Ω.

2.2.2. Spaces of Banach space-valued functions

Let X be a real Banach space with a norm $\|\cdot\|_X$. We denote by $C([0,T];X)$ the space of continuous X-valued functions on $[0,T]$ and denote by $C^1([0,T];X)$ the space of continuous differentiable X-valued functions on $[0,T]$. For $p \in [1,\infty)$, we denote by $L^p((0,T);X)$ the space of the strongly measurable X-valued functions on $(0,T)$ with

$$\int_0^T \|u(t)\|_X^p dt < \infty,$$

and this space is a real Banach space with the norm

$$\|u\|_{L^p((0,T);X)} := \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p}.$$

In the case of $p = \infty$, we also refer the Banach space $L^\infty((0,T);X)$ with the usual modification. On the other hand, $L^2((0,T);X)$ is a real Hilbert space with the inner product

$$\langle u, v\rangle_{L^2((0,T);X)} := \int_0^T \langle u(t), v(t)\rangle_X dt$$

when X is a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle_X$.

Given $u \in L^p((0,T);X)$, we say that u is weakly differentiable if there exists $\dfrac{du}{dt} \in L^p((0,T);X)$ such that

$$\int_0^T \frac{du}{dt}(t)\varphi(t)dt = -\int_0^T u(t)\frac{d\varphi}{dt}(t)dt$$

for all smooth test $\varphi : (0,T) \to \mathbf{R}$ with compact support. We denote by $W^{1,p}((0,T);X)$ the space of functions in $L^p((0,T);X)$ which are weak differentiable.

Proposition 2.1 ([3], Theorem 1.20, Aubin–Lions compact criterion)**.** *Let X_0, X_1 and X_2 be three Banach spaces with $X_0 \subset X_1 \subset X_2$. Let $1 \le p, q \le \infty$. Suppose that X_0 is compactly embedded in X_1 and that X_1 is continuously embedded in X_2. Then the set*

$$W := \left\{u \in L^p((0,T);X_0) \mid \frac{du}{dt} \in L^q((0,T);X_2).\right\}$$

is embedded compactly into $\begin{cases} L^p((0,T);X_1) & \text{if } p < \infty \\ C([0,T];X_1) & \text{if } p = \infty \text{ and } q > 1 \end{cases}.$

2.3. Monotone operators and subdifferential of convex functionals

2.3.1. Monotone operators

Let X be a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle_X$. Let $A : X \to 2^X$ be a multi-valued operator in X. We often write $(u,v) \in A$ as $v \in A(u)$ and denote by $D(A)$

the effective domain of A, $D(A) := \{u \in X \mid (u,v) \in A\}(= \{u \in X \mid A(u) \neq \emptyset.\})$. A multi-valued operator $A : X \to 2^X$ is said to be monotone if the inequalities

$$\langle u - v, \zeta - \eta \rangle_X \geq 0$$

hold for all $(u,\zeta), (v,\eta) \in A$. In particular, A is said to be maximal if A satisfies that $A = \hat{A}$ if $\hat{A}$ is a monotone operator such that $A \subset \hat{A}$. If X is a real Hilbert space, it is well known that maximal monotone operators in X are characterized as follows:

Proposition 2.2 ([3], Proposition 2.2, Hilbert space version)**.** *Let A be a monotone operator in a real Hilbert space X. Then A is maximal if and only if for any $\lambda > 0$ and any $f \in X$ there exists unique $u_{\lambda,f} \in D(A)$ such that*

$$u_{\lambda,f} + \lambda A u_{\lambda,f} \ni f.$$

By above proposition, for $\lambda > 0$, we define the resolvent operator $J_\lambda^A : X \to X$ given by the formula :

$$J_\lambda^A f := u_{\lambda,f},$$

and each resolvent operator is contractive on X, i.e.,

$$\|J_\lambda^A(u) - J_\lambda^A(v)\|_X \leq \|u - v\|_X$$

for all $(u,v) \in X \times X$([3], Proposition 2.3). On the other hand, maximal monotone operators have the following strong-weak closed property.

Proposition 2.3 ([3], Proposition 2.1, Hilbert space version)**.** *Let A be a maximal monotone operator in a real Hilbert space X and let $\{(u_j,\zeta_j)\}$ be a sequence in A. If $\{u_j\}$ converges strongly to u in X and $\{\zeta_j\}$ converges weakly to ζ in X, then $(u,\zeta) \in A$.*

2.3.2. subdifferential of convex functionals

One of important classes of monotone operators is given by the subdifferential of functionals. We recall two notations, (Fréchet) differential and subdifferential, since subdifferential is one of generalizations of differential of convex functions. Let $I : X \to (-\infty, \infty]$ be convex on X, i.e., I satisfies

$$I(\theta u + (1-\theta)v) \leq \theta I(u) + (1-\theta) I(v)$$

for any $(u,v) \in X \times X$ and any $\theta \in [0,1]$. We say that I is differentiable at $u \in X$ if there exists $\zeta \in X$ such that

$$\lim_{\|h\|_X \to 0} \frac{|I(u+h) - I(u) - \langle \zeta, h \rangle_X|}{\|h\|_X} = 0,$$

and $\nabla_X I(u) := \zeta$ is called (Fréchet) gradient of I at u and denote by $D(\nabla_X I)$ the effective domain of $\nabla_X I$:

$$D(\nabla_X I) := \{u \in X \mid I \text{ is differentiable at } u\}.$$

On the other hand, we say that I is subdifferentiable at u if there exists $\zeta \in X$ such that

$$I(u+h) \geq I(u) + \langle \zeta, h \rangle_X \text{ for any } h \in X,$$

and ζ is called subgradient of I at u and

$$\partial_X I(u) := \{\zeta \in X \mid \zeta \text{ is subgradient of } I \text{ at } u\}$$

and denote by $D(\partial_X I)$ the effective domain of $\partial_X I$:

$$D(\partial_X I) := \{u \in X \mid I \text{ is subdifferentiable at } u\}.$$

Since I is convex on X, the inclusion $D(\nabla_X I) \subset D(\partial_X I)$ holds. In particular, if $u \in D(\nabla_X I)$, then $\partial_X I(u) = \{\nabla_X I(u)\}$.

Proposition 2.4 ([3], Theorem 2.8)**.** *Let $I : X \to (-\infty, \infty]$ be a proper, lower semi-continuous and convex, i.e., I satisfies that*
(i)(proper) The effective domain of I, $D(I) := \{u \in X \mid I(u) < \infty\}$, is not empty.
(ii)(lower semi-continuity) If $\{u_j\}_{j=1}^{\infty} \subset X$ converges to u strongly in X, then

$$I(u) \leq \liminf_{j\to\infty} I(u_j).$$

(iii)(convexity) For any $(u, v) \in X \times X$ and any $\theta \in [0, 1]$, following inequality holds:

$$I(\theta u + (1-\theta)v) \leq \theta I(u) + (1-\theta)I(v).$$

Then $\partial_X I$ is maximal monotone in $X \times X$.

In this paper, we often consider functionals on space-time spaces. Let X be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle_X$. Given functional $I : X \to (-\infty, \infty]$ which is proper, lower semi-continuous and convex in X, we define a functional I^T on the real Hilbert space $L^2((0, T); X)$ associated with I by

$$I^T(u) := \int_0^T I(u(t))dt.$$

Proposition 2.5. *Let $I : X \to [0, \infty]$ be a proper, convex and lower semicontinuous in X. Then*
(i) I^T is proper, lower semicontinuous and convex on $L^2((0, T); X)$.
(ii) $\partial_{L^2((0,T);X)} I^T$ is maximal monotone in $L^2((0, T); X)$ and is characterized as follows:

$$\partial_{L^2((0,T);X)} I^T(u) = \{\zeta \in L^2((0, T); X) \mid \zeta(t) \in \partial_X I(u(t)) \text{ for a.e. } t \in (0, T)\}$$

for all $u \in L^2((0, T); X)$.

Proof of Proposition 2.5 Let $A := \partial_{L^2((0,T);X)} I^T$ and

$$Bu := \{\zeta \in L^2((0, T); X) \mid \zeta(t) \in \partial_X I(u(t)) \text{ for a.e. } t \in (0, T)\}.$$

(i) This is immediate consequence of Fatou's lemma and lower semi-continuity of I.
(ii) Monotonicity of A is immediate consequence of monotonicity of $\partial_X I$. So, we prove $A = B$. We split this into $A \subset B$ and $B \subset A$. First, we prove $B \subset A$. Let $(u, \zeta) \in B$. Since $\zeta(t) \in \partial_X I(u(t))$ for a.e. $t \in (0, T)$, we have

$$\langle \zeta(t), h(t) \rangle_X \leq I(u(t) + h(t)) - I(u(t))$$

for any $h \in L^2((0, T); X)$ and a.e. $t \in (0, T)$. Taking integration on $(0, T)$, we have

$$\langle \zeta, h \rangle_{L^2((0,T);X)} \leq I^T(u + h) - I^T(u).$$

This means that $(u, \zeta) \in A$. Next, we prove that $A \subset B$. We prove that B is maximal in $L^2((0, T); X)$. Proposition 2.2 implies that it is sufficient to prove that for any $f \in L^2((0, T); X)$ and $\lambda > 0$, there exits unique $u_{\lambda,f}$ such that $u_{\lambda,f} + \lambda B u_{\lambda,f} \ni f$. Such $u_{\lambda,f}$ is given by $v := \mathbf{J}^T_{B,\lambda} f$, where

$$\mathbf{J}^T_{B,\lambda} : L^2((0, T); X) \to L^2((0, T); X);\ u(t) \mapsto J^B_\lambda u(t).$$

We check this. By definition of resolvent, it follows that v is unique X-valued function on $(0, T)$ satisfying

$$v(t) + \lambda \zeta(t) = f(t) \quad \text{in} \ \ X$$

for a.e. $t \in (0, T)$, where $\zeta(t) \in Bv(t)$ for a.e. $t \in (0, T)$. Hence, it is sufficient to check to prove that $v \in L^2((0, T); X)$ and $\zeta \in L^2((0, T); X)$. Since J^B_λ is contractive in X, the operator $\mathbf{J}^T_{B,\lambda}$ is also contractive, and $v (= \mathbf{J}^T_{B,\lambda} f) \in L^2((0, T); X)$. On the other hand, $\zeta \in L^2((0, T); X)$ since $\zeta = \lambda^{-1}(f - v)$. $\square$

2.4. Moreau–Yosida regularization

Let X be a real Hilbert space and let $I : X \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex functional on X and let $J^I_\tau := J^{\partial_X I}_\tau, \tau > 0$ be the resolvent operators of $\partial_X I$. Then the Yosida regularizations $(\partial_X I)_\tau : X \to X$ of $\partial_X I$ are defined as

$$(\partial_X I)_\tau := \frac{1}{\tau}(\mathbf{Id}_X - J^I_\tau), \tag{1}$$

where $\mathbf{Id}_X$ is the identity operator in X. Then it is known that the following holds:

Proposition 2.6 ([3], Proposition 2.3)**.** *Let $I : X \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous in X. Then the following holds:*
(i) $(\partial_X I)_\tau(u) \in \partial_X I(J^I_\tau(u))$ for all $\tau > 0$ and all $u \in X$.
(ii) Each J^I_τ is contractive on X, i.e.,

$$\|J^I_\tau(u) - J^I_\tau(v)\|_X \leq \|u - v\|_X$$

for all $(u, v) \in X \times X$.
(iii) Each $(\partial_X I)_\tau$ is Lipschitz continuous in X with constant τ^{-1}, i.e.,

$$\sup_{\|u-v\|_X \neq 0} \frac{\|(\partial_X I)_\tau(u) - (\partial_X I)_\tau(v)\|_X}{\|u - v\|_X} = \frac{1}{\tau}.$$

In particular, $D((\partial_X I)_\tau) = X$.

