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QUASILINEAR TYPE KOBAYASHI–WARREN–CARTER SYSTEM INCLUDING DYNAMIC BOUNDARY CONDITION

Ryota Nakayashiki
Department of Mathematics and Informatics, Graduate School of Science, Chiba University,
1-33, Yayoi-cho, Inage-ku, 263-8522, Chiba, Japan
(E-mail: `nakayashiki1108@chiba-u.jp`)

Abstract. In this paper, we consider a coupled system of two parabolic type initial-boundary value problems, which is called *Kobayashi–Warren–Carter system*. The system is known as a mathematical model of grain boundary motion in a polycrystal, which reproduces the crystalline dynamics by means of a type of quasilinear (singular) diffusion equation. Recently, the theoretical results of the Kobayashi–Warren–Carter systems have been established by a number of researchers under the simple setting of boundary conditions. However, the variety of the boundary conditions, such as inhomogeneous and more dynamic cases, were not focused so much. The main issue of this paper is to study the qualitative properties of the Kobayashi–Warren–Carter systems including dynamic boundary conditions. Consequently, we prove two Main Theorems, concerned with the existence of the solution, and upper semi-continuity among solution classes.

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Introduction

Let $\nu > 0$ be a fixed constant. Let $(0, T)$ be the time-interval with a constant $T > 0$. Let Ω be a bounded spatial domain with a dimension $1 < N \in \mathbb{N}$. We denote by Γ the boundary of Ω, we suppose C^∞-regularity for Γ. We denote by $n_\Gamma \in \mathbb{S}^{N-1}$ the unit outer normal on Γ, and set $Q := (0, T) \times \Omega$ and $\Sigma := (0, T) \times \Gamma$.

In this paper, we take a nonnegative constant $\varepsilon \geq 0$ to consider the following coupled system of parabolic type PDEs, denoted by $(\mathrm{KWC})_\varepsilon$.

$(\mathrm{KWC})_\varepsilon$:

$$\partial_t \eta - \Delta \eta + g(\eta) + \alpha'(\eta)|\nabla \theta| = 0 \text{ in } Q, \tag{0.1}$$

$$\alpha_0(\eta)\partial_t \theta - \mathrm{div}\left(\alpha(\eta)\frac{\nabla \theta}{|\nabla \theta|} + \nu^2 \nabla \theta\right) = 0 \text{ in } Q, \tag{0.2}$$

$$\nabla \eta_{|_\Gamma} \cdot n_\Gamma = 0 \text{ on } \Sigma, \tag{0.3}$$

$$\partial_t \theta_\Gamma - \varepsilon^2 \Delta_\Gamma \theta_\Gamma + (\alpha(\eta)\tfrac{\nabla \theta}{|\nabla \theta|} + \nu^2 \nabla \theta)_{|_\Gamma} \cdot n_\Gamma = 0, \text{ and } \theta_{|_\Gamma} = \theta_\Gamma \text{ on } \Sigma, \tag{0.4}$$

$$\eta(0, \cdot) = \eta_0, \ \theta(0, \cdot) = \theta_0 \text{ in } \Omega, \text{ and } \theta_\Gamma(0, \cdot) = \theta_{\Gamma,0} \text{ on } \Gamma. \tag{0.5}$$

The above system is known as "*Kobayashi–Warren–Carter system*", named after R. Kobayashi, J. A. Warren, and W. C. Carter, who proposed a phase-field model of grain boundary motion in a polycrystal (cf. [12, 13]). Our system $(\mathrm{KWC})_\varepsilon$ is one of modified versions of the original system and the principal modifications are in the point that:

- the quasilinear diffusion in (0.2), with singularity includes the regularization term $\nu^2 \nabla \theta$ with a small constant $\nu > 0$;
- the boundary data θ_Γ treated as an unknown variable, which is governed by the dynamic boundary conditions (0.4), consisting of the parabolic type PDE and the transmission condition $\theta_{|_\Gamma} = \theta_\Gamma$, on Σ.

In the original model [12,13], the spatial domain Ω is settled as a two-dimensional domain $(N = 2)$, and the main focus is to reproduce the dynamics of the crystalline orientation by the time and spatial variation of a vector field:

$$(t, x) \in Q \mapsto \varpi(t, x) := \eta(t, x)^{\mathrm{t}}\big[\cos \theta(t, x), \sin \theta(t, x)\big],$$

consisting of two order parameters $\eta = \eta(t, x)$ and $\theta = \theta(t, x)$. The variation of $\varpi = \varpi(t, x)$ is supposed to be governed by gradient flow of the following energy functional, called *free-energy*, and for any $\varepsilon \geq 0$, the free-energy for $(\mathrm{KWC})_\varepsilon$ is provided as follows.

$$\begin{aligned}
[\eta, \theta, \theta_\Gamma] \in D(\mathscr{F}_\varepsilon) \mapsto \mathscr{F}_\varepsilon(\eta, \theta, \theta_\Gamma) &:= \frac{1}{2}\int_\Omega |\nabla \eta|^2 \, dx + \int_\Omega \hat{g}(\eta) \, dx \\
+ \int_\Omega \alpha(\eta)|\nabla \theta| \, dx + \frac{\nu^2}{2}\int_\Omega |\nabla \theta|^2 \, dx &+ \frac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon \theta_\Gamma)|^2 \, d\Gamma \in [0, \infty],
\end{aligned} \tag{0.6}$$

with the effective domain:

$$D(\mathscr{F}_\varepsilon) := \left\{ [\eta, \theta, \theta_\Gamma] \;\middle|\; \begin{array}{l} \eta \in H^1(\Omega), \, \theta \in H^1(\Omega), \, \theta_\Gamma \in H^{\frac{1}{2}}(\Gamma), \\ \varepsilon \theta_\Gamma \in H^1(\Gamma) \text{ and } \theta_{|_\Gamma} = \theta_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma) \end{array} \right\}.$$

In the context, the unknowns $\eta = \eta(t,x)$ and $\theta = \theta(t,x)$ are supposed to be order parameters which correspond to the orientation order and the orientation angle in a polycrystal, respectively. η is supposed to take values on $[0,1]$ and the threshold values 1 and 0 indicate the completely oriented phase and the disoriented phase of orientation, respectively. $g = g(\eta)$ in (0.1) is a given Lipschitz continuous function on $\mathbb{R}$, and $\hat{g} = \hat{g}(\eta)$ is its nonnegative primitive. $0 < \alpha_0 = \alpha_0(\eta)$ in (0.2) is a given Lipschitz function, $0 < \alpha = \alpha(\eta)$ in (0.1)–(0.2) is a given C^2-convex function, and $\alpha' = \alpha'(\eta)$ is the differential of α. Besides, "$_{|\Gamma}$" denotes the trace on Γ for a Sobolev function, $d\Gamma$ denotes the area element on Γ, ∇_Γ denotes the surface gradient on Γ, and $\mathit{\Delta}_\Gamma$ denotes the Laplacian on the surface, i.e. the so-called Laplace–Beltrami operator (cf. [18]). The equations in (0.5) are initial conditions with given initial data η_0, θ_0, and $\theta_{\Gamma,0}$, for the components η, θ, and θ_Γ, respectively.

The objective of this study is to develop the mathematical analysis for the Kobayashi–Warren–Carter systems, and the main issue of this paper is concerned with the qualitative properties of the systems $(\mathrm{KWC})_\varepsilon$, for any $\varepsilon \geq 0$. The theoretical results of the Kobayashi–Warren–Carter systems have been established by a number of researchers (cf. [6–8,11,15,16,19–22]), and a number of qualitative results for L^2-based solutions have been obtained from various viewpoints. However, for the settings of boundary conditions, most of previous works supposed only a few simple cases, such as homogeneous Dirichlet / Neumann cases, and the variety of the boundary conditions, such as inhomogeneous and more dynamic cases, were not focused so much.

On the other hand, there is a previous work [4] which dealt with the system, consisting of quasilinear diffusion Allen–Cahn type equations with the dynamic boundary conditions. Then, the qualitative results of the systems, such as the existence of the unique solution and some continuous dependences, were obtained by means of the theory of nonlinear evolution equation.