On the other hand, the Moreau–Yosida regularizations of I are defined by

$$I_\tau(u) := \inf_{v\in X}\left\{I(v) + \frac{1}{2\tau}\|v-u\|_X^2\right\}.$$

Then it is known that following holds:

Proposition 2.7 ([3], Theorem 2.9)**.** *Let $I : X \to (-\infty,\infty]$ be a proper, lower semicontinuous and convex on X. Then the Moreau–Yosida regularizations $\{I_\tau\}_{\tau>0}$ of I is also a proper, continuous and convex on X, and the following holds for all $u \in X$ and all $\tau > 0$:*
(i) $I_\tau(u) < \infty$,
(ii) $I_\tau(u) = I(J_\tau^I u) + \dfrac{1}{2\tau}\|u - J_\tau^I u\|_X^2$,
(iii) I_τ is differentiable at u, and $\nabla_X I_\tau(u) = (\partial_X I)_\tau u$,
(iv) $I(J_\tau^I u) \le I_\tau(u) \le I(u)$ and $\lim\limits_{\tau\to 0} I_\tau(u) = I(u)$.

3. Setting and Main Result

In this section, first we will recall the notion of solutions proposed in [19] to the following the Neumann problem for one-harmonic map flow equations:

$$(\mathrm{HF}^{\Omega,M,T}; u_0)\begin{cases} \dfrac{\partial u}{\partial t} = -\pi_u\left(-\mathrm{div}\left(\dfrac{\nabla u}{|\nabla u|}\right)\right) & \text{in} \quad \Omega\times(0,T), \\ \left(\dfrac{\nabla u}{|\nabla u|}\right)\cdot\nu = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded Lipschitz domain in $\mathbf{R}^d$, $\partial\Omega$ is the boundary of Ω and ν is the unit outer vector of Ω, M is a compact manifold embedded into $\mathbf{R}^n$, $\pi_p : \mathbf{R}^n \to T_pM$ be the orthogonal projection from $\mathbf{R}^n$ to the tangent space T_pM of M at $p \in M$. Next, we recall the notion of solutions to a discretized problem of $(\mathrm{HF}^{\Omega,M,T}; u_0)$ proposed in [22] and state our main result. In the case of Dirichlet problems, their discretized problem were proposed in [23].

3.1. Giga–Kobayashi solutions

We recall the formulation of one-harmonic map flow equation proposed by Y. Giga and R. Kobayashi in [19], which formulate $(\mathrm{HF}^{\Omega,M,T}; u_0)$ as an evolution inclusion in a space-time Hilbert space whose law is the composition of the subdifferential of total variation and the projection associated with the orthogonal projections $\pi_p : \mathbf{R}^n \to T_pM, p \in M$.

First, we explain notations of spaces to describe $(\mathrm{HF}^{\Omega,M,T}; u_0)$ as an evolution inclusion:

- $\mathbf{H} := L^2(\Omega; \mathbf{R}^n)$,
- $\mathbf{H}^T := L^2((0,T); \mathbf{H})$,
- $\mathbf{M} := \{u \in \mathbf{H} \mid u(x) \in M \text{ for a.e. } x \in \Omega\}$,

- $\mathbf{M}^T := \{u \in \mathbf{H}^T \mid u(t) \in \mathbf{M}$ for a.e. $t \in (0,T)\}$.

Next, we explain notations to describe the law of the evolution inclusions. Let

$$\Phi(u) := \int_\Omega |Du|, \qquad u \in \mathbf{H},$$

$$\Phi^T(u) := \int_0^T \int_\Omega |Du(t)| dt, \qquad u \in \mathbf{H}^T.$$

Since the total variation is L^1 lower semicontinuous ([1]) and convex on BV, Φ is lower semicontinuous and convex on $\mathbf{H}$. Moreover, Proposition 2.5 (i) implies Φ^T is lower semicontinuous and convex on $\mathbf{H}^T$. Hence its subdifferential $\partial_{\mathbf{H}^T}\Phi^T(u)$ is maximal monotone in $\mathbf{H}^T$, and corresponds to $-\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ with the Neumann boundary condition. On the other hand, for $v \in \mathbf{M}^T$, $P_v^T : \mathbf{H}^T \to \mathbf{H}^T$ denotes the projections associated with the orthogonal projections $\pi_p : \mathbf{R}^n \to T_pM$ defined by

$$P_v^T \eta(x,t) := \pi_{v(x,t)}\eta(x,t), \qquad (x,t) \in \Omega \times (0,T),$$

where $\eta \in \mathbf{H}^T$. Of course, P_v^T corresponds to π_v.

Using these notations, Giga–Kobayashi solutions to the Neumann problem $(\mathrm{HF}^{\Omega,M,T}; u_0)$ are defined as follows:

Definition 3.1 (Giga–Kobayashi solutions)**.** Let $T > 0$. Let $u_0 \in \mathbf{M}$. Then $u \in C([0,T]; \mathbf{H})$ is called a Giga–Kobayashi solution to $(\mathrm{HF}^{\Omega,M,T}; u_0)$ if $u(t) \in \mathbf{M}$ for all $t \in [0,T]$ and $u \in W^{1,2}((0,T); \mathbf{H}) \cap D(\partial_{\mathbf{H}^T}\Phi^T)$ solves

$$(\mathrm{HF}^{\Omega,M,T}_{GK}; u_0) \begin{cases} \dfrac{du}{dt} \in -P_u^T \partial_{\mathbf{H}^T}\Phi^T(u) & \text{in} \quad \mathbf{H}^T, \\ u(0) = u_0 & \text{in} \quad \mathbf{H}. \end{cases}$$

3.2. Discrete Giga–Kobayashi solutions and main result

We recall the discretization, proposed by Y. Giga, H. Kuroda and N. Yamazaki in [22], of Giga–Kobayashi solutions, and state main theorem in this paper.

3.2.1. Discrete Giga–Kobayashi solutions

Y. Giga and his collaborators formulated discrete solutions to $(\mathrm{HF}^{\Omega,M,T}; u_0)$ as a Giga–Kobayashi solutions whose associated energy Φ^T replaced by the discrete energy associated with a rectangle discretization. Note this notion of discrete solutions may not coincide one of the original problems except for single-variable case. For related works, see [4], [19] and [23].

Let us begin with explanation of a rectangle decomposition. Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$. We call a **finite** family $\Omega_\Delta := \{\Omega_\alpha\}_{\alpha\in\Delta}$ of subsets of Ω a *rectangle decomposition* of Ω indexed by Δ if Ω_Δ satisfies

- $\bigcup_{\alpha\in\Delta} \Omega_\alpha = \Omega$,

- $\Omega_\alpha \cap \Omega_\beta = \emptyset$ for $\alpha \neq \beta$, $(\alpha, \beta) \in \Delta \times \Delta$,
- For each $\alpha \in \Delta$, there exists a rectangle R_α in $\mathbf{R}^d$ such that $\Omega_\alpha = R_\alpha \cap \Omega$.

For each $\Omega_\alpha \in \Omega_\Delta$, the symbol $\mathbf{1}_{\Omega_\alpha}$ denotes the characteristic function on Ω_α in Ω:

$$\mathbf{1}_{\Omega_\alpha}(x) := \begin{cases} 1 & \text{for} \quad x \in \Omega_\alpha, \\ 0 & \text{for} \quad x \in \Omega \setminus \Omega_\alpha. \end{cases}$$

By the definition of Ω_Δ, each Ω_α has finite perimeter. Figure 1 shows an example of Ω_Δ.

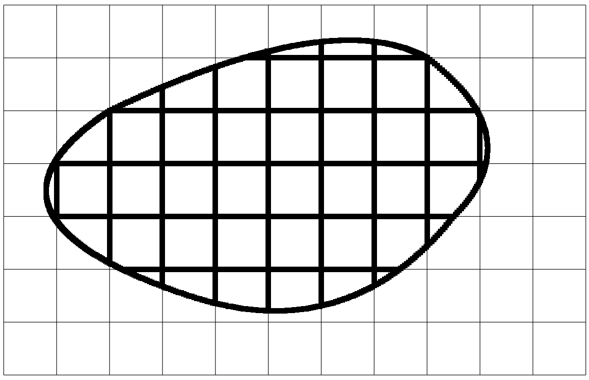

Figure 1: Example of a decomposition

Next, we define the discretized versions of $\mathbf{H}$ and $\mathbf{H}^T$ associated with Ω_Δ as follows:

- $\mathbf{H}_\Delta := \left\{ \sum_{\alpha \in \Delta} u_\alpha \mathbf{1}_{\Omega_\alpha} \mid u_\alpha \in \mathbf{R}^n. \right\} \subset \mathbf{H}$,
- $\mathbf{H}_\Delta^T := L^2((0,T); \mathbf{H}_\Delta) \subset \mathbf{H}^T$.

Note that the inner product of these two discretized spaces is given by the formula:

$$\langle u, v \rangle_{\mathbf{H}_\Delta} = \sum_{\alpha \in \Delta} \langle u_\alpha, v_\alpha \rangle_{\mathbf{R}^n} \mathcal{L}^d(\Omega_\alpha), \qquad u, v \in \mathbf{H}_\Delta,$$

$$\langle u^T, v^T \rangle_{\mathbf{H}_\Delta^T} = \int_0^T \langle u^T(t), v^T(t) \rangle_{\mathbf{H}_\Delta} dt, \qquad u^T, v^T \in \mathbf{H}_\Delta^T.$$

We also define the discretized versions of $\mathbf{M}$ and $\mathbf{M}^T$ associated with Ω_Δ as follows:

- $\mathbf{M}_\Delta := \left\{ \sum_{\alpha \in \Delta} u_\alpha \mathbf{1}_{\Omega_\alpha} \mid u_\alpha \in M \right\} \subset \mathbf{M}$,
- $\mathbf{M}_\Delta^T := \left\{ u \in \mathbf{H}^T \mid u(t) \in \mathbf{M}_\Delta \text{ for a.e. } t \in (0,T) \right\} \subset \mathbf{M}^T$.

Next, we define the discrete functionals Φ_Δ and Φ_Δ^T of Φ and Φ^T associated with Ω_Δ given by the formula, respectively:

$$\Phi_\Delta(u) := \begin{cases} \displaystyle\int_\Omega |Du| & \text{if} \quad u \in \mathbf{H}_\Delta, \\ +\infty, & \text{otherwise}. \end{cases}$$

$$\Phi_\Delta^T(u) := \begin{cases} \displaystyle\int_0^T \int_\Omega |Du(t)|dt & \text{if} \quad u \in \mathbf{H}_\Delta^T, \\ +\infty, & \text{otherwise}. \end{cases}$$

Note that for $u := \sum_{\alpha\in\Delta} u_\alpha \mathbf{1}_{\Omega_\alpha} \in \mathbf{H}_\Delta$, its isotropic total variation $\int_\Omega |Du|$ is given by the formula:

$$\int_\Omega |Du| = \sum_{(\alpha,\beta)\in\Delta\times\Delta} \|u_\alpha - u_\beta\|_{\mathbf{R}^n}\omega_{\alpha,\beta},$$

where $\omega_{\alpha,\beta} := \mathcal{H}^{d-1}(\partial\Omega_\alpha \cap \partial\Omega_\beta \cap \Omega)$ for each $(\alpha, \beta) \in \Delta \times \Delta$. On the other hand, Φ_Δ (Φ_Δ^T, respectively) is proper, convex and lower semicontinuous on $\mathbf{H}_\Delta$ ($\mathbf{H}_\Delta^T$, respectively).