In view of this, we set the goal in this paper to show the following two Main Theorems.

Main Theorem 1: the existence of solution to $(\mathrm{KWC})_\varepsilon$, and the uniqueness in the constant case of α_0.

Main Theorem 2: upper semi-continuity of solution classes with respect to $\varepsilon \geq 0$.

Here is the contents of this paper. In Section 1, we list some preliminaries and some specific notations which are used throughout this paper. In Section 2, we state Main Theorems, with the assumptions for $(\mathrm{KWC})_\varepsilon$, and the definition of the solutions. The proofs of Main Theorems are stated in Section 5, and these are discussed on the basis of the Key-properties, demonstrated in Sections 3 and 4, respectively.

1 Preliminaries

In this Section, we outline some notations and known facts, as preliminaries of our study.

Notation 1 (Notations in real analysis)**.** For arbitrary $a, b \in [-\infty, \infty]$, we define:

$$a \vee b := \max\{a, b\} \text{ and } a \wedge b := \min\{a, b\},$$

and especially, we write $[a]^+ := a \vee 0$ and $[b]^- := -(b \wedge 0)$, respectively.

Let $d \in \mathbb{N}$ be any fixed dimension. Then, we simply denote by $|x|$ and $x \cdot y$ the Euclidean norm of $x \in \mathbb{R}^d$ and the standard scalar product of $x, y \in \mathbb{R}^d$, respectively. Also, we denote by $\mathbb{B}^d$ and $\mathbb{S}^{d-1}$ the d-dimensional unit open ball centered at the origin, and its boundary, respectively, i.e.:

$$\mathbb{B}^d := \left\{ \ x \in \mathbb{R}^d \ \middle| \ |x| < 1 \ \right\} \text{ and } \mathbb{S}^{d-1} := \left\{ \ x \in \mathbb{R}^d \ \middle| \ |x| = 1 \ \right\}.$$

In particular, when $d > 1$, we write $x \leq y$, if $x_i \leq y_i$, for all $i = 1, \cdots, d$, and define:

$$\begin{cases} x \vee y := [x_1 \vee y_1, \ldots, x_d \vee y_d], \ x \wedge y := [x_1 \wedge y_1, \ldots, x_d \wedge y_d], \\ [x]^+ := \bigl[[x_1]^+, \ldots, [x_d]^+\bigr] \text{ and } [y]^- := \bigl[[y_1]^-, \ldots, [y_d]^-\bigr], \end{cases} \quad \text{for all } x, y \in \mathbb{R}^d.$$

For any $d \in \mathbb{N}$, the d-dimensional Lebesgue measure is denoted by $\mathcal{L}^d$. Unless otherwise specified, the measure theoretical phrases, such as "a.e.", "dt", "dx", and so on, are with respect to the Lebesgue measure in each corresponding dimension. Also, in the observations on a smooth surface S, the phrase "a.e." is with respect to the Hausdorff measure in each corresponding Hausdorff dimension, and the area element on S is denoted by dS.

Additionally, we note the following elementary fact.

(Fact 0) Let $m \in \mathbb{N}$ be a fixed finite number. If $\{\alpha_1, \ldots, \alpha_m\} \subset \mathbb{R}$ and $\{a_n^k\}_{n=1}^\infty$, $k = 1, \ldots, m$, fulfill that:

$$\varliminf_{n \to \infty} a_n^k \geq \alpha_k, \ k = 1, \ldots, m, \text{ and } \varlimsup_{n \to \infty} \sum_{k=1}^m a_n^k \leq \sum_{k=1}^m \alpha_k.$$

Then, it holds that:

$$\lim_{n \to \infty} a_n^k = \alpha_k, \ k = 1, \ldots, m.$$

Notation 2 (Notations of functional analysis)**.** For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X, and denote by $\langle \, \cdot \, , \, \cdot \, \rangle_X$ the duality pairing between X and the dual space X^* of X. Let $\mathcal{I}_X : X \to X$ be the identity map from X onto X. In particular, when X is a Hilbert space, we denote by $(\, \cdot \, , \, \cdot \,)_X$ the inner product in X.

For two Banach spaces X, Y, let $\mathcal{L}(X, Y)$ be the Banach space of bounded linear operators from X onto Y.

For two Banach spaces X, Y, and a set-valued operator $\tilde{\mathcal{A}}$, we denote by $D(\tilde{\mathcal{A}})$ the domain of $\tilde{\mathcal{A}}$, i.e. $D(\tilde{\mathcal{A}}) := \{\tilde{z} \in X \,|\, \tilde{\mathcal{A}}z \neq \emptyset\}$, and we often say "$[z_0, z_0^*] \in \tilde{\mathcal{A}}$ in $X \times Y$", to mean "$z_0 \in D(\tilde{\mathcal{A}})$ and $z_0^* \in \tilde{\mathcal{A}}z_0$", by identifying the operator $\tilde{\mathcal{A}}$ with its graph in $X \times Y$.

For Banach spaces $X_1, \cdots, X_d$ with $1 < d \in \mathbb{N}$, let $X_1 \times \cdots X_d$ be the product Banach space endowed with the norm $|\cdot|_{X_1 \times \cdots \times X_d} := |\cdot|_{X_1} + \cdots + |\cdot|_{X_d}$. However, when all $X_1, \cdots, X_d$ are Hilbert spaces, $X_1 \times \cdots \times X_d$ denotes the product Hilbert space endowed with the inner product $(\, \cdot \, , \, \cdot \,)_{X_1 \times \cdots \times X_d}$ and the norm $|\cdot|_{X_1 \times \cdots \times X_d} := (|\cdot|_{X_1}^2 + \cdots + |\cdot|_{X_d}^2)^{\frac{1}{2}}$.

Notation 3. Throughout this paper, let $T > 0$, $1 < N \in \mathbb{N}$, and $\nu > 0$ be fixed constants. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, such that:

($\boldsymbol{\omega}$ 1) Ω has a C^∞-boundary $\Gamma := \partial\Omega$;

($\boldsymbol{\omega}$ 2) the function $d_\Gamma : x \in \overline{\Omega} \mapsto d_\Gamma(x) := \inf_{y\in\Gamma} |x-y| \in [0,\infty)$ forms a C^∞-function as a neighborhood of Γ.

Let Δ_N be the operator of Laplacian, subject to the zero-Neumann boundary condition, which is defined as:

$$\begin{aligned} \Delta_N : v \in D(\Delta_N) \ &:= \ \left\{ \ z \in H^2(\Omega) \ \middle| \ \nabla z_{|_\Gamma} \cdot n_\Gamma = 0 \text{ in } H^{\frac{1}{2}}(\Gamma) \ \right\} \subset L^2(\Omega) \\ &\mapsto \ \Delta_N v := \Delta v \in L^2(\Omega). \end{aligned}$$

In this paper, we identify the operator $-\Delta_N$ as an linear and continuous operator from $H^1(\Omega)$ into $H^1(\Omega)^*$, via the following Green-type formula (cf. [1, Proposition 5.6.2]):

$$-\int_\Omega \Delta_N z \, w \, dx = \int_\Omega \nabla z \cdot \nabla w \, dx, \quad \text{for all } [z,w] \in D(\Delta_N) \times L^2(\Omega).$$

Notation 4 (Notations of surface-differentials)**.** On this basis of Notation 3, let $[\,\cdot, n_\Gamma]$ be a linear operator from $C^\infty(\overline{\Omega})$ into $\mathcal{D}'(\Gamma)$, which is defined as:

$$\begin{gathered} \langle [\varpi, n_\Gamma], \varphi \rangle := \int_\Omega \operatorname{div} \varpi \, \varphi^{\rm ex} \, dx + \int_\Omega \varpi \cdot \nabla \varphi^{\rm ex} \, dx, \\ \text{for all } \varpi \in C^\infty(\overline{\Omega})^N \text{ and } \varphi \in C^\infty(\Gamma), \end{gathered}$$

by using the extension $\varphi^{\rm ex} \in C^\infty(\overline{\Omega})$ of each $\varphi \in C^\infty(\Gamma)$.