Next, for $u \in \mathbf{M}$ and $v \in \mathbf{M}^T$, we define the two projections $P_u : \mathbf{H} \to \mathbf{H}$ and $P_v^T : \mathbf{H}^T \to \mathbf{H}^T$ by

$$\begin{aligned} P_u\zeta(x) &:= \pi_{u(x)}\zeta(x), & x &\in \Omega, \\ P_v^T\eta(x,t) &:= \pi_{v(x,t)}\eta(x,t), & (x,t) &\in \Omega \times (0,T), \end{aligned}$$

where $\zeta \in \mathbf{H}$ and $\eta \in \mathbf{H}^T$. Using these notation, the discrete solutions of $(\mathrm{HF}^{\Omega,M,T}; u_0)$ are defined as follows:

Definition 3.2 (Discrete Giga–Kobayashi solution)**.** Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ and let Ω_Δ be a rectangle decomposition of Ω. Let M be a manifold embedded into $\mathbf{R}^n$. Let $T > 0$. Let $u_0 \in \mathbf{M}_\Delta$. Then a map $u \in C([0,T]; \mathbf{H})$ is a discrete Giga–Kobayashi solution associated with Ω_Δ of $(\mathrm{HF}^{\Omega,M,T}; u_0)$ if $u(t) \in \mathbf{M}_\Delta$ for all $t \in [0,T]$ and $u \in W^{1,2}((0,T); \mathbf{H}) \cap D(\partial_{\mathbf{H}^T}\Phi_\Delta^T)$ solves

$$(\mathrm{DHF}_{GK}^{\Omega_\Delta,M,T}; u_0) \begin{cases} \dfrac{du}{dt} \in -P_u^T \partial_{\mathbf{H}^T}\Phi_\Delta^T(u) & \text{in} \quad \mathbf{H}^T, \\ u(0) = u_0 & \text{in} \quad \mathbf{H}. \end{cases}$$

Remark 3.1. For a.e. $t \in (0,T)$, $u(t) \in \mathbf{H}_\Delta$ since $u \in D(\partial_{\mathbf{H}^T}\Phi_\Delta^T) \subset \mathbf{H}_\Delta^T$.

Remark 3.2. In the case of $d = 1$, Giga–Kobayashi solutions and discrete Giga–Kobayashi solutions coincide([19]).

3.2.2. Main result

Theorem 3.1. *Let $T > 0$ and M be an m-dimensional compact and path-connected C^2 manifold embedded into $\mathbf{R}^n$. Let Ω be a bounded Lipschitz domain in $\mathbf{R}^d$ and let $\Omega_\Delta := \{\Omega_\alpha\}_{\alpha\in\Delta}$ be a rectangle decomposition of Ω. Then*

(i)*(Existence) For every $u_0 \in \mathbf{M}_\Delta$, there exists a discrete Giga–Kobayashi solution to* $(\mathrm{HF}^{\Omega,M,T}; u_0)$ *associated with* Ω_Δ.
(ii)*(Uniqueness) Let* u_0^1, u_0^2 *be in* $\mathbf{M}_\Delta$. *Let* $u^1, u^2 \in C([0,T];\mathbf{H})$ *be discrete Giga–Kobayashi solutions to* $(\mathrm{HF}^{\Omega,M,T}; u_0^1)$ *and* $(\mathrm{HF}^{\Omega,M,T}; u_0^2)$ *associated with* Ω_Δ*, respectively. Then*

$$\|u^1(t) - u^2(t)\|_{\mathbf{H}}^2 \le e^{\int_0^T \lambda_{\Omega_\Delta,M}(s)ds} \|u_0^1 - u_0^2\|_{\mathbf{H}}^2 \tag{2}$$

holds for a.e. $t \in (0,T)$*, where* $\lambda_{\Omega_\Delta,M}$ *is a positive and square-integrable function on* $[0,T]$ *depending only on* Ω_Δ *and* M*. In particular, if* $u_0^1 = u_0^2$ *in* $\mathbf{H}_\Delta$*, then* $u^1 = u^2$ *in* $\mathbf{H}_\Delta^T$.

Remark 3.3. Y. Giga et al. ([22]) proved global existence of discrete Giga–Kobayashi solutions when $n = 3$, $d = 2$ and $M = \mathbf{S}^2$. Here, we shall explain their approach breifly. First, they considered the the regularized energies of Φ_Δ^T given by

$$\tilde{\Phi}_\Delta^{T,\epsilon}(u) := \begin{cases} \displaystyle\int_0^T \sum_{(\alpha,\beta)\in\Delta\times\Delta} \sqrt{\|u_\alpha - u_\beta\|_{\mathbf{R}^3}^2 + \epsilon^2}\, \omega_{\alpha,\beta} dt, & \text{if} \quad u := \displaystyle\sum_{\alpha\in\Delta} u_\alpha \mathbf{1}_{\Omega_\alpha} \in \mathbf{H}_\Delta^T, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $\omega_{\alpha,\beta} := \mathcal{H}^1(\partial\Omega_\alpha \cap \partial\Omega_\beta \cap \Omega)$ for each $(\alpha,\beta) \in \Delta\times\Delta$. Then, one can compute

$$\partial_{\mathbf{H}^T} \tilde{\Phi}_\Delta^{T,\epsilon}(u) := \left\{ \sum_{(\alpha,\beta)\in\Delta\times\Delta} \frac{\omega_{\alpha,\beta}}{\mathcal{L}^2(\Omega_\alpha)} \frac{u_\alpha - u_\beta}{\sqrt{\|u_\alpha - u_\beta\|_{\mathbf{R}^3}^2 + \epsilon^2}} \right\}$$

for $u := \sum_{\alpha\in\Delta} u_\alpha \mathbf{1}_{\Omega_\alpha} \in \mathbf{H}_\Delta^T$. Subsequently, they considered the regularized problem associated with $\Phi_\Delta^{T,\epsilon}$:

$$(\mathrm{DHF}_{GK}^{\Omega_\Delta,\mathbf{S}^2,T}; u_0)_\epsilon \begin{cases} \dfrac{du_\epsilon}{dt} \in -P_{u_\epsilon}^T \partial_{\mathbf{H}^T} \tilde{\Phi}_\Delta^{T,\epsilon}(u_\epsilon) & \text{in} \quad \mathbf{H}^T, \\ u_\epsilon(0) = u_0 & \text{in} \quad \mathbf{H}. \end{cases}$$

These problems are regard as ordinary differential equations in S_Δ^2 and admit solutions thanks to the computations of $\partial_{\mathbf{H}^T} \tilde{\Phi}_\Delta^{T,\epsilon}$. Finally, they applied the abstract convergence result established by Y. Giga et al. ([17]) to prove that the solutions to $(\mathrm{DHF}_{GK}^{\Omega_\Delta,\mathbf{S}^2,T}; u_0)_\epsilon$ converge to a discrete Giga–Kobayashi solution.

Their approach is still available to our setting. In this paper, we prove that we can also construct discrete Giga–Kobayashi solutions by the Moreau–Yosida regularizations. A difference of their regularizations and the Moreau–Yosida regularizations of Φ_Δ is smoothness of them. The Moreau–Yosida regularizations are not smoother than their regularizations. Therefore, the Moreau–Yosida regularizations are considered to be preserve the singularity of Φ_Δ. On the other hand, we do not use the abstract convergence result and prove directy.

Remark 3.4. In the case of Dirichlet problem, Giga–Kuroda–Yamazaki ([23]) proved global existence of discrete Giga–Kobayashi solutions when $M = \mathbf{S}^2$ and $d = 2$. we can obtain a similar result.

4. Proof of Theorem 3.1.

Here a proof of main theorem in this paper is given. We split a proof into Theorem 3.1 (i)(existence) and Theorem 3.1 (ii)(uniqueness).

4.1. Proof of Theorem 3.1.(i) (Existence).

We shall split a proof of existence to five steps again.

Step 1 : Reduction of the Problem

The multi-valued operator $\partial_{\mathbf{H}^T}\Phi_\Delta^T$ may take vales in $\mathbf{H}^T$. A purpose of this step is to consider a reduced problem of $(\mathrm{DHF}_{GK}^{\Omega_\Delta,M,T}; u_0)$ whose $\partial_{\mathbf{H}_\Delta}\Phi_\Delta^T$ replaced with the subdifferential of the below reduced functionals on $\mathbf{H}_\Delta^T$, and we prove that solutions to the reduced problem are also solutions to the original problem. Existence of solutions of the reduced problem is proved in Step 4. Let us begin with next lemma:

Lemma 4.1. *Let $\Psi_\Delta : \mathbf{H}_\Delta \to \mathbf{R}$ and $\Psi_\Delta^T : \mathbf{H}_\Delta^T \to \mathbf{R}$ be functionals defined as the formulae:*

$$\Psi_\Delta(u) := \Phi_\Delta(u), \qquad u \in \mathbf{H}_\Delta,$$

$$\Psi_\Delta^T(u) := \Phi_\Delta^T(u) = \int_0^T \Psi_\Delta(u(s))ds, \qquad u \in \mathbf{H}_\Delta^T.$$

Then
(i) Ψ_Δ is convex on $\mathbf{H}_\Delta$ and Lipschitz continuous on $\mathbf{H}_\Delta$, i.e.,

$$\mathrm{Lip}(\Psi_\Delta) := \sup_{\|u-v\|_{\mathbf{H}_\Delta}\neq 0} \frac{|\Psi_\Delta(u) - \Psi_\Delta(v)|}{\|u-v\|_{\mathbf{H}_\Delta}} < \infty.$$

(ii) Ψ_Δ^T is also convex and Lipschitz continuous on $\mathbf{H}_\Delta^T$.

Proof of Lemma 4.1 (i) and (ii) are clear because of the definitions of Φ_Δ and Φ_Δ^T. $\square$

Next, proposition states that it is enough to consider the problem associated with $\partial_{\mathbf{H}_\Delta}\Psi_\Delta$ instead of $(\mathrm{DHF}_{GK}^{\Omega_\Delta,M,T}; u_0)$.

Proposition 4.1. *Assume the assumption in Theorem 3.1. If $u \in C([0,T]; \mathbf{H}_\Delta)$ is such that $u(t) \in \mathbf{M}_\Delta$ for all $t \in (0,T)$ and $u \in D(\partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T) \cap W^{1,2}((0,T); \mathbf{H}_\Delta)$ solves*

$$(\mathrm{DHF}_{GK,red}^{\Omega_\Delta,M,T}; u_0) \begin{cases} \dfrac{du}{dt} \in -P_u^T \partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T(u) & \text{in } \mathbf{H}_\Delta^T, \\ u(0) = u_0 & \text{in } \mathbf{H}_\Delta, \end{cases}$$

then u solves $(\mathrm{DHF}_{GK}^{\Omega_\Delta,M,T}; u_0)$. In particular, u is a discrete Giga–Kobayashi solution associated with Ω_Δ to $(\mathrm{HF}^{\Omega,M,T}; u_0)$.

This proposition is an immediately consequence of next lemma:

Lemma 4.2. *Let Ψ_Δ and Ψ_Δ^T be as in Lemma 4.1. Then the following inclusions hold:*
(i) $\partial_{\mathbf{H}_\Delta}\Psi_\Delta(u) \subset \partial_{\mathbf{H}}\Phi_\Delta(u)$ for all $u \in D(\partial_{\mathbf{H}_\Delta}\Psi_\Delta)$.
(ii) $\partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T(u) \subset \partial_{\mathbf{H}^T}\Phi_\Delta^T(u)$ for all $u \in D(\partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T)$.

Proof of Lemma 4.2. (i) Let $(u,\zeta) \in \partial_{\mathbf{H}_\Delta}\Psi_\Delta$. We shall prove that $(u,\zeta) \in \partial_{\mathbf{H}}\Phi_\Delta$, i.e.,

$$\langle \zeta, h\rangle_{\mathbf{H}} + \Phi_\Delta(u) \le \Phi_\Delta(u+h)$$

for all $h \in \mathbf{H}$.

First, we assume $h \in \mathbf{H}_\Delta$. Then convexity of Φ_Δ in $\mathbf{H}_\Delta$(Lemma 4.1. (i)) implies

$$\begin{aligned}\langle \zeta, h\rangle_{\mathbf{H}} &= \langle \zeta, h\rangle_{\mathbf{H}_\Delta}\\ &\le \Psi_\Delta(u+h) - \Psi_\Delta(u)\\ &= \Phi_\Delta(u+h) - \Phi_\Delta(u).\end{aligned}$$

Next, we assume $h \in \mathbf{H}\backslash\mathbf{H}_\Delta$. Then $\langle \zeta, h\rangle_{\mathbf{H}} + \Phi_\Delta(u) < \infty$. On the other hand, $\Phi_\Delta(u+h) = \infty$ since $u+h \notin D(\Phi_\Delta)$. Hence, $\langle \zeta, h\rangle_{\mathbf{H}} + \Phi_\Delta(u) \le \Phi_\Delta(u+h)$. Therefore, $(u,\zeta) \in \partial_{\mathbf{H}}\Phi_\Delta$.