Let ∇_Γ be the operator of surface-gradient on Γ, which is defined as:

$$\nabla_\Gamma : \varphi \in C^\infty(\Gamma) \mapsto \nabla_\Gamma \varphi := \nabla \varphi^{\rm ex} - (\nabla d_\Gamma \otimes \nabla d_\Gamma) \nabla \varphi^{\rm ex} \in C^\infty(\Gamma)^N, \tag{1.1}$$

by using the extension $\varphi^{\rm ex} \in C^\infty(\overline{\Omega})$ of each $\varphi \in C^\infty(\Gamma)$.

Let $\operatorname{div}_\Gamma$ be the operator of surface-divergence, which is defined as:

$$\operatorname{div}_\Gamma : \omega \in C^\infty(\Gamma)^N \mapsto \operatorname{div}_\Gamma \omega := \operatorname{div} \omega^{\rm ex} - \nabla(\omega^{\rm ex} \cdot \nabla d_\Gamma) \cdot \nabla d_\Gamma \in C^\infty(\Gamma), \tag{1.2}$$

by using the extension $\omega^{\rm ex} \in C^\infty(\overline{\Omega})^N$ of each $\omega \in C^\infty(\Gamma)^N$.

It is known that the definition formulas (1.1) and (1.2) are well-defined, and the values $\nabla_\Gamma \varphi$ and $\operatorname{div}_\Gamma \omega$ are settled independently of the choices of extensions $\varphi^{\rm ex} \in C^\infty(\overline{\Omega})$ and $\omega^{\rm ex} \in C^\infty(\overline{\Omega})^N$ of $\varphi \in C^\infty(\Gamma)$ and $\omega \in C^\infty(\Gamma)^N$, respectively.

On the basis of (1.1) and (1.2), the Laplace–Beltrami operator Δ_Γ, i.e. the surface-Laplacian on Γ is defined as follows:

$$\Delta_\Gamma : \varphi \in C^\infty(\Gamma) \mapsto \Delta_\Gamma \varphi := \operatorname{div}_\Gamma(\nabla_\Gamma \varphi) \in C^\infty(\Gamma).$$

Remark 1. Let us define a closed subspace $\boldsymbol{L}^2_{\rm div}(\Omega)$ in $L^2(\Omega)^N$, and a closed subspace $\boldsymbol{L}^2_{\rm tan}(\Gamma)$ in $L^2(\Gamma)^N$, by putting:

$$\begin{gathered} \boldsymbol{L}^2_{\rm div}(\Omega) := \left\{ \ \omega \in L^2(\Omega)^N \ \middle| \ \operatorname{div} \omega \in L^2(\Omega) \ \right\}, \\ \text{and } \ \boldsymbol{L}^2_{\rm tan}(\Gamma) := \left\{ \ \omega \in L^2(\Gamma)^N \ \middle| \ \omega \cdot n_\Gamma = 0 \text{ a.e. on } \Gamma \ \right\}, \ \text{ respectively.} \end{gathered}$$

Then, on account of the general theories as in [10, 18], we can see the following facts.

(Fact 1) (cf. [10]) The mapping $\nu \in H^1(\Omega)^N \mapsto \nu_{|_\Gamma} \cdot n_\Gamma \in H^{\frac{1}{2}}(\Gamma)$ can be extended as a linear and continuous operator $[(\,\cdot\,)_{|_\Gamma} \cdot n_\Gamma]$ from $\boldsymbol{L}^2_{\mathrm{div}}(\Omega)$ into $H^{-\frac{1}{2}}(\Gamma)$, such that:

$$\langle [\nu_{|_\Gamma} \cdot n_\Gamma], z_{|_\Gamma} \rangle_{H^{\frac{1}{2}}(\Gamma)} = \int_\Omega \mathrm{div}\, \nu\, z\, dx + \int_\Omega \nu \cdot \nabla z\, dx,$$
$$\text{for all } \nu \in \boldsymbol{L}^2_{\mathrm{div}}(\Omega) \text{ and } z \in H^1(\Omega).$$

(Fact 2) The surface gradient ∇_Γ can be extended as a linear and continuous operator from $H^1(\Gamma)$ into $\boldsymbol{L}^2_{\mathrm{tan}}(\Gamma)$. The extension is derived in the definition process of the space $H^1(\Gamma)$ as the completion of $C^\infty(\Gamma)$. Then, the topology of the completion is taken with respect to the norm, induced by the following bi-linear form:

$$[\varphi, \psi] \in C^\infty(\Gamma)^2 \mapsto \int_\Gamma \bigl(\varphi\psi + \nabla_\Gamma \varphi \cdot \nabla_\Gamma \psi\bigr)\, d\Gamma.$$

The inner product in $(\cdot,\cdot)_{H^1(\Gamma)}$ is given as the extension of the above bi-linear form. Hence, in this paper, we identify the operator ∇_Γ with the extension from $H^1(\Gamma)$ into $\boldsymbol{L}^2_{\mathrm{tan}}(\Gamma)$. Additionally, since the boundary $\Gamma = \partial\Omega$ has no boundary as $(N-1)$-dimensional surface, the dual space $H^1(\Gamma)^*$ of $H^1(\Gamma)$ coincides with $H^{-1}(\Gamma)$ $(= \overline{C^\infty_{\mathrm{c}}(\Gamma)})$, i.e. the closure of the class $C^\infty_{\mathrm{c}}(\Gamma)$ of smooth functions with compact supports on Γ in the topology of $H^1(\Gamma)^*$.

(Fact 3) The surface divergence div_Γ can be defined as a linear and continuous operator from $\boldsymbol{L}^2_{\mathrm{tan}}(\Gamma)$ into $H^{-1}(\Gamma)$ $(= H^1(\Gamma)^*)$, via the following Green-type formula (cf. [18, Section 2]):

$$-\int_\Gamma \mathrm{div}_\Gamma w\, z\, d\Gamma = \int_\Gamma w \cdot \nabla_\Gamma z\, d\Gamma, \text{ for any } z \in H^1(\Gamma) \text{ and}$$
$$\text{any } w \in \boldsymbol{L}^2_{\mathrm{tan}}(\Gamma) \text{ satisfying } \mathrm{div}_\Gamma w \in L^2(\Gamma).$$

Hence, in this paper, we regard the Laplace–Beltrami operator $\Delta_\Gamma = \mathrm{div}_\Gamma \circ \nabla_\Gamma$ as the linear and continuous operator from $H^1(\Gamma)$ into $H^{-1}(\Gamma)$.

Notation 5 (Notations in convex analysis)**.** For any proper lower semi-continuous (l.s.c. from now on) and convex function $\Psi : X \to (-\infty, \infty]$ defined on a Hilbert space X, we denote by $D(\Psi)$ its effective domain, and denote by $\partial\Psi$ its subdifferential. The subdifferential $\partial\Psi$ is a set-valued map corresponding to a weak differential of Ψ, and it has a maximal monotone graph in the product space $X^2 := X \times X$ (cf. [3, Chapter 2]). More precisely, for each $z_0 \in X$, the value $\partial\Psi(z_0)$ is defined as a set of all elements $z_0^* \in X$ which satisfy the following variational inequality:

$$(z_0^*, z - z_0)_X \le \Psi(z) - \Psi(z_0), \text{ for any } z \in D(\Psi).$$

For Hilbert spaces $X_1, \cdots, X_d$ with $1 < d \in \mathbb{N}$, let us consider a proper l.s.c. and convex function on the product space $X_1 \times \cdots \times X_d$:

$$\tilde{\Psi} : [z_1, \cdots, z_d] \in X_1 \times \cdots \times X_d \mapsto \tilde{\Psi}(z) = \tilde{\Psi}(z_1, \cdots, z_d) \in (-\infty, \infty].$$

Then, for any $i \in \{1, \ldots, d\}$, we denote by $\partial_{z_i}\tilde{\Psi} : X_1 \times \cdots \times X_d \to X_i$ a set-valued operator, which maps any $z = [z_1, \ldots, z_i, \ldots, z_d] \in X_1 \times \cdots \times X_i \times \cdots \times X_d$ to a subset in X_i:

$$\begin{aligned}\partial_{z_i}\tilde{\Psi}(z) &= \partial_{z_i}\tilde{\Psi}(z_1, \cdots, z_i, \cdots, z_d) \\ &:= \left\{ \tilde{z}^* \in X_i \;\middle|\; \begin{array}{r} (\tilde{z}^*, \tilde{z} - z_i)_{X_i} \le \tilde{\Psi}(z_1, \cdots, \tilde{z}, \cdots, z_d) \\ -\tilde{\Psi}(z_1, \cdots, z_i, \cdots, z_d), \text{ for any } \tilde{z} \in X_i \end{array} \right\}.\end{aligned}$$

Remark 2 (Examples of the subdifferentials)**.** As one of representatives of the subdifferentials, we exemplify the following set-valued function $\mathrm{Sgn} : \mathbb{R}^N \to 2^{\mathbb{R}^N}$, given as:

$$\omega \in \mathbb{R}^N \mapsto \mathrm{Sgn}(\omega) := \begin{cases} \dfrac{\omega}{|\omega|}, & \text{if } \omega \neq 0, \\ \overline{\mathbb{B}^N}, & \text{otherwise.} \end{cases}$$

It is known that the set-valued function Sgn coincides with the subdifferential of the Euclidean norm $|\cdot| : \omega \in \mathbb{R}^N \mapsto |\omega| = \sqrt{\omega \cdot \omega} \in [0, \infty)$, i.e.:

$$\partial|\cdot|(\omega) = \mathrm{Sgn}(\omega), \text{ for any } \omega \in D(\partial|\cdot|) = \mathbb{R}^N.$$

Also, it is known that (cf. [2, Section 2 in Chapter 2], [3, Chapter 2]) the operator $-\varDelta_N : z \in D(\varDelta_N) \subset L^2(\Omega) \mapsto -\varDelta z \in L^2(\Omega)$ coincides with the subdifferential of a proper l.s.c. and convex function Ψ_N on $L^2(\Omega)$, defined as:

$$z \in L^2(\Omega) \mapsto \Psi_N(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 \, dx, \text{ if } z \in H^1(\Omega), \\ \infty, \quad \text{otherwise.} \end{cases}$$

More precisely:

$$\partial\Psi_N(z) = \{-\varDelta_N z\} \text{ in } L^2(\Omega), \text{ for any } z \in D(\partial\Psi_N) = D(\varDelta_N).$$

Remark 3. As is easily checked:

$$\partial\tilde{\Psi} \subset \partial_{z_1}\tilde{\Psi} \times \cdots \times \partial_{z_d}\tilde{\Psi} \text{ in } [X_1 \times \cdots \times X_d]^2,$$

but it should be noted that the converse inclusion is not true, necessarily. In fact, the monotonicity of $\partial_{z_1}\tilde{\Psi} \times \cdots \times \partial_{z_d}\tilde{\Psi}$ in $[X_1 \times \cdots \times X_d]^2$ is not so obvious.

Finally, we note that notions of functional-convergence.

Definition 1 (Mosco-convergence: cf. [17])**.** Let X be an abstract Hilbert space. Let $\Psi : X \to (-\infty, \infty]$ be a proper l.s.c. and convex function, and let $\{\Psi_n\}_{n=1}^\infty$ be a sequence of proper l.s.c. and convex functions $\Psi_n : X \to (-\infty, \infty]$, $n \in \mathbb{N}$. Then, it is said that $\Psi_n \to \Psi$ on X, in the sense of Mosco, as $n \to \infty$, iff. the following two conditions are fulfilled.

(M1) Lower-bound condition: $\varliminf_{n\to\infty} \Psi_n(\check{z}_n) \geq \Psi(\check{z})$, if $\check{z} \in X$, $\{\check{z}_n\}_{n=1}^{\infty} \subset X$, and $\check{z}_n \to \check{z}$ weakly in X as $n \to \infty$.

(M2) Optimality condition: for any $\hat{z} \in D(\Psi)$, there exists a sequence $\{\hat{z}_n\}_{n=1}^{\infty} \subset X$ such that $\hat{z}_n \to \hat{z}$ in X and $\Psi_n(\hat{z}_n) \to \Psi(\hat{z})$, as $n \to \infty$.

Remark 4. As a basic matter of the Mosco-convergence, we can see the following fact (see [1, Theorem 3.66], [9, Chapter 2], and so on).

(Fact 4) Let X, Ψ and $\{\Psi_n\}_{n=1}^{\infty}$ be as in Definition 1. Besides, let us assume that:

$$\Psi_n \to \Psi \text{ on } X\text{, in the sense of Mosco, as } n \to \infty,$$

and

$$\begin{cases} [z, z^*] \in X^2,\ [z_n, z_n^*] \in \partial\Psi_n \text{ in } X^2,\ n \in \mathbb{N}, \\ z_n \to z \text{ in } X \text{ and } z_n^* \to z^* \text{ weakly in } X, \text{ as } n \to \infty. \end{cases}$$

Then, it holds that:

$$[z, z^*] \in \partial\Psi \text{ in } X^2, \text{ and } \Psi_n(z_n) \to \Psi(z), \text{ as } n \to \infty.$$

2 Statements of Main Theorems

Next, we state Main Theorems in this paper. First, let us set the product spaces:

- $H := L^2(\Omega) \times L^2(\Gamma)$, $\mathscr{H} := L^2(\Omega) \times H$,
- $$V_\varepsilon := \left\{ w = [\xi, \xi_\Gamma] \in H \;\middle|\; \begin{array}{l} \xi \in H^1(\Omega),\ \xi_\Gamma \in H^{\frac{1}{2}}(\Gamma),\ \varepsilon\xi_\Gamma \in H^1(\Gamma), \\ \text{and } \xi_{|\Gamma} = \xi_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma) \end{array} \right\}, \tag{2.1}$$
- $$\mathscr{V}_\varepsilon := \left\{ z = [\zeta, w] \in \mathscr{H} \;\middle|\; \zeta \in H^1(\Omega) \text{ and } w = [\xi, \xi_\Gamma] \in V_\varepsilon \right\}, \text{ for any } \varepsilon \geq 0. \tag{2.2}$$

Remark 5. If $\varepsilon > 0$ (resp. $\varepsilon = 0$), then V_ε and $\mathscr{V}_\varepsilon$ (resp. V_0 and $\mathscr{V}_0$), given in (2.1) and (2.2), are closed linear spaces in $H^1(\Omega) \times H^1(\Gamma)$ and $H^1(\Omega) \times (H^1(\Omega) \times H^1(\Gamma))$ (resp. $H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$ and $H^1(\Omega) \times (H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma))$), and hence, they are Hilbert spaces endowed with the inner products of $H^1(\Omega) \times H^1(\Gamma)$ and $H^1(\Omega) \times (H^1(\Omega) \times H^1(\Gamma))$ (resp. $H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$ and $H^1(\Omega) \times (H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma))$), respectively.

Next, we prescribe the assumptions in this study.

(A0) $1 < N \in \mathbb{N}$, $T > 0$, $\nu > 0$, and $\Omega \subset \mathbb{R}^N$ be the fixed constants, and the bounded domain as in Notation 3. Besides, let $n_\Gamma : \Gamma \to \mathbb{S}^{N-1}$ be the unit outer normal on $\Gamma := \partial\Omega$.

(A1) $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, such that:

$$g(0) \leq 0 \text{ and } g(1) \geq 0.$$

Also, g is supposed to have a nonnegative potential $\hat{g} : \mathbb{R} \to [0, \infty)$.

(A2) $\alpha_0 : \mathbb{R} \to (0, \infty)$ is a Lipschitz continuous function on $\mathbb{R}$.

(A3) $\alpha : \mathbb{R} \to (0, \infty)$ is a C^2-function such that $\alpha'(0) = 0$, $\alpha' \in L^\infty(\mathbb{R})$ and $\alpha'' \geq 0$ on $\mathbb{R}$. Also, $\alpha\alpha'$ is a Lipschitz continuous function on $\mathbb{R}$.