(ii) Let $(u,\zeta) \in \partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T$. By Proposition 2.5 (ii), we see that $(u(t),\zeta(t)) \in \partial_{\mathbf{H}_\Delta}\Psi_\Delta$ for a.e. $t \in (0,T)$. Lemma 4.2 (i) implies that $(u(t),\zeta(t)) \in \partial_{\mathbf{H}}\Phi_\Delta$ for a.e. $t \in (0,T)$. Applying Proposition 2.5 (ii) again, we have $(u,\zeta) \in \partial_{\mathbf{H}^T}\Phi_\Delta^T$. □

Proof of Proposition 4.1. Let u be solution to $(\mathrm{DHF}_{GK,red}^{\Omega_\Delta,M,T}; u_0)$. Then, by the assumption of u, it is trivial that u satisfies the condition of discrete Giga–Kobayashi solution except

$$\frac{du}{dt} \in -P_u^T\partial_{\mathbf{H}^T}\Phi_\Delta^T(u) \ \text{ in } \mathbf{H}^T.$$

This condition is immediate conclusion of Lemma 4.2. Indeed, Lemma 4.2 and $u \in \mathbf{M}_\Delta$ imply $-P_u^T\partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T(u) \subset -P_u^T\partial_{\mathbf{H}^T}\Phi_\Delta^T(u)$, and this and $(\mathrm{DHF}_{GK,red}^{\Omega_\Delta,M,T}; u_0)$ imply that

$$\frac{du}{dt} \in -P_u^T\partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T(u) \subset -P_u^T\partial_{\mathbf{H}^T}\Phi_\Delta^T(u) \ \text{ in } \mathbf{H}^T.$$

Hence, u is a discrete Giga–Kobayashi solution associated with Ω_Δ to $(\mathrm{HF}^{\Omega,M,T}; u_0)$. □

Step 2 : Approximation of the Reduced Problem

A purpose of this step, we consider the approximation problems of $(\mathrm{DHF}_{GK,red}^{\Omega_\Delta,M,T}; u_0)$ with the Moreau–Yosida approximations of Ψ_Δ and construct solutions to their approximation problems.

Let us begin with the notations to state approximation problems. Let $\{J_\tau^{\Psi_\Delta}\}_{\tau>0}$ be the resolvents of $\partial_{\mathbf{H}_\Delta}\Psi_\Delta$ where Ψ_Δ is as in Lemma 4.1. Let $\Psi_\Delta^\tau := (\Psi_\Delta)_\tau, \tau > 0$ be the Moreau–Yosida approximations of Ψ_Δ(See Section 2.4):

$$\Psi_\Delta^\tau(u) := \inf_{v\in\mathbf{H}_\Delta}\left\{\Psi_\Delta(v) + \frac{1}{2\tau}\|u-v\|_{\mathbf{H}_\Delta}^2\right\}, \qquad \tau > 0.$$

We define the space-time energy $\Psi_\Delta^{T,\tau}$ on $\mathbf{H}_\Delta^T$ associated with Ψ_Δ^τ by

$$\Psi_\Delta^{T,\tau}(u) = \int_0^T \Psi_\Delta^\tau(u(t))dt, \qquad u \in \mathbf{H}_\Delta^T.$$

Then our approximation problems of the reduced problem are as follows:

Proposition 4.2. *Assume the assumption in Theorem 3.1. Let $\tau > 0$. Then there exists a map $u_\tau \in C^1([0,T];\mathbf{H}_\Delta)$ such that $u_\tau(t) \in \mathbf{M}_\Delta$ for all $t \in [0,T]$ and u_τ solves*

$$(\mathrm{DHF}_{GK,red}^{\Omega_\Delta,M,T}; u_0)_\tau \begin{cases} \dfrac{du_\tau}{dt}(t) = -P_{u_\tau(t)}\nabla_{\mathbf{H}_\Delta}\Psi_\Delta^\tau(u_\tau(t)), & \textit{in } \ \mathbf{H}_\Delta, \ t \in (0,T), \\ u_\tau(0) = u_0 & \textit{in } \ \mathbf{H}_\Delta. \end{cases}$$

In particular, u_τ satisfies that the energy estimate, i.e.,

$$\int_0^t \left\| \frac{du_\tau}{dt}(s) \right\|_{\mathbf{H}_\Delta}^2 ds + \Psi_\Delta^\tau(u_\tau(t)) \le \Psi_\Delta(u_0) \tag{3}$$

for all $t \in [0,T]$.

Proof of Proposition 4.2. Proposition 2.7 (iii) and Proposition 2.6 (iii) imply that $\nabla_{\mathbf{H}_\Delta}\Psi_\Delta^\tau$ is Lipschitz continuous on $\mathbf{H}_\Delta$. Moreover, for $u \in \mathbf{M}_\Delta$, $P_u\nabla_{\mathbf{H}_\Delta}\Psi_\Delta^\tau(u)$ is tangent vector at u since P_u is the orthogonal projection from $\mathbf{H}_\Delta$ to the tangent space $T_u\mathbf{M}_\Delta$. Hence, the problems $(\mathrm{DHF}_{GK,red}^{\Omega_\Delta,M,T}; u_0)_\tau$ are considered as the ordinary differential equations with the continuous law in finite dimensional compact manifold $\mathbf{M}_\Delta$. Hence each problem $(\mathrm{DHF}_{GK,red}^{\Omega_\Delta,M,T}; u_0)_\tau$ admits at least one global solution u_τ.

Next, we establish the energy estimate (3). Since Ψ_Δ^τ is Fréchet differentiable on $\mathbf{H}_\Delta$(Proposition 2.7 (iii)) and $u_\tau \in C^1([0,T];\mathbf{H}_\Delta)$, $\Psi_\Delta^\tau(u_\tau)$ is differentiable on $(0,T)$. Hence,

$$\begin{aligned} \frac{d}{dt}\Psi_\Delta^\tau(u_\tau(t)) &= \left\langle \nabla_{\mathbf{H}_\Delta}\Psi_\Delta^\tau(u_\tau(t)), \frac{du_\tau}{dt}(t) \right\rangle_{\mathbf{H}_\Delta} \\ &= \left\langle \nabla_{\mathbf{H}_\Delta}\Psi_\Delta^\tau(u_\tau(t)), -P_{u_\tau(t)}\nabla_{\mathbf{H}_\Delta}\Psi_\Delta^\tau(u_\tau(t)) \right\rangle_{\mathbf{H}_\Delta} \\ &= -\left\langle P_{u_\tau(t)}\nabla_{\mathbf{H}_\Delta}\Psi_\Delta^\tau(u_\tau(t)), P_{u_\tau(t)}\nabla_{\mathbf{H}_\Delta}\Psi_\Delta^\tau(u_\tau(t)) \right\rangle_{\mathbf{H}_\Delta} \\ &= -\left\| \frac{du_\tau}{dt}(t) \right\|_{\mathbf{H}_\Delta}^2 \end{aligned}$$

for $t \in (0,T)$. Taking integral on $[0,t]$ of the above identity, we have the energy identity

$$\int_0^t \left\| \frac{du_\tau}{dt}(s) \right\|_{\mathbf{H}_\Delta}^2 ds + \Psi_\Delta^\tau(u_\tau(t)) = \Psi_\Delta^\tau(u_0).$$

By Proposition 2.7 (iv), we have

$$\int_0^t \left\| \frac{du_\tau}{dt}(s) \right\|_{\mathbf{H}_\Delta}^2 ds + \Psi_\Delta^\tau(u_\tau(t)) \le \Psi_\Delta(u_0).$$

□

Step 3 : Estimates for approximation solutions

In this step, we establish inequalities for approximation solutions.

Lemma 4.3. *Let $\{u_\tau\}_{\tau>0}$ be as in Proposition 4.2. Then the following inequalities hold for a.e. $t \in (0,T)$:*

$$\sup_{\tau>0} \|\nabla_{\mathbf{H}_\Delta^T} \Psi_\Delta^{T,\tau}(u_\tau)\|_{\mathbf{H}_\Delta^T} \leq T^{1/2}\mathrm{Lip}(\Psi_\Delta).$$

Proof of Lemma 4.3. Set $\zeta_\tau := \nabla_{\mathbf{H}_\Delta^T} \Psi_\Delta^{T,\tau}(u_\tau)$. By Proposition 2.5 (ii) and Proposition 2.6 (i) and convexity of Ψ_Δ in $\mathbf{H}_\Delta$, the following inequalities hold for a.e. $t \in (0,T)$:

$$\begin{aligned}
\|\zeta_\tau(t)\|_{\mathbf{H}_\Delta} &= \|\nabla_{\mathbf{H}_\Delta} \Psi_\Delta^\tau(u_\tau(t))\|_{\mathbf{H}_\Delta} \\
&= \sup_{\|h\|_{\mathbf{H}_\Delta}\leq 1} \langle \nabla_{\mathbf{H}_\Delta} \Psi_\Delta^\tau(u_\tau(t)), h\rangle_{\mathbf{H}_\Delta} \\
&\leq \sup_{\|h\|_{\mathbf{H}_\Delta}\leq 1} \left(\Psi_\Delta(J_\tau^{\Psi_\Delta}(u_\tau(t)) + h) - \Psi_\Delta(J_\tau^{\Psi_\Delta}(u_\tau(t))) \right) \\
&\leq \sup_{0<\|h\|_{\mathbf{H}_\Delta}\leq 1} \frac{|\Psi_\Delta(J_\tau^{\Psi_\Delta}(u_\tau(t)) + h) - \Psi_\Delta(J_\tau^{\Psi_\Delta}(u_\tau(t)))|}{\|h\|_{\mathbf{H}_\Delta}} \|h\|_{\mathbf{H}_\Delta} \\
&\leq \mathrm{Lip}(\Psi_\Delta).
\end{aligned}$$

Taking the $L^2((0,T);\mathbf{R})$ norm in the above inequalities, we have

$$\|\zeta_\tau\|_{\mathbf{H}_\Delta^T} \leq T^{1/2}\ \mathrm{Lip}(\Psi_\Delta).$$

□

Lemma 4.4. *Let $u_\tau, \tau > 0$ be as in Proposition 4.2. Then there exists a positive constant C such that*

$$\sup_{t\in[0,T]} \|u_\tau(t)\|_{\mathbf{H}_\Delta} + \|u_\tau\|_{W^{1,2}((0,T);\mathbf{H}_\Delta)} \leq C.$$

Proof of Lemma 4.4. Since $u_0 \in \mathbf{M}_\Delta$ and $u_\tau(t) \in \mathbf{M}_\Delta$ for all $t \in [0,T]$, we see that

$$\sup_{t\in[0,T]} \|u_\tau(t)\|_{\mathbf{H}_\Delta} \leq \sup_{t\in[0,T]} \|u_\tau(t)\|_{L^\infty(\Omega;\mathbf{R}^n)}\mathcal{L}^d(\Omega)^{1/2} \leq \sup_{p\in M} \|p\|_{\mathbf{R}^n}\mathcal{L}^d(\Omega)^{1/2} \tag{4}$$

for all $\tau > 0$. Since M is compact and Ω is bounded, the right-hand side of (4) is finite. On the other hand, the energy estimate (3) implies that

$$\left\|\frac{du_\tau}{dt}\right\|_{\mathbf{H}_\Delta^T} \leq \Psi_\Delta(u_0)^{1/2} \tag{5}$$

for all $\tau > 0$. Of course, $\Psi_\Delta(u_0) < \infty$ by the definition of Ψ_Δ. We plug (4) and (5) to obtain

$$\begin{aligned}
\sup_{t\in[0,T]} \|u_\tau(t)\|_{\mathbf{H}_\Delta} + \|u_\tau\|_{W^{1,2}((0,T);\mathbf{H}_\Delta)} &\leq \sup_{t\in[0,T]} \|u_\tau(t)\|_{\mathbf{H}_\Delta} + T^{1/2} \sup_{t\in(0,T)} \|u_\tau(t)\|_{\mathbf{H}_\Delta} + \left\|\frac{du_\tau}{dt}\right\|_{\mathbf{H}_\Delta^T} \\
&\leq (1+T^{1/2}) \sup_{p\in M} \|p\|_{\mathbf{R}^n}\mathcal{L}^d(\Omega)^{1/2} + \Psi_\Delta(u_0)^{1/2}.
\end{aligned}$$

Taking

$$C := (1 + T^{1/2}) \sup_{p \in M} \|p\|_{\mathbf{R}^n} \mathcal{L}^d(\Omega)^{1/2} + \Psi_\Delta(u_0)^{1/2},$$

we obtain the desired inequality. □

Step 4 : Convergence of Approximation Solutions

Proposition 4.3. *Assume the assumption in Theorem 3.1. There exists a map* $u \in C([0,T]; \mathbf{H}_\Delta)$ *such that* $u(t) \in \mathbf{M}_\Delta$ *for all* $t \in (0,T)$ *and* $u \in W^{1,2}((0,T); \mathbf{H}_\Delta) \cap D(\partial_{\mathbf{H}_\Delta^T} \Psi_\Delta^T)$ *and* u *solves the reduced problem of* $(\mathrm{DHF}_{GK}^{\Omega_\Delta, M, T}; u_0)$*:*

$$(\mathrm{DHF}_{GK,red}^{\Omega_\Delta, M, T}; u_0) \begin{cases} \dfrac{du}{dt} \in -P_u^T \partial_{\mathbf{H}_\Delta^T} \Psi_\Delta^T(u) & in \ \ \mathbf{H}_\Delta^T, \\ u(0) = u_0 & in \ \ \mathbf{H}_\Delta. \end{cases}$$

We need four lemmas to prove Proposition 4.3.