(A4) There exists a positive constant $\delta_\alpha > 0$, such that:

$$\alpha_0(\sigma) \geq \delta_\alpha \text{ and } \alpha(\sigma) \geq \delta_\alpha, \text{ for any } \sigma \in \mathbb{R}.$$

(A5) There are two fixed constants m_0 and M_0, and for any $\varepsilon \geq 0$, the initial data $u_0 = [\eta_0, v_0] = [\eta_0, \theta_0, \theta_{\Gamma,0}]$ belongs to a class $\mathscr{D}_\varepsilon \subset \mathscr{H}$, defined as:

$$\mathscr{D}_\varepsilon := \left\{ \begin{array}{l|l} z = [\zeta, w] = [\zeta, \xi, \xi_\Gamma] \in \mathscr{V}_\varepsilon & 0 \leq \zeta \leq 1,\, m_0 \leq \xi \leq M_0, \text{ a.e. in } \Omega, \\ \text{with } w = [\xi, \xi_\Gamma] \in V_\varepsilon & \text{and } m_0 \leq \xi_\Gamma \leq M_0, \text{ a.e. on } \Gamma \end{array} \right\}.$$

Remark 6 (Possible choices of given functions)**.** Referring to [12, 13], the settings

$$g(\sigma) = \sigma - 1 \text{ with } \hat{g}(\sigma) := \frac{1}{2}(\sigma - 1)^2, \text{ and } \alpha_0(\sigma) = \alpha(\sigma) = \frac{\sigma^2}{2} + \delta_\alpha, \text{ for any } \sigma \in \mathbb{R},$$

provide possible functions that fulfill the assumptions (A1)–(A4).

On this basis, we define the solution to $(\mathrm{KWC})_\varepsilon$, for any $\varepsilon \geq 0$, as follows.

Definition 2 (Definition of solution)**.** For any $\varepsilon \geq 0$, a triplet of functions $u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in L^2(0, T; \mathscr{H})$ with $v = [\theta, \theta_\Gamma] \in L^2(0, T; H)$ is called a solution to $(\mathrm{KWC})_\varepsilon$, iff. the following items hold.

(S0) $u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in W^{1,2}(0, T; \mathscr{H}) \cap L^\infty(0, T; \mathscr{V}_\varepsilon)$,

$$0 \leq \eta \leq 1,\ m_0 \leq \theta \leq M_0, \text{ a.e. in } Q, \text{ and } m_0 \leq \theta_\Gamma \leq M_0 \text{ a.e. on } \Sigma.$$

(S1) η solves the following variational equality:

$$\int_\Omega (\partial_t \eta(t) + g(\eta(t)) + \alpha'(\eta(t))|\nabla\theta(t)|)\varphi \, dx + \int_\Omega \nabla\eta(t) \cdot \nabla\varphi \, dx = 0,$$
$$\text{for any } \varphi \in H^1(\Omega), \text{ a.e. } t \in (0, T),$$

subject to the initial condition $\eta(0) = \eta_0$ in $L^2(\Omega)$.

(S2) A pair of functions $v = [\theta, \theta_\Gamma]$ solves the following variational inequality:

$$\begin{aligned} &\int_\Omega \alpha_0(\eta(t))\partial_t\theta(t)(\theta(t) - \psi) \, dx + \nu^2 \int_\Omega \nabla\theta(t) \cdot \nabla(\theta(t) - \psi) \, dx \\ &+ \int_\Gamma \partial_t\theta_\Gamma(t)(\theta_\Gamma(t) - \psi_\Gamma) \, d\Gamma + \int_\Gamma \nabla_\Gamma(\varepsilon\theta_\Gamma(t)) \cdot \nabla_\Gamma(\varepsilon(\theta_\Gamma(t) - \psi_\Gamma)) \, d\Gamma \\ &+ \int_\Omega \alpha(\eta(t))|\nabla\theta(t)| \, dx \leq \int_\Omega \alpha(\eta(t))|\nabla\psi| \, dx, \end{aligned}$$
$$\text{for any } [\psi, \psi_\Gamma] \in V_\varepsilon, \text{ a.e. } t \in (0, T),$$

subject to the initial conditions $v(0) = [\theta(0), \theta_\Gamma(0)] = v_0 = [\theta_0, \theta_{\Gamma,0}]$ in H.

In this paper, the keypoints of our mathematical analysis are to reformulate the free-energy, given in (0.6), to the following form:

$$\mathscr{F}_\varepsilon(\eta,\theta,\theta_\Gamma) = \Phi_\varepsilon(\eta,\theta,\theta_\Gamma) + \int_\Omega \hat{G}(\eta)\,dx,$$

where

$$\eta \in \mathbb{R} \mapsto \hat{G}(\eta) := \hat{g}(\eta) - \frac{1}{2\nu^2}\bigl(\alpha(\eta)\bigr)^2 \in \mathbb{R},$$

with the convex function $\Phi_\varepsilon : \mathscr{H} \to [0,\infty]$, defined as:

$$\begin{aligned} u &= [\eta, v] = [\eta,\theta,\theta_\Gamma] \in \mathscr{H} \mapsto \Phi_\varepsilon(u) = \Phi_\varepsilon(\eta, v) = \Phi_\varepsilon(\eta,\theta,\theta_\Gamma) \\ &:= \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\eta|^2\,dx + \dfrac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2\,d\Gamma + \dfrac{1}{2}\int_\Omega \left(\nu|\nabla\theta| + \dfrac{1}{\nu}\alpha(\eta)\right)^2 dx, \\ \qquad \text{if } u = [\eta, v] = [\eta,\theta,\theta_\Gamma] \in \mathscr{V}_\varepsilon, \text{ with } v = [\theta,\theta_\Gamma] \in V_\varepsilon, \\ \infty, \quad \text{otherwise.} \end{cases} \\ &\text{for } \varepsilon \geq 0. \end{aligned} \tag{2.3}$$

Note that the function $\hat{G}$ is a primitive of:

$$\eta \in \mathbb{R} \mapsto G(\eta) := g(\eta) - \nu^{-2}\alpha(\eta)\alpha'(\eta) \in \mathbb{R}.$$

Moreover, by the assumptions (A1) and (A3), G is Lipschitz continuous function on $\mathbb{R}$, and for the primitive $\hat{G} \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ of G, it holds that:

$$\bigl|\hat{G}(\tilde{\eta}) - \hat{G}(\eta) - G(\eta)(\tilde{\eta} - \eta)\bigr| \leq \frac{|G'|_{L^\infty(\mathbb{R})}}{2}|\tilde{\eta} - \eta|^2, \text{ for all } \eta, \tilde{\eta} \in \mathbb{R}. \tag{2.4}$$

On the basis of this reformulation, we associate the system (KWC)$_\varepsilon$ with the following Cauchy problem of an evolution equation:

$$\begin{cases} A_0(u(t))u'(t) + \partial\Phi_\varepsilon(u(t)) + \mathcal{G}(u(t)) \ni 0 \text{ in } \mathscr{H}, \text{ a.e. } t \in (0,T), \\ u(0) = u_0 \text{ in } \mathscr{H}, \end{cases} \tag{2.5}$$

which is governed by the subdifferential $\partial\Phi_\varepsilon$ of the convex function Φ_ε on $\mathscr{H}$. In the context, the unknown $u \in C([0,T];\mathscr{H})$ is associated with the solution $[\eta,\theta,\theta_\Gamma]$ of the system (KWC)$_\varepsilon$, i.e.:

$$\begin{cases} u(t) = [\eta(t), v(t)] = [\eta(t),\theta(t),\theta_\Gamma(t)] \text{ in } \mathscr{H} \\ \qquad\qquad \text{with } v(t) = [\theta(t),\theta_\Gamma(t)] \in H, \text{ for any } t \in [0,T], \\ u_0 = [\eta_0, v_0] = [\eta_0,\theta_0,\theta_{\Gamma,0}] \text{ in } \mathscr{H} \text{ with } v_0 = [\theta_0,\theta_{\Gamma,0}] \in H. \end{cases}$$