Lemma 4.5. *Let* $\{u_\tau\}_{\tau > 0}$ *be as in Proposition 4.2. There exists a positive sequence* $\{\tau_l\}_{l=1}^\infty$ *converging to* 0 *such that there exists* $u \in C([0,T]; \mathbf{H}_\Delta) \cap W^{1,2}((0,T); \mathbf{H}_\Delta)$ *and* $\zeta \in \mathbf{H}_\Delta^T$ *satisfying*

$$\lim_{\tau_l \to 0} u_{\tau_l} = u \ \textit{strongly in} \ C([0,T]; \mathbf{H}_\Delta), \tag{6}$$

$$\lim_{\tau_l \to 0} \nabla_{\mathbf{H}_\Delta^T} \Psi_\Delta^{T,\tau_l}(u_{\tau_l}) = \zeta \ \textit{weakly in} \ \mathbf{H}_\Delta^T. \tag{7}$$

Proof of Lemma 4.5. Note that $\mathbf{H}_\Delta$ is embedded compactly into itself since $\mathbf{H}_\Delta$ is a finite dimensional space. Hence the Aubin–Lions compact criterion (Proposition 2.1 with $p = \infty$, $q = 2$ and $X_0 = X_1 = X_2 := \mathbf{H}_\Delta$) implies that the space

$$\left\{ u \in L^2((0,T); \mathbf{H}_\Delta) \mid \frac{du}{dt} \in L^\infty((0,T); \mathbf{H}_\Delta) \right\}$$

is compactly embedded into $C([0,T]; \mathbf{H}_\Delta)$. Hence Lemma 4.4 implies that we can take a sequence $\{\tau_{l_1}\}_{l_1=1}^\infty$ tending to 0 and $u \in C([0,T]; \mathbf{H}_\Delta) \cap W^{1,2}((0,T); \mathbf{H}_\Delta)$ such that

$$\lim_{l_1 \to \infty} u_{\tau_{l_1}} = u \text{ strongly in } C([0,T]; \mathbf{H}_\Delta).$$

On the other hand, Lemma 4.3 and weak compactness in the Hilbert space $\mathbf{H}_\Delta$ imply that there exists a subsequence $\{\tau_{l_2}\}_{l_2=1}^\infty$ of $\{\tau_{l_1}\}$ such that

$$\lim_{\tau_{l_2} \to 0} \nabla_{\mathbf{H}_\Delta^T} \Psi_\Delta^{T,\tau_{l_2}}(u_{\tau_{l_2}}) = \zeta \text{ weakly in } \mathbf{H}_\Delta^T.$$

Therefore, $\{\tau_{l_2}\}_{l_2=1}^\infty$ is a desired positive sequence. □

Lemma 4.6. *Let* $\{u_{\tau_l}\}_{l=1}^\infty$ *and* u *be as in Lemma 4.5. Let* $\mathbf{J}_{\tau_l}^{\Psi_\Delta} : \mathbf{H}_\Delta^T \to \mathbf{H}_\Delta^T$ *be the operator defined by*

$$\mathbf{J}_{\tau_l}^{\Psi_\Delta}(v)(t) := J_{\tau_l}^{\Psi_\Delta}(v(t)) \quad \textit{for a.e. } t \in (0,T).$$

Then the sequence $\{\mathbf{J}_{\tau_l}^{\Psi_\Delta}(u_{\tau_l})\}_{l=1}^\infty$ *converges strongly to* u *in* $\mathbf{H}_\Delta^T$*.*

Proof of Lemma 4.6. Fix $t \in (0,T)$. Since $J_\tau^{\Psi_\Delta}$ is contractive on $\mathbf{H}_\Delta$ and the definition of the Yosida regularization (see (1).), we have

$$\begin{aligned}\|J_{\tau_l}^{\Psi_\Delta}(u_{\tau_l}(t)) - u(t)\|_{\mathbf{H}_\Delta} &\le \|J_{\tau_l}^{\Psi_\Delta}(u_{\tau_l}(t)) - u_{\tau_l}(t)\|_{\mathbf{H}_\Delta} + \|u_{\tau_l}(t) - u(t)\|_{\mathbf{H}_\Delta}\\ &= \tau_l \ \|(\partial_{\mathbf{H}_\Delta}\Psi_\Delta)_{\tau_l}(u_{\tau_l}(t))\|_{\mathbf{H}_\Delta} + \|u_{\tau_l}(t) - u(t)\|_{\mathbf{H}_\Delta}\\ &\le \mathrm{Lip}(\Psi_\Delta)\tau_l + \|u_{\tau_l}(t) - u(t)\|_{\mathbf{H}_\Delta}.\end{aligned}$$

Taking the norm $L^2(0,T)$ of above inequalities, a triangle inequality implies that we have

$$\begin{aligned}\|\mathbf{J}_{\tau_l}^{\Psi_\Delta}(u_{\tau_l}) - u\|_{\mathbf{H}_\Delta^T} &\le \|\mathrm{Lip}(\Psi_\Delta)\tau_l + \|u_{\tau_l} - u\|_{\mathbf{H}_\Delta}\|_{L^2(0,T;\mathbf{R})}\\ &\le \|\mathrm{Lip}(\Psi_\Delta)\tau_l\|_{L^2(0,T;\mathbf{R})} + \|\|u_{\tau_l} - u\|_{\mathbf{H}_\Delta}\|_{L^2(0,T;\mathbf{R})}\\ &= T^{1/2}\mathrm{Lip}(\Psi_\Delta)\tau_l + \|u_{\tau_l} - u\|_{\mathbf{H}_\Delta^T}.\end{aligned}$$

By Lemma 4.5, we see that $\lim_{\tau_l\to 0} \|\mathbf{J}_{\tau_l}^{\Psi_\Delta}(u_{\tau_l}) - u\|_{\mathbf{H}_\Delta^T} = 0$. □

Lemma 4.7. *Let u and ζ be as in Lemma 4.5. Then $\zeta \in \partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T(u)$.*

Proof of Lemma 4.7. Let $\{u_{\tau_l}\}$ be as in Lemma 4.5. Set $\zeta_{\tau_l} := \nabla_{\mathbf{H}_\Delta^T}\Psi_\Delta^{T,\tau_l}(u_{\tau_l})$. Then Proposition 2.5 (ii) and Proposition 2.6 (i) imply that

$$(J_{\tau_l}^{\Psi_\Delta}(u_{\tau_l}(t)), \zeta_{\tau_l}(t)) \in \partial_{\mathbf{H}_\Delta}\Psi_\Delta$$

for a.e. $t \in (0,T)$ and every τ_l. This can be rewritten as

$$(\mathbf{J}_{\tau_l}^{\Psi_\Delta}(u_{\tau_l}), \zeta_{\tau_l}) \in \partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T,$$

where $\mathbf{J}_{\tau_l}^{\Psi_\Delta}$ is as in Lemma 4.6. Hence, Lemma 4.6, Lemma 4.5 and the maximal monotonicity of $\partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T$ (see Proposition 2.5 (i) and Proposition 2.3) imply that we have $\zeta \in \partial_{\mathbf{H}_\Delta^T}\Psi_\Delta^T(u)$. □

Lemma 4.8. *Let $\{\zeta_{\tau_l}\}_{l=1}^\infty$, $\{u_{\tau_l}\}_{l=1}^\infty$, u and ζ be as in Lemma 4.5. Then $\{P_{u_{\tau_l}}^T\zeta_{\tau_l}\}_{l=1}^\infty$ converges weakly to $P_u^T\zeta$ in $\mathbf{H}_\Delta^T$.*

Proof of Lemma 4.8. Let $\eta \in \mathbf{H}_\Delta^T$. Write

$$u_{\tau_l}(t) := \sum_{\alpha\in\Delta}(u_{\tau_l}(t))_\alpha \mathbf{1}_{\Omega_\alpha}, \quad u(t) := \sum_{\alpha\in\Delta}(u(t))_\alpha \mathbf{1}_{\Omega_\alpha}, \quad \eta(t) := \sum_{\alpha\in\Delta}(\eta(t))_\alpha \mathbf{1}_{\Omega_\alpha}$$

for a.e. $t \in (0,T)$. Lemma 4.5 implies that

$$\lim_{\tau_l\to 0}(u_{\tau_l}(t))_\alpha = (u(t))_\alpha \ \text{ in } \mathbf{R}^n \text{ for all } t \in [0,T] \ \text{ and all } \alpha \in \Delta. \tag{8}$$

A smoothness of $\pi : M \to \mathbf{R}^{n\times n}$ and (8) imply that $\pi_{(u_{\tau_l}(t))_\alpha}(\eta(t))_\alpha$ converges to $\pi_{(u(t))_\alpha}(\eta(t))_\alpha$ for a.e. $t \in (0,T)$ and all $\alpha \in \Delta$. Since each $\pi_{(u_{\tau_l}(t))_\alpha}$ is the orthogonal projection, we have $\|\pi_{(u_{\tau_l}(t))_\alpha}\|_{\mathbf{R}^{n\times n}} \le 1$, and then $\|\pi_{(u_{\tau_l}(t))_\alpha}(\eta(t))_\alpha\|_{\mathbf{R}^n} \le \|(\eta(t))_\alpha\|_{\mathbf{R}^n}$ for a.e. $t \in (0,T)$

and all $\alpha \in \Delta$. Hence (8), $\eta \in \mathbf{H}_\Delta^T$ and the Lebesgue convergence theorem imply that $P_{u_{\tau_l}}^T \eta$ converges strongly to $P_{u_{\tau_l}}^T \eta$ in $\mathbf{H}_\Delta^T$. Since $\{\zeta_{\tau_l}\}$ converges weakly in $\mathbf{H}_\Delta^T$, we have

$$\lim_{l\to\infty} \langle P_{u_{\tau_l}}^T \zeta_{\tau_l}, \eta\rangle_{\mathbf{H}_\Delta^T} = \lim_{l\to\infty} \langle \zeta_{\tau_l}, P_{u_{\tau_l}}^T \eta\rangle_{\mathbf{H}_\Delta^T} = \langle \zeta, P_u^T \eta\rangle_{\mathbf{H}_\Delta^T} = \langle P_u^T \zeta, \eta\rangle_{\mathbf{H}_\Delta^T}.$$

□

Now, we return to a proof of Proposition 4.3.