Besides, A_0 is an operator, defined as:

$$u = [\eta, v] = [\eta,\theta,\theta_\Gamma] \in \mathscr{D}_\varepsilon \mapsto A_0(u) = A_0(\eta) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_0(\eta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{L}(\mathscr{H};\mathscr{H}),$$

and $\mathcal{G} : \mathscr{H} \to \mathscr{H}$ is a Lipschitz operator, defined as:

$$u = [\eta, v] = [\eta,\theta,\theta_\Gamma] \in \mathscr{H} \mapsto \mathcal{G}(u) = \mathcal{G}(\eta) := {}^{\mathrm{t}}[G(\eta), 0, 0] \in \mathscr{H}.$$

Remark 7. We can easily show that the functional Φ_ε, given in (2.3), is a proper, l.s.c. and convex function on $\mathscr{H}$. Therefore, for each $\varepsilon \geq 0$, the subdifferential $\partial\Phi_\varepsilon$ is a maximal monotone graph in $\mathscr{H}^2$. However, the presence of $A_0(u(t))$ does not allow us to apply the general theories for nonlinear evolution equations, e.g. [2, 3].

Remark 8. Notice that:

$$\begin{aligned}\Phi_\varepsilon(\eta, \tilde{v}) - \Phi_\varepsilon(\eta, v) &= \int_\Omega \alpha(\eta)|\nabla\tilde{\theta}|\, dx + \frac{\nu^2}{2}\int_\Omega |\nabla\tilde{\theta}|^2\, dx + \frac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon\tilde{\theta}_\Gamma)|^2\, d\Gamma \\ &\quad - \int_\Omega \alpha(\eta)|\nabla\theta|\, dx - \frac{\nu^2}{2}\int_\Omega |\nabla\theta|^2\, dx - \frac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2\, d\Gamma,\end{aligned}$$

for all $\eta \in L^2(\Omega)$, $v = [\theta, \theta_\Gamma] \in V_\varepsilon$, and $\tilde{v} = [\tilde{\theta}, \tilde{\theta}_\Gamma] \in V_\varepsilon$.

So, putting:

$$\tilde{A}_0(\tilde{\eta}) := \begin{bmatrix} \alpha_0(\tilde{\eta}) & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{L}(H; H), \text{ for any } \tilde{\eta} \in L^\infty(\Omega),$$

it is easily seen that the condition (S2) in Definition 2 is equivalent to the following Cauchy problem:

$$\begin{cases} \tilde{A}_0(\eta)\partial_t v(t) + \partial_v \Phi_\varepsilon(\eta(t), v(t)) \ni 0 \text{ in } H, \text{ a.e. } t \in (0, T), \\ v(0) = v_0 \text{ in } H. \end{cases}$$

Based on these, our Main Theorems are stated as follows.

Main Theorem 1 (Existence of solutions and uniqueness)**.** *Under the all assumptions (A0)–(A5), the following two items hold.*

(A) *The Cauchy problem* (2.5) *admits at least one solution* $u \in L^2(0, T; \mathscr{H})$. *In particular, if* α_0 *is a constant, then the solution is unique.*

(B) *The solution* $u = [\eta, v] = [\eta, \theta, \theta_\Gamma]$ *to the Cauchy problem* (2.5)*, with* $v = [\theta, \theta_\Gamma]$ *is a solution to the Kobayashi–Warren–Carter system (KWC)*$_\varepsilon$*.*

Remark 9. Note that the uniqueness in nonconstant case of α_0 is still open. Hence, the item (B) is not sufficient to show the equivalence between (2.5) and (KWC)$_\varepsilon$, and the class of solutions to (KWC)$_\varepsilon$ may not be a singleton, for any $\varepsilon \geq 0$.

Main Theorem 2 (ε-upper semi-continuity of solution classes)**.** *Under the all assumptions (A0)–(A5), let us fix any* $\varepsilon_0 \geq 0$*, and take a sequence of initial data* $\{u_{0,\varepsilon}\}_{\varepsilon>0} = \{[\eta_{0,\varepsilon}, \theta_{0,\varepsilon}, \theta_{\Gamma,0,\varepsilon}]\}_{\varepsilon>0} \subset \mathscr{V}_\varepsilon$*, such that:*

$$[\eta_{0,\varepsilon}, \theta_{0,\varepsilon}, \varepsilon\theta_{\Gamma,0,\varepsilon}] \to [\eta_{0,\varepsilon_0}, \theta_{0,\varepsilon_0}, \varepsilon_0\theta_{\Gamma,0,\varepsilon_0}] \text{ weakly in } H^1(\Omega)^2 \times H^1(\Gamma), \text{ as } \varepsilon \to \varepsilon_0. \tag{2.6}$$

Also, for any $\varepsilon \geq 0$*, we denote by* $\mathfrak{S}_\varepsilon(u_{0,\varepsilon})$ *the class of all solutions* $u = [\eta, \theta, \theta_\Gamma]$ *to (KWC)*$_\varepsilon$ *subject to the initial condition* $u(0) = u_{0,\varepsilon}$ *in* $\mathscr{H}$*. Besides, we define the* ω*-limit set* $\overline{\lim}_{\varepsilon\to\varepsilon_0} \mathfrak{S}_\varepsilon(u_{0,\varepsilon})$ *of the sequence of solution classes* $\{\mathfrak{S}_\varepsilon(u_{0,\varepsilon})\}_{\varepsilon\geq 0}$*, as* $\varepsilon \to \varepsilon_0$*, by letting:*

$$\overline{\lim_{\varepsilon\to\varepsilon_0}}\, \mathfrak{S}(u_{0,\varepsilon}) := \left\{ u = [\eta, \theta, \theta_\Gamma] \;\middle|\; \begin{array}{l} \text{there exists } \{\varepsilon_n\}_{n=1}^\infty \subset [0, \infty) \text{ and} \\ \{u_n = [\eta_n, \theta_n, \theta_{\Gamma,n}] \in \mathfrak{S}_{\varepsilon_n}(u_{0,\varepsilon_n})\}_{n=1}^\infty \\ \text{such that } u_n \to u \text{ in } \mathscr{H}, \text{ as } n \to \infty. \end{array} \right\}.$$

Then, the following two items hold.

(C) $\overline{\lim}_{\varepsilon\to\varepsilon_0} \mathfrak{S}_\varepsilon(u_{0,\varepsilon})$ *is nonempty and compact in* $\mathscr{H}$.

(D) $\overline{\lim}_{\varepsilon\to\varepsilon_0} \mathfrak{S}_\varepsilon(u_{0,\varepsilon}) \subset \mathfrak{S}_{\varepsilon_0}(u_{0,\varepsilon_0})$.

Remark 10. The smoothness of Γ and the Green type formula (cf. [1, Proposition 5.6.2]) allows us to derive

$$\theta_{\Gamma,0,\varepsilon} = \theta_{0,\varepsilon|\Gamma} \to \theta_{0,\varepsilon_0|\Gamma} = \theta_{\Gamma,0,\varepsilon_0} \text{ weakly in } H^{\frac{1}{2}}(\Gamma), \text{ as } \varepsilon \to \varepsilon_0,$$

from the assumption (2.6).

3 Key-Lemmas

In this Section, we prove several Key-Lemmas that are vital for our Main Theorems.

We begin by prescribing a class of relaxed convex functions. For every $\varepsilon \geq 0$ and $0 \leq \delta \leq 1$, let us define:

$$\begin{aligned} u &= [\eta, v] = [\eta, \theta, \theta_\Gamma] \in \mathscr{H} \mapsto \Phi_\varepsilon^\delta(u) = \Phi_\varepsilon^\delta(\eta, v) = \Phi_\varepsilon^\delta(\eta, \theta, \theta_\Gamma) \\ &:= \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\eta|^2\,dx + \dfrac{1}{2}\displaystyle\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2\,d\Gamma + \dfrac{1}{2}\displaystyle\int_\Omega \left(\nu f_\delta(\nabla\theta) + \dfrac{1}{\nu}\alpha(\eta)\right)^2 dx, \\ \qquad \text{if } u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in \mathscr{V}_\varepsilon, \text{ with } v = [\theta, \theta_\Gamma] \in V_\varepsilon, \\ \infty, \quad \text{otherwise}, \end{cases} \end{aligned}$$

with use of the following real convex function:

$$f_\delta : \omega \in \mathbb{R}^N \mapsto f_\delta(\omega) := \sqrt{\delta^2 + |\omega|^2} \in [0, \infty). \tag{3.1}$$

As is easily checked, the functional Φ_ε^δ, for every $\varepsilon \geq 0$ and $0 \leq \delta \leq 1$, is proper, l.s.c. and convex on $\mathscr{H}$. Especially, for any $\varepsilon \geq 0$, the class of convex functions $\{\Phi_\varepsilon^\delta \,|\, 0 \leq \delta \leq 1\}$ forms a relaxation sequence for Φ_ε^0, i.e. the convex function Φ_ε^δ when $\delta = 0$, and Φ_ε^0 coincides with the convex function Φ_ε, given in (2.3).

On this basis, we can prove the following Key-Lemmas.

Key-Lemma 1 (Representation of $\partial_\eta\Phi_\varepsilon^\delta$)**.** *For every $\varepsilon \geq 0$ and $0 \leq \delta \leq 1$, it holds that:*

$$D(\partial_\eta\Phi_\varepsilon^\delta) = D(\varDelta_N) \times V_\varepsilon,$$

and

$$\begin{gathered} \partial_\eta\Phi_\varepsilon^\delta(\eta, v) = -\varDelta_N\eta + \alpha'(\eta) f_\delta(\nabla\theta) + \nu^{-2}\alpha(\eta)\alpha'(\eta) \text{ in } L^2(\Omega), \\ \text{for any } \eta \in D(\varDelta_N), \text{ and any } v = [\theta, \theta_\Gamma] \in V_\varepsilon. \end{gathered}$$

Proof. By virtue of (A3), we can verify this Key-Lemma 1 as a straightforward consequence of the general theories of subdifferentials, e.g. [2, Section 2 in Chapter 2], [3, Chapter 2], and so on. $\square$

Key-Lemma 2 (Representation of $\partial_v \Phi_\varepsilon^\delta$)**.** *For every $\varepsilon \geq 0$, and $0 < \delta \leq 1$, let us set:*

$$D(\mathcal{A}_\varepsilon^\delta) := \left\{ [\eta, \theta, \theta_\Gamma] \in \mathscr{V}_\varepsilon \;\middle|\; \begin{array}{l} \alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta \in \boldsymbol{L}^2_{\mathrm{div}}(\Omega), \\ -\varDelta_\Gamma(\varepsilon^2\theta_\Gamma) + \bigl[(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta)_{|_\Gamma} \cdot n_\Gamma\bigr] \in L^2(\Gamma) \end{array} \right\}, \tag{3.2}$$

and let us define a single-valued operator $\mathcal{A}_\varepsilon^\delta : D(\mathcal{A}_\varepsilon^\delta) \subset \mathscr{H} \to H$, by putting:

$$\begin{aligned} u =& [\eta, \theta, \theta_\Gamma] \in D(\mathcal{A}_\varepsilon^\delta) \subset \mathscr{H} \mapsto \mathcal{A}_\varepsilon^\delta u = \mathcal{A}_\varepsilon^\delta [\eta, \theta, \theta_\Gamma] \\ :=& \left[\begin{array}{c} -\mathrm{div}\bigl(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta\bigr) \\ -\varDelta_\Gamma(\varepsilon^2\theta_\Gamma) + \bigl[(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta)_{|_\Gamma} \cdot n_\Gamma\bigr] \end{array} \right] \in H. \end{aligned} \tag{3.3}$$

Then, it holds that:

$$\partial_v \Phi_\varepsilon^\delta = \mathcal{A}_\varepsilon^\delta \text{ in } \mathscr{H} \times H, \text{ for every } \varepsilon \geq 0, \text{ and } 0 < \delta \leq 1.$$

Proof. First, we show that $\mathcal{A}_\varepsilon^\delta \subset \partial_v \Phi_\varepsilon^\delta$ in $\mathscr{H} \times H$. Let us assume that:

$$\begin{gathered} u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in D(\mathcal{A}_\varepsilon^\delta) \text{ with } v = [\theta, \theta_\Gamma] \in V_\varepsilon, \\ \text{and } v^* = [\theta^*, \theta_\Gamma^*] = \mathcal{A}_\varepsilon^\delta u = \mathcal{A}_\varepsilon^\delta [\eta, v] = \mathcal{A}_\varepsilon^\delta [\eta, \theta, \theta_\Gamma] \text{ in } H. \end{gathered} \tag{3.4}$$

Then, by using Remark 1 (Fact 1)–(Fact 3), (3.2), (3.3), [4, Key-Lemma 1], and Young's inequality, we obtain that:

$$\begin{aligned} (v^*, w - v)_H &= (\theta^*, \xi - \theta)_{L^2(\Omega)} + (\theta_\Gamma^*, \xi_\Gamma - \theta_\Gamma)_{L^2(\Gamma)} \\ &= \int_\Omega \bigl(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta\bigr) \cdot \nabla(\xi - \theta)\, dx \\ &\quad + \int_\Gamma \nabla_\Gamma(\varepsilon\theta_\Gamma) \cdot \nabla_\Gamma(\varepsilon(\xi_\Gamma - \theta_\Gamma))\, d\Gamma \\ &\leq \int_\Omega \alpha(\eta)(f_\delta(\nabla\xi) - f_\delta(\nabla\theta))\, dx + \frac{\nu^2}{2} \int_\Omega (|\nabla\xi|^2 - |\nabla\theta|^2)\, dx \\ &\quad + \frac{1}{2} \int_\Gamma (|\nabla_\Gamma(\varepsilon\xi_\Gamma)|^2 - |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2)\, d\Gamma \\ &= \Phi_\varepsilon^\delta(\eta, w) - \Phi_\varepsilon^\delta(\eta, v), \text{ for any } w = [\xi, \xi_\Gamma] \in V_\varepsilon, \end{aligned}$$

which implies that:

$$\begin{gathered} u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in D(\partial_v \Phi_\varepsilon^\delta) \text{ with } v = [\theta, \theta_\Gamma] \in V_\varepsilon, \\ \text{and } v^* = [\theta^*, \theta_\Gamma^*] \in \partial_v \Phi_\varepsilon^\delta(u) = \partial_v \Phi_\varepsilon^\delta(\eta, v) = \partial_v \Phi_\varepsilon^\delta(\eta, \theta, \theta_\Gamma) \text{ in } H. \end{gathered} \tag{3.5}$$

Conversely, if (3.5) holds, i.e.:

$$\begin{gathered} (v^*, \tilde{w} - v)_H = (\theta^*, \tilde{\xi} - \theta)_{L^2(\Omega)} + (\theta_\Gamma^*, \tilde{\xi}_\Gamma - \theta_\Gamma)_{L^2(\Gamma)} \leq \Phi_\varepsilon^\delta(\eta, \tilde{w}) - \Phi_\varepsilon^\delta(\eta, v), \\ \text{for any } \tilde{w} = [\tilde{\xi}, \tilde{\xi}_\Gamma] \in V_\varepsilon. \end{gathered} \tag{3.6}$$