Proof of Proposition 4.3. Let $\{u_{\tau_l}\}$, $\{\zeta_{\tau_l}\}$, u and ζ be as in Lemma 4.5. We prove that u satisfies the required equation. Let $v \in \mathbf{H}_\Delta^T$. Then

$$\int_0^T \left\langle \frac{du_\tau}{dt}(t), v(t) \right\rangle_{\mathbf{H}_\Delta} dt = \int_0^T \langle -P_{u_{\tau_l}(t)} \nabla_{\mathbf{H}_\Delta} \Psi_\Delta^{\tau_l}(u_{\tau_l}(t)), v(t)\rangle_{\mathbf{H}_\Delta} dt.$$

Taking the limit $\tau_l \to 0$, Lemma 4.5 and Lemma 4.8 imply that we have

$$\int_0^T \left\langle \frac{du}{dt}(t), v(t) \right\rangle_{\mathbf{H}_\Delta} dt = \int_0^T \langle -P_{u(t)} \zeta(t), v(t)\rangle_{\mathbf{H}_\Delta} dt.$$

Hence,

$$\frac{du}{dt} = -P_u^T \zeta \quad \text{in} \quad \mathbf{H}_\Delta^T.$$

Therefore, Lemma 4.7 implies that we have

$$\frac{du}{dt} \in -P_u^T \partial_{\mathbf{H}_\Delta^T} \Psi_\Delta^T(u) \quad \text{in } \mathbf{H}_\Delta^T.$$

□

Step 5 : Connect the previous steps

In this step, we finish a proof of existence.

Proof of Theorem 3.1 (i). Let u be as in Proposition 4.3. Then Proposition 4.1 implies that u is a discrete Giga–Kobayashi solution to $(\mathrm{HF}^{\Omega,M,T}; u_0)$ associated with Ω_Δ. Our proof of Theorem 3.1 (i) is completed. □

4.2. Proof of Theorem 3.1 (ii) (Uniqueness).

We use so-called evolution variational inequalities to establish the inequality (2) for discrete Giga–Kobayashi solutions. First we state an evolution variational inequality in Hilbert spaces:

Lemma 4.9 (Evolution variational inequality). *Let X be a real Hilbert space. Let $I : X \to (-\infty, \infty]$ be a functional on X. Let $x^1, x^2 \in W^{1,2}((0,T); X)$. Suppose that there exists a subset Y of $D(I)$ and $\lambda \in L^2(0,T)$ such that $x^1(t), x^2(t) \in Y$ and*

$$(\mathrm{EVI}; x^j(t), y, I, \lambda) \qquad \frac{1}{2}\frac{d}{dt}\|x^j(t) - y\|_X^2 \le I(y) - I(x^j(t)) + \frac{\lambda(t)}{2}\|x^j(t) - y\|_X^2$$

for all $y \in Y$ and a.e. $t \in (0,T)$ and $j = 1,2$. Then

$$\|x^1(t) - x^2(t)\|_X^2 \leq e^{\int_0^t \lambda(s)ds}\|x^1(0) - x^2(0)\|_X^2$$

for all $t \in [0,T]$. In particular, if $x^1(0) = x^2(0)$ in X, then $x^1 = x^2$ in $L^2((0,T);X)$.

Proof of Lemma 4.9. Fix $t \in (0,T)$. Inserting $y = x^2(t)$ in (EVI; $x^1(t), y, I, \lambda$) and $y = x^1(t)$ in (EVI; $y, x^2(t), I, \lambda$), we obtain

$$\frac{d}{dt}\|x^1(t) - x^2(t)\|_X^2 \leq \lambda(t)\|x^1(t) - x^2(t)\|_X^2.$$

By the Gronwall inequality, we have

$$\|x^1(t) - x^2(t)\|_X^2 \leq e^{\int_0^t \lambda(s)ds}\|x^1(0) - x^2(0)\|_X^2$$

for all $t \in [0,T]$. □

Moreover, we need two lemmas in order to prove uniqueness:

Lemma 4.10. *Let X be a compact and path-connected C^2-manifold embedded into $\mathbf{R}^n$. Let p and q be points in X and let ν be a unit vector in N_pX. Then we have*

$$|\langle p, \nu\rangle_{\mathbf{R}^n} - \langle q, \nu\rangle_{\mathbf{R}^n}| \leq \frac{1}{2} \cdot \mathrm{curv}(X) \cdot \mathrm{dist}_X^2(p,q).$$

Proof of Lemma 4.10. Let $\gamma : [0, \mathrm{dist}_X(p,q)] \to X$ be an arc-length parametrized geodesic with $\gamma(0) = p$ and $\gamma(\mathrm{dist}_X(p,q)) = q$. Then, by the Taylor expansion of $\langle \gamma, \nu\rangle_{\mathbf{R}^n}$ at 0 with second order, we have

$$\langle q, \nu\rangle_{\mathbf{R}^n} = \langle \gamma(0), \nu\rangle_{\mathbf{R}^n} + \langle \gamma'(0), \nu\rangle_{\mathbf{R}^n}\mathrm{dist}_X(p,q) + \int_0^{\mathrm{dist}_X(p,q)} (\mathrm{dist}_X(p,q) - s)\langle \gamma''(s), \nu\rangle_{\mathbf{R}^n} ds.$$

Here, $\gamma(0) = p$ and $\gamma'(0) \in T_pX$ imply that

$$\langle q, \nu\rangle_{\mathbf{R}^n} = \langle p, \nu\rangle_{\mathbf{R}^n} + \int_0^{\mathrm{dist}_X(p,q)} (\mathrm{dist}_X(p,q) - s)\langle \gamma''(s), \nu\rangle_{\mathbf{R}^n} ds.$$

Hence, we have

$$|\langle p, \nu\rangle_{\mathbf{R}^n} - \langle q, \nu\rangle_{\mathbf{R}^n}| = \left|\int_0^{\mathrm{dist}_X(p,q)} (\mathrm{dist}_X(p,q) - s)\langle \gamma''(s), \nu\rangle_{\mathbf{R}^n} ds.\right|$$

We apply the triangle inequality, the Cauchy–Schwarz inequality and Hölder inequality to obtain that

$$\begin{aligned}
&\left|\int_0^{\mathrm{dist}_X(p,q)} (\mathrm{dist}_X(p,q) - s)\langle \gamma''(s), \nu\rangle_{\mathbf{R}^n} ds\right| \\
&\leq \sup_{s \in [0, \mathrm{dist}_X(p,q)]} \|\gamma''(s)\|_{\mathbf{R}^n} \left(\int_0^{\mathrm{dist}_X(p,q)} (\mathrm{dist}_X(p,q) - s) ds\right).
\end{aligned}$$

Since $\pi^{\perp}_{\gamma(t)}\gamma''(t) = \gamma''(t)$ for all t, we have

$$\begin{aligned}
&\sup_{s\in[0,\mathrm{dist}_X(p,q)]} \|\gamma''(s)\|_{\mathbf{R}^n} \left(\int_0^{\mathrm{dist}_X(p,q)} (\mathrm{dist}_X(p,q)-s)ds\right)\\
&= \sup_{s\in[0,\mathrm{dist}_X(p,q)]} \|\pi^{\perp}_{\gamma(t)}\gamma''(s)\|_{\mathbf{R}^n} \cdot \frac{\mathrm{dist}^2_X(p,q)}{2}.
\end{aligned}$$

The definition of the curvature of M implies that

$$\begin{aligned}
\sup_{s\in[0,\mathrm{dist}_X(p,q)]} \|\pi^{\perp}_{\gamma(t)}\gamma''(s)\|_{\mathbf{R}^n} \cdot \frac{\mathrm{dist}^2_X(p,q)}{2} &= \sup_{s\in[0,\mathrm{dist}_X(p,q)]} \kappa(\gamma(s),\gamma'(s)) \cdot \frac{\mathrm{dist}^2_X(p,q)}{2}\\
&\le \mathrm{curv}(X)\cdot \frac{\mathrm{dist}^2_X(p,q)}{2}.
\end{aligned}$$

□

Lemma 4.11. *Under the assumption in Lemma 4.10, the following inequality holds:*

$$\|p-q\|_{\mathbf{R}^n} \le \mathrm{dist}_X(p,q) \le 2\max\left\{1, \frac{\mathrm{diam}(X)}{\mathrm{lfs}(X)}\right\} \|p-q\|_{\mathbf{R}^n}$$

for each point p and q in X.

Proof of Lemma 4.11. Let p and q be in X. The inequality that $\|p-q\|_{\mathbf{R}^n} \le \mathrm{dist}_X(p,q)$ is trivial because of the definition of an intrinsic distance dist_X. We need the next lemma to prove the opposite inequality:

Lemma 4.12 ([28], Proposition 6.3)**.** *Let Y be a compact C^2 manifold embedded into $\mathbf{R}^n$. Then, the inequality*

$$\mathrm{dist}_Y(P,Q) \le \mathrm{lfs}(Y)\left(1 - \left(1 - \frac{2\|P-Q\|_{\mathbf{R}^n}}{\mathrm{lfs}(Y)}\right)^{1/2}\right)$$

holds for every point P and Q in Y with $\|P-Q\|_{\mathbf{R}^n} \le \mathrm{lfs}(Y)/2$.

First, we assume $\|p-q\|_{\mathbf{R}^n} \le \mathrm{lfs}(X)/2$. By Lemma 4.12 with $Y := X$, $P := p$ and $Q := q$ and by the inequality $1-x \le \sqrt{1-x}$ for all $x \in [0,1]$, we have

$$\begin{aligned}
\mathrm{dist}_X(p,q) &\le \mathrm{lfs}(X)\left(1 - \left(1 - \frac{2\|p-q\|_{\mathbf{R}^n}}{\mathrm{lfs}(X)}\right)^{1/2}\right)\\
&\le \mathrm{lfs}(X)\left(1 - \left(1 - \frac{2\|p-q\|_{\mathbf{R}^n}}{\mathrm{lfs}(X)}\right)\right) = 2\|p-q\|_{\mathbf{R}^n}.
\end{aligned}$$

Next, we assume that $\|p-q\|_{\mathbf{R}^n} > \mathrm{lfs}(X)/2$. Then we have

$$\mathrm{dist}_X(p,q) = \frac{\mathrm{dist}_X(p,q)}{\|p-q\|_{\mathbf{R}^n}}\|p-q\|_{\mathbf{R}^n} \le 2\frac{\mathrm{diam}(X)}{\mathrm{lfs}(X)}\|p-q\|_{\mathbf{R}^n}.$$

Hence, we have

$$\mathrm{dist}_X(p,q) \le 2\max\left\{1, \frac{\mathrm{diam}(X)}{\mathrm{lfs}(X)}\right\}\|p-q\|_{\mathbf{R}^n}.$$

$\square$

Now, we return to prove of uniqueness.

Proof of Theorem 3.1.(ii). For $j=1,2$, let u^j be discrete Giga–Kobayashi solutions to $(\mathrm{HF}^{\Omega,M,T}; u_0^j)$ associated with Ω_Δ. Let $\zeta^j \in \partial_{\mathbf{H}_\Delta^T}\Phi_\Delta^T(u^j)$ for $j=1,2$, and let

$$\lambda_{\Omega_\Delta,M}(t) := 4(n-m)\cdot\max_{\alpha\in\Delta}\mathcal{L}^d(\Omega_\alpha)^{-1/2}\cdot\mathrm{curv}(M)\cdot\max\left(\left\{1,\frac{\mathrm{diam}(M)}{\mathrm{lfs}(M)}\right\}\right)^2\max_{j=1,2}\|\zeta^j(t)\|_{\mathbf{H}}.$$

We shall check that u^1 and u^2 satisfy the assumption in Lemma 4.9 with $X := \mathbf{H}$, $I := \Phi_\Delta$, $Y := \mathbf{M}_\Delta$ and $\lambda := \lambda_{\Omega_\Delta,M}$. By the definition of discrete Giga–Kobayashi solutions, it is trivial that u^1 and u^2 satisfy the assumption in Lemma 4.9 except evolution variational inequalities. Hence, in the rest of the proof, we shall focus on these issues. First we consider u^1. For $v\in\mathbf{M}_\Delta$, we compute $\frac{d}{dt}\|u^1-v\|_{\mathbf{H}}^2$ to split the monotone term $\mathbf{I}$ and the non-monotone term $\mathbf{II}$. In order to do this, write

$$u^1(t) := \sum_{\alpha\in\Delta}u_\alpha^1(t)\mathbf{1}_{\Omega_\alpha}\in\mathbf{M}_\Delta,\quad v := \sum_{\alpha\in\Delta}v_\alpha\mathbf{1}_{\Omega_\alpha}\in\mathbf{M}_\Delta\quad\text{for } t\in[0,T].$$

Ont the other hand, set

$$\nu_k^1(t) := \sum_{\alpha\in\Delta}\nu_{k,\alpha}^1(t)\mathbf{1}_{\Omega_\alpha},\qquad t\in[0,T],\qquad k=1,\dots,n-m,$$

where each family $\{\nu_{k,\alpha}^1(t)\}_{k=1}^{n-m}$ is orthogonal bases of $N_{u_\alpha^1(t)}M$ for all $t\in[0,T]$ and $\alpha\in\Delta$. Then we can write

$$P_{u^1(t)}(u^1(t)-v) = (u^1(t)-v) - \sum_{k=1}^{n-m}\langle u^1(t)-v,\nu_k^1(t)\rangle_{\mathbf{R}^n}\nu_k^1(t),\qquad t\in[0,T].$$

We argue for fixed time $t\in[0,T]$ and we do not specify the dependence on time for notational convenience.