Then, taking arbitrary $\sigma > 0$, $w = [\xi, \xi_\Gamma]$, putting $\tilde{w} = [\tilde{\xi}, \tilde{\xi}_\Gamma] = [\theta + \sigma\xi, \theta_\Gamma + \sigma\xi_\Gamma]$ in (3.6), and invoking (3.1), we compute that:

$$
\begin{aligned}
(v^*, w)_H \leq\ & \frac{1}{2\sigma} \int_\Gamma (|\nabla_\Gamma(\varepsilon(\theta_\Gamma + \sigma\xi_\Gamma))|^2 - |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2)\, d\Gamma \\
& + \frac{1}{2\sigma} \int_\Omega \left[\left(\nu f_\delta(\nabla(\theta + \sigma\xi)) + \frac{1}{\nu}\alpha(\eta) \right)^2 - \left(\nu f_\delta(\nabla\theta) + \frac{1}{\nu}\alpha(\eta) \right)^2 \right] dx \\
& \to \int_\Gamma \nabla_\Gamma(\varepsilon\theta_\Gamma) \cdot \nabla_\Gamma(\varepsilon\xi_\Gamma)\, d\Gamma + \int_\Omega \left(\nu f_\delta(\nabla\theta) + \frac{1}{\nu}\alpha(\eta) \right) \nu \nabla f_\delta(\nabla\theta) \cdot \nabla\xi\, dx \\
& = \int_\Gamma \nabla_\Gamma(\varepsilon\theta_\Gamma) \cdot \nabla_\Gamma(\varepsilon\xi_\Gamma)\, d\Gamma + \int_\Omega (\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta) \cdot \nabla\xi\, dx, \text{ as } \sigma \downarrow 0,
\end{aligned}
$$

and therefore

$$
\begin{gathered}
(v^*, w)_H = \int_\Omega (\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta) \cdot \nabla\xi\, dx + \int_\Gamma \nabla_\Gamma(\varepsilon\theta_\Gamma) \cdot \nabla_\Gamma(\varepsilon\xi_\Gamma)\, d\Gamma, \\
\text{for any } w = [\xi, \xi_\Gamma] \in V_\varepsilon.
\end{gathered} \tag{3.7}
$$

Here, taking any $\xi_0 \in H^1_0(\Omega)$ and putting $w = [\xi_0, 0]$ in (3.7), we deduce that:

$$
\theta^* = -\mathrm{div}(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta) \in L^2(\Omega) \text{ in } \mathscr{D}'(\Omega). \tag{3.8}
$$

Additionally, from Remark 1 (Fact 1)–(Fact 3), (3.7), and (3.8), one can see that:

$$
\begin{aligned}
(\theta^*_\Gamma, \xi_\Gamma)_{L^2(\Gamma)} =\ & \int_\Omega (\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta) \cdot \nabla\xi\, dx - (\theta^*, \xi)_{L^2(\Omega)} \\
& + \int_\Gamma \nabla_\Gamma(\varepsilon\theta_\Gamma) \cdot \nabla_\Gamma(\varepsilon\xi_\Gamma) d\,\Gamma \\
=\ & \left\langle \left[(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta)_{|_\Gamma} \cdot n_\Gamma \right], \xi_{|_\Gamma} \right\rangle_{H^{\frac{1}{2}}(\Gamma)} \\
& + \left\langle -\Delta_\Gamma(\varepsilon\theta_\Gamma), \varepsilon\xi_\Gamma \right\rangle_{H^1(\Gamma)}, \text{ for any } w = [\xi, \xi_\Gamma] \in V_\varepsilon.
\end{aligned}
$$

This identity implies that:

$$
\theta^*_\Gamma = -\Delta_\Gamma(\varepsilon^2\theta_\Gamma) + \left[(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta)_{|_\Gamma} \cdot n_\Gamma \right] \in L^2(\Gamma) \text{ in } H^{-1}(\Gamma). \tag{3.9}
$$

As a consequence of (3.8) and (3.9), we verify (3.4).

Thus, we conclude this Key-Lemma 2. □

Key-Lemma 3 (Representation of $\partial\Phi^\delta_\varepsilon$)**.** *For every $\varepsilon \geq 0$, $0 < \delta \leq 1$, it holds that:*

$$
\partial\Phi^\delta_\varepsilon = \partial_\eta\Phi^\delta_\varepsilon \times \partial_v\Phi^\delta_\varepsilon \text{ in } \mathscr{H}^2,
$$

more precisely:

$$
D(\partial\Phi^\delta_\varepsilon) = D(\partial_\eta\Phi^\delta_\varepsilon) \times D(\partial_v\Phi^\delta_\varepsilon) \text{ in } L^2(\Omega) \times H,
$$

and

$$
\begin{gathered}
\partial\Phi^\delta_\varepsilon(u) = \partial_\eta\Phi^\delta_\varepsilon(\eta, v) \times \partial_v\Phi^\delta_\varepsilon(\eta, v) \text{ in } L^2(\Omega) \times H, \\
\text{for any } u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in L^2(\Omega) \times H \text{ with } v = [\theta, \theta_\Gamma] \in V_\varepsilon.
\end{gathered}
$$

Proof. From Remark 3, it is sufficient to show $\partial\Phi_\varepsilon^\delta \supset \partial_\eta\Phi_\varepsilon^\delta \times \partial_v\Phi_\varepsilon^\delta$ in $\mathscr{H}^2$. Let us assume:

$$\begin{gathered} u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in D(\partial_\eta\Phi_\varepsilon^\delta) \times D(\partial_v\Phi_\varepsilon^\delta) \text{ with } v = [\theta, \theta_\Gamma] \in V_\varepsilon, \\ \text{and } u^* = [\eta^*, v^*] = [\eta^*, \theta^*, \theta_\Gamma^*] \in \partial_\eta\Phi_\varepsilon^\delta(\eta, v) \times \partial_v\Phi_\varepsilon^\delta(\eta, v) \text{ in } L^2(\Omega) \times H. \end{gathered}$$

Then, with Key-Lemmas 1 and 2 in mind, it holds that:

$$\eta^* = \partial_\eta\Phi_\varepsilon^\delta(\eta, v) = -\mathit{\Delta}_N\eta + \alpha'(\eta)f_\delta(\nabla\theta) + \nu^{-2}\alpha(\eta)\alpha'(\eta) \text{ in } L^2(\Omega), \tag{3.10}$$

$$v^* = \partial_v\Phi_\varepsilon^\delta(\eta, v) = \begin{bmatrix} -\mathrm{div}\big(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta\big) \\ -\mathit{\Delta}_\Gamma(\varepsilon^2\theta_\Gamma) + \big[(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta)_{|\Gamma} \cdot n_\Gamma\big] \end{bmatrix} \text{ in } H. \tag{3.11}$$

Here, since the subdifferential (gradient) of the real convex function:

$$[\tilde{\eta}, \tilde{\omega}] \in \mathbb{R} \times \mathbb{R}^N \mapsto \frac{1}{2}\left(\nu f_\delta(\tilde{\omega}) + \frac{1}{\nu}\alpha(\tilde{\eta})\right)^2,$$

coincides with the vectorial function:

$$[\tilde{\eta}, \tilde{\omega}] \in \mathbb{R} \times \mathbb{R}^N \mapsto \begin{bmatrix} \alpha'(\tilde{\eta})f_\delta(\tilde{\omega}) + \nu^{-2}\alpha(\tilde{\eta})\alpha'(\tilde{\eta}) \\ \alpha(\tilde{\eta})\nabla f_\delta(\tilde{\omega}) + \nu^2\tilde{\omega} \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^N, \tag{3.12}$$

we can observe that:

$$\begin{aligned} &(u^*, z - u)_{\mathscr{H}} = (\eta^*, \zeta - \eta)_{L^2(\Omega)} + (v^*, w - v)_H \\ &= \int_\Omega \nabla\eta \cdot \nabla(\xi - \eta)\,dx + \int_\Gamma \nabla_\Gamma(\varepsilon\theta_\Gamma) \cdot \nabla_\Gamma(\varepsilon(\xi_\Gamma - \theta_\Gamma))\,d\Gamma \\ &\quad + \int_\Omega \big(\alpha'(\eta)f_\delta(\nabla\theta) + \nu^{-2}\alpha(\eta)\alpha'(\eta)\big)(\zeta - \eta)\,dx \\ &\quad + \int_\Omega \big(\alpha(\eta)\nabla f_\delta(\nabla\theta) + \nu^2\nabla\theta\big) \cdot \nabla(\xi - \theta)\,dx \\ &\le \frac{1}{2}\int_\Omega (|\nabla\zeta|^2 - |\nabla\eta