By Proposition 2.5 (ii), $\frac{du^1}{dt} = -P_{u^1}\zeta^1$ and we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u^1-v\|_{\mathbf{H}}^2 &= \left\langle\frac{du^1}{dt},u^1-v\right\rangle_{\mathbf{H}} = \langle\zeta^1,-P_{u^1}(u^1-v)\rangle_{\mathbf{H}}\\
&= \left\langle\zeta^1,(v-u^1)-\sum_{k=1}^{n-m}(\langle v,\nu_k^1\rangle_{\mathbf{R}^n}-\langle u^1,\nu_k^1\rangle_{\mathbf{R}^n})\nu_k^1\right\rangle_{\mathbf{H}}\\
&= \langle\zeta^1,v-u^1\rangle_{\mathbf{H}} + \left\langle\zeta^1,\sum_{k=1}^{n-m}(\langle u^1,\nu_k^1\rangle_{\mathbf{R}^n}-\langle v,\nu_k^1\rangle_{\mathbf{R}^n})\nu_k^1\right\rangle_{\mathbf{H}}\\
&=: \mathbf{I}+\mathbf{II}.
\end{aligned}$$

Next, we estimate the terms **I** and **II**, respectively:

• Estimate for the term **I**: Since $\zeta \in \partial_{\mathbf{H}_\Delta}\Phi_\Delta(u^1)$, the definition of the subdifferential implies that

$$\mathbf{I} \leq \Phi_\Delta(v) - \Phi_\Delta(u^1).$$

• Estimate for the term **II**: The Cauchy–Schwarz inequality implies that

$$\begin{aligned}\mathbf{II} &\leq \max_{j=1,2}\|\zeta^j\|_{\mathbf{H}} \sum_{k=1}^{n-m} \|\langle u^1, \nu_k^1\rangle_{\mathbf{R}^n} - \langle v, \nu_k^1\rangle_{\mathbf{R}^n}\|_{\mathbf{H}} \\ &= \max_{j=1,2}\|\zeta^j\|_{\mathbf{H}} \sum_{k=1}^{n-m} \left(\sum_{\alpha\in\Delta} |\langle u_\alpha^1, \nu_{k,\alpha}^1\rangle_{\mathbf{R}^n} - \langle v_\alpha, \nu_{k,\alpha}^1\rangle_{\mathbf{R}^n}|^2 \mathcal{L}^d(\Omega_\alpha)\right)^{\frac{1}{2}}.\end{aligned}$$

Lemma 4.10 and Lemma 4.11 imply that

$$\begin{aligned}&\max_{j=1,2}\|\zeta^j\|_{\mathbf{H}} \sum_{k=1}^{n-m} \left(\sum_{\alpha\in\Delta} |\langle u_\alpha^1, \nu_{k,\alpha}^1\rangle_{\mathbf{R}^n} - \langle v_\alpha, \nu_{k,\alpha}^1\rangle_{\mathbf{R}^n}|^2 \mathcal{L}^d(\Omega_\alpha)\right)^{\frac{1}{2}} \\ &\leq \max_{j=1,2}\|\zeta^j\|_{\mathbf{H}} \sum_{k=1}^{n-m} \left(\sum_{\alpha\in\Delta} 4\ \mathrm{curv}(M)^2 \left(\max\left\{1, \frac{\mathrm{diam}(M)}{\mathrm{lfs}(M)}\right\}\right)^4 \|u_\alpha^1 - v_\alpha\|_{\mathbf{R}^n}^4 \mathcal{L}^d(\Omega_\alpha)\right)^{\frac{1}{2}} \\ &= 2(n-m)\max_{j=1,2}\|\zeta^j\|_{\mathbf{H}}\mathrm{curv}(M)\left(\max\left\{1, \frac{\mathrm{diam}(M)}{\mathrm{lfs}(M)}\right\}\right)^2 \\ &\qquad\qquad \cdot \left\{\left(\sum_{\alpha\in\Delta}(\|u_\alpha^1 - v_\alpha\|_{\mathbf{R}^n}^{\frac{1}{4}}\mathcal{L}^d(\Omega_\alpha)^{\frac{1}{2}})^4 \mathcal{L}^d(\Omega_\alpha)^{-1}\right)^{\frac{1}{4}}\right\}^2 \\ &\leq \frac{\lambda_{\Omega_\Delta,M}}{2} \cdot \left\{\left(\sum_{\alpha\in\Delta}(\|u_\alpha^1 - v_\alpha\|_{\mathbf{R}^n}\mathcal{L}^d(\Omega_\alpha)^{\frac{1}{2}})^4\right)^{\frac{1}{4}}\right\}^2.\end{aligned}$$

The monotonicity of the sequence p-norms for exponents $p \in [1,\infty]$ implies

$$\begin{aligned}\frac{\lambda_{\Omega_\Delta,M}}{2}\left\{\left(\sum_{\alpha\in\Delta}(\|u_\alpha^1 - v_\alpha\|_{\mathbf{R}^n}\mathcal{L}^d(\Omega_\alpha)^{\frac{1}{2}})^4\right)^{\frac{1}{4}}\right\}^2 &\leq \frac{\lambda_{\Omega_\Delta,M}}{2}\left\{\left(\sum_{\alpha\in\Delta}(\|u_\alpha^1 - v_\alpha\|_{\mathbf{R}^n}\mathcal{L}^d(\Omega_\alpha)^{\frac{1}{2}})^2\right)^{\frac{1}{2}}\right\}^2 \\ &= \frac{\lambda_{\Omega_\Delta,M}}{2}\|u^1 - v\|_{\mathbf{H}}^2.\end{aligned}$$

To connect the inequalities for **II**, we have $\mathbf{II} \leq \dfrac{\lambda_{\Omega_\Delta,M}}{2}\|u^1 - v\|_{\mathbf{H}}^2$.

Therefore, we combine the estimates of **I** and **II** to obtain

$$\frac{1}{2}\frac{d}{dt}\|u^1 - v\|_{\mathbf{H}}^2 \leq \Phi_\Delta(v) - \Phi_\Delta(u^1) + \frac{\lambda_{\Omega_\Delta,M}}{2}\|u^1 - v\|_{\mathbf{H}}^2.$$

By similar argument for u^1, we also have an evolution variational inequality for u^2:

$$\frac{1}{2}\frac{d}{dt}\|u^2 - v\|^2_{\mathbf{H}} \le \Phi_\Delta(v) - \Phi_\Delta(u^2) + \frac{\lambda_{\Omega_\Delta,M}}{2}\|u^2 - v\|^2_{\mathbf{H}}.$$

Therefore, Lemma 4.9 implies that

$$\|u^1(t) - u^2(t)\|^2_{\mathbf{H}} \le e^{\int_0^t \lambda_{\Omega_\Delta,M}(s)ds}\|u^1_0 - u^2_0\|^2_{\mathbf{H}}$$

for all $t \in [0,T]$. Our proof of Theorem 3.1 (ii) is completed. □

Remark 4.1. Since a discrete Giga–Kobayashi solution constructed in Theorem 3.1 is unique and Ψ_Δ is Lipschitz on $\mathbf{H}_\Delta$, we see that the square-integrable function $\lambda_{\Omega_\Delta,M}$ is bounded by some positive constant depending only on Ω_Δ and M.

5. Non-uniqueness for piecewise constant Giacomelli–Mazón–Moll solutions

Let $\widehat{\Omega} := \{0 < x < 1\}$, $\widehat{\Omega}_l := \{0 < x < 2^{-1}\}$, $\widehat{\Omega}_r := \{2^{-1} < x < 1\}$, $\widehat{T} := \pi/4$,

$$\widehat{u}_0(x) := (1,0)\mathbf{1}_{\widehat{\Omega}_l}(x) + (-1,0)\mathbf{1}_{\widehat{\Omega}_r}(x) \in \mathbf{S}^1 \quad \text{for} \quad x \in \widehat{\Omega}.$$

In this section, we consider the Neumann problem of one-harmonic map flow equation from the interval $\widehat{\Omega}$ into $\mathbf{S}^1$:

$$(\mathrm{HF}^{\widehat{\Omega},\mathbf{S^1},\widehat{T}};\widehat{u}_0)\begin{cases} \partial_t u - \pi_u \partial_x\left(\dfrac{\partial_x u}{|\partial_x u|}\right) = 0 & \text{in} \quad \widehat{\Omega}\times(0,\widehat{T}), \\ \left(\dfrac{\partial_x u}{|\partial_x u|}\right) = 0 & \text{in} \quad \{0,1\}\times(0,\widehat{T}), \\ u(0) = \widehat{u}_0 & \text{on} \quad \widehat{\Omega}, \end{cases}$$

and our purpose is to compare with Giga–Kobayashi solutions and Giacomelli–Mazón–Moll solutions to $(\mathrm{HF}^{\widehat{\Omega},\mathbf{S^1},\widehat{T}};\widehat{u}_0)$. In the case of Giga–Kobayashi solutions, one can prove that $\widehat{u_0}$ is a unique stationary solution in a space of piecewise constant functions by similar argument in [15, Subsections 4.1 and 4.2]. [1] On the other hand, in the case of Giacomelli–Mazón–Moll solutions, $\widehat{u_0}$ is not a stationary solution, and there exist at least two piecewise constant solutions. This occurs due to the fact that there exist two geodesic midpoints in $\mathbf{S}^1$ between $(1,0)$ and $(-1,0)$ and Giacomelli–Mazón–Moll solutions is related to the geodesic distance of $\mathbf{S}^1$. We shall prove the result about Giacomelli–Mazón–Moll solutions.

First, we recall their notion of solution in our setting. For more general setting, see [13] and [14]. Before giving the notion of solution, we recall additional fundamental properties

[1] Y. Giga et al. considered the Dirichlet boundary problem ([19]). Their argument is still valid even in the case of the Neumann boundary conditions. Indeed, by similar argument in [19], we can calculate as $\partial_x\left(\frac{\partial_x\widehat{u}_0}{|\partial_x\widehat{u}_0|}\right) = (-2,0)\mathbf{1}_{\widehat{\Omega}_l} + (2,0)\mathbf{1}_{\widehat{\Omega}_r} =: v$ in the sense of Giga–Kobayashi solutions. In addition, $\pi_{\widehat{u}_0}v = 0$ since $\widehat{u}_0$ and v are orthogonal. Hence, $\widehat{u}_0$ is a stationary Giga–Kobayashi solution.

of functions of bounded variation on an interval. For fundamental of functions in BV space, we refer to the monographs by [1].

Let Ω be an open interval. Let $u := (u^1, u^2) \in BV(\Omega; \mathbf{R}^2)$. Then there exists a finite Radon measure $\mu := (\mu^1, \mu^2) \in M(\Omega; \mathbf{R}^2)$ such that

$$-\int_\Omega \frac{d\varphi}{dx} u^j d\mathcal{L}^1 = \int_\Omega \varphi d\mu^j \quad \text{for } j = 1, 2$$

for all smooth tests $\varphi : \Omega \to \mathbf{R}$ with compact support. We write $Du := \mu$. The Radon measure Du is decomposed into three mutually orthogonal measures:

$$Du = \frac{du}{dx}\mathcal{L}^1 + D^c u + D^j u,$$

where du/dx denotes the Radon–Nikodým derivative of the measure Du w.r.t the Lebesgue measure $\mathcal{L}^1$. $D^c u$ is the Cantor part of u, which is supported on the set, denoted by $\Omega \setminus J_u$, of the Lebesgue points of u, i.e., these points $x \in \Omega$ for which there exists a $\widehat{u(x)} \in \mathbf{R}^2$ such that

$$\lim_{r \to 0} \frac{1}{\mathcal{L}^1(B_r(x))} \int_{B_r(x)} \|\widehat{u(x)} - u(y)\|_{\mathbf{R}^2} dy = 0,$$

where $B_r(x) := \{y \in \mathbf{R} \mid |x - y| \leq r\}$. $D^j u$ is the jump part of u, which is supported on the set, denoted by J_u, of the jump points of u, i.e., these points $x \in \Omega$ for which there exist $u(x)^+, u(x)^- \in \mathbf{R}^2$ and $\nu_u(x) \in \{\pm 1\}$ such that

$$\lim_{r \to 0} \frac{1}{\mathcal{L}^1(B_r^\pm(x, \nu_u(x)))} \int_{B_r^\pm(x, \nu_u(x))} \|u(x)^\pm - u(y)\|_{\mathbf{R}^2} dy = 0,$$

where $B_r^\pm(x, \nu_u(x)) := \{y \in \mathbf{R} \mid (y - x)\nu_u(x) \gtrless 0, |x - y| \leq r\}$. The jump set J_u is a Borel set. To keep this fact in mind, we denote by u^* the precise representation of u which is defined by

$$u^*(x) := \begin{cases} \widehat{u(x)} & \text{if } \quad x \in \Omega \setminus J_u, \\ (u(x)^+ + u(x)^-)/2 & \text{if } \quad x \in J_u, \end{cases}$$

In what follows, we identify $u = \widehat{u} = u^*$ on $\Omega \setminus J_u$.

Now, we return to state the definition of Giacomelli–Mazón–Moll solutions. Note that the equation

$$\partial_t u - \pi_u \partial_x \left(\frac{\partial_x u}{|\partial_x u|} \right) = 0$$

can be rewritten as

$$\partial_t u - \partial_x \left(\frac{\partial_x u}{|\partial_x u|} \right) - |\partial_x u| u = 0.$$

Giacomelli–Mazón–Moll solutions are defined for this form, and we need to pay attention to the interpretation of $|\partial_x u| u$ because u may be a BV function.

Definition 5.1 (representation of $|\partial_x u|u$)**.** Let $u \in BV(\Omega; \mathbf{R}^2)$ with $u(x) \in \mathbf{S}^1$ for a.e. $x \in \Omega$. We write $\nu \in u^g|Du|$ whenever ν can be written as

$$\nu = u(\|\partial_x u\|_{\mathbf{R}^2}\mathcal{L}^1 + |D^c u|) + u^g\|u^+ - u^-\|_{\mathbf{R}^2}\mathcal{H}^0\lfloor_{J_u},$$

where $u^g : J_u \to \mathbf{S}^1$ maps $x \in J_u$ to a geodesic midpoint [2] $u^g(x) \in \mathbf{S}^1$ between $u^+(x)$ and $u^-(x)$ and $u^g\|u^+ - u^-\|_{\mathbf{R}^2}$ is $\mathcal{H}^0$-measurable.

Definition 5.2 (Giacomelli–Mazón–Moll solution)**.** Let $a < b$. Let $\Omega := \{a < x < b\}$. Let $T > 0$. Let $u_0 \in L^2(\Omega; \mathbf{R}^2)$ with $u_0 \in \mathbf{S}^1$ in a.e. Ω. A map

$$u \in C([0,T]; L^2(\Omega; \mathbf{R}^2)) \cap L^1((0,T); BV(\Omega; \mathbf{R}^2)) \cap W^{1,2}((0,T); L^2(\Omega; \mathbf{R}^2)) \qquad (9)$$

is a Giacomelli–Mazón–Moll solution, GMM solution for short, to $(\mathrm{HF}^{\Omega,\mathbf{S}^1,T}; u_0)$ if $u(0) = u_0$, $u \in \mathbf{S}^1$ for a.e. in $\Omega \times (0,T)$, and there exists a map $z \in L^\infty(\Omega \times (0,T); \mathbf{R}^2)$ with $\|z\|_{L^\infty(\Omega\times(0,T);\mathbf{R}^2)} \leq 1$ such that $z(t) := z(\cdot,t) \in BV(\Omega; \mathbf{R}^2)$ in a.e. $(0,T)$,

$$\langle z(x,t), u(x,t)\rangle_{\mathbf{R}^2} = 0 \text{ for a.e. } (x,t) \in \Omega \times (0,T),$$

and the following holds for a.e. $t \in (0,T)$:

$$\begin{cases} \partial_t u(t) - \partial_x z(t) \in (u(t))^g|Du(t)| & \text{in} \quad M(\Omega; \mathbf{R}^2), \\ \partial_t u(t) \wedge u(t) = \partial_x(z(t) \wedge u(t)) & \text{in} \quad L^2(\Omega; \mathbf{R}), \\ z(a,t) = z(b,t) = 0 & \text{in} \quad \mathbf{R}^2, \end{cases}$$

where

$$z(a,t) := \lim_{x\to+a} z(x,t), \qquad z(b,t) := \lim_{x\to b-} z(x,t)$$

and $\wedge$ is the wedge product, i.e., $x \wedge y := x^1y^2 - x^2y^1$ for $x := (x^1, x^2)$ and $y := (y^1, y^2)$.

Theorem 5.1 (Non-uniqueness)**.** *Let* $\widehat{\Omega} := \{0 < x < 1\}$, $\widehat{\Omega}_l := \{0 < x < 2^{-1}\}$, $\widehat{\Omega}_r := \{2^{-1} < x < 1\}$, $\widehat{T} := \pi/4$,

$$\widehat{u}_0 := (1,0)\mathbf{1}_{\widehat{\Omega}_l} + (-1,0)\mathbf{1}_{\widehat{\Omega}_r}.$$

Let θ_+ *and* θ_- *be functions on* $\widehat{\Omega} \times [0, \pi/4)$ *defined by the forms:*

$$\theta_\pm(\cdot,t) := \pm 2t\mathbf{1}_{\widehat{\Omega}_l} \pm (\pi - 2t)\mathbf{1}_{\widehat{\Omega}_r}.$$

Then $u_\pm := (\cos\theta_\pm, \sin\theta_\pm)$ *are Giacomelli–Mazón–Moll solutions to* $(\mathrm{HF}^{\widehat{\Omega},\mathbf{S}^1,\widehat{T}}; \widehat{u}_0)$*, and* $u_+(\cdot,t) \neq u_-(\cdot,t)$ *in* $L^2(\widehat{\Omega}; \mathbf{R}^2)$ *for all* $t \in (0,\widehat{T})$*.*

Proof of Theorem 5.1. By the definition of $u_\pm$, we see that $u_\pm$ satisfy (9), $u_\pm(\cdot,0) = \widehat{u_0}$ and $u_+(\cdot,t) \neq u_-(\cdot,t)$ in $L^2(\widehat{\Omega}; \mathbf{R}^2)$ for a.e. $t \in (0,\widehat{T})$. We shall prove that $u_\pm$ are GMM solutions to $(\mathrm{HF}^{\widehat{\Omega},\mathbf{S}^1,\widehat{T}}; \widehat{u}_0)$. Let

$$\eta_\pm(x,t) := \pm 2x\mathbf{1}_{\widehat{\Omega}_l} \pm (2 - 2x)\mathbf{1}_{\widehat{\Omega}_r}, \qquad (x,t) \in \widehat{\Omega} \times (0,\widehat{T}),$$

[2] Geodesically midpoints : A point $m \in \mathbf{S}^1$ is called a geodesically midpoint between $p \in \mathbf{S}^1$ and $q \in \mathbf{S}^1$ if $d_{\mathbf{S}^1}(p,m) = d_{\mathbf{S}^1}(m,q)$, where $d_{\mathbf{S}^1}(P,Q)$ denotes the geodesic distance between $P \in \mathbf{S}^1$ and $Q \in \mathbf{S}^1$.

$$z_\pm(x,t) := \boldsymbol{\gamma}_\pm(x,t)\eta_\pm(x,t), \qquad (x,t) \in \widehat{\Omega} \times (0,\widehat{T}),$$

where $\boldsymbol{\gamma}_\pm = (-\sin(\theta_\pm), \cos(\theta_\pm))$. Then by the definition, we see that $z_\pm(\cdot,t) \in BV(\widehat{\Omega};\mathbf{R}^2)$ and $z_\pm(0,t) = z_\pm(1,t) = 0$ for a.e. $t \in (0,\widehat{T})$, and $\|z_\pm(x,t)\|_{\mathbf{R}^2} \leq 1$ and $\langle u_\pm(x,t), z_\pm(x,t)\rangle_{\mathbf{R}^2} = 0$ for a.e. $(x,t) \in \widehat{\Omega} \times (0,\widehat{T})$. By calculations, we have the following equality for a.e. $t \in (0,\widehat{T})$:

$$\partial_t u_\pm(\cdot,t) = \pm 2\boldsymbol{\gamma}_\pm(\cdot,t)\mathbf{1}_{\widehat{\Omega}_l} \mp 2\boldsymbol{\gamma}_\pm(\cdot,t)\mathbf{1}_{\widehat{\Omega}_r} \ \text{ in } L^2(\widehat{\Omega};\mathbf{R}^2),$$

$$\partial_x z_\pm(\cdot,t) = \pm 2\boldsymbol{\gamma}_\pm(\cdot,t)\mathbf{1}_{\widehat{\Omega}_l} \mp 2\boldsymbol{\gamma}_\pm(\cdot,t)\mathbf{1}_{\widehat{\Omega}_r} + (0, \mp 2\cos(2t))\delta_{\frac{1}{2}} \ \text{ in } M(\widehat{\Omega};\mathbf{R}^2),$$

$$|Du_\pm(\cdot,t)| = \left|2\sin\left(\frac{\theta_\pm^+(\cdot,t) - \theta_\pm^-(\cdot,t)}{2}\right)\right|\delta_{\frac{1}{2}} = 2\cos(2t)\delta_{\frac{1}{2}} \ \text{ in } M(\widehat{\Omega};\mathbf{R}^2),$$

where $\delta_{1/2}$ denotes the Dirac delta at $1/2$. Since the geodesically midpoints between $u_\pm(1/2,t)^+$ and $u_\pm(1/2,t)^-$ are $\{(0,\pm 1)\}$ for $t \in (0,\widehat{T})$, we see that $u_\pm$ and $z_\pm$ satisfy

$$\partial_t u_\pm(\cdot,t) - \partial_x z_\pm(\cdot,t) \in u(\cdot,t)^g |Du_\pm(\cdot,t)| \quad \text{in} \quad M(\widehat{\Omega};\mathbf{R}^2) \text{ a.e in } (0,\widehat{T}).$$

On the other hand, since

$$\partial_x(u_\pm(\cdot,t) \wedge z_\pm(\cdot,t)) = \partial_x \eta_\pm(\cdot,t),\ \partial_t u_\pm(\cdot,t) \wedge u_\pm(\cdot,t) = \partial_t \theta_\pm(\cdot,t),\ \partial_x \eta_\pm(\cdot,t) = \partial_t \theta_\pm(\cdot,t)$$

in $L^2(\widehat{\Omega};\mathbf{R}^2)$ for a.e. $t \in (0,\widehat{T})$, we have

$$\partial_x(u_\pm(\cdot,t) \wedge z_\pm(\cdot,t)) = \partial_t u_\pm(\cdot,t) \wedge u_\pm(\cdot,t) \ \text{ in } L^2(\widehat{\Omega};\mathbf{R}^2) \ \text{ for a.e. } t \in (0,\widehat{T}).$$

Therefore, $u_\pm$ are GMM solutions to $(\mathrm{HF}^{\widehat{\Omega},\mathbf{S^1},\widehat{T}}; \widehat{u}_0)$. □

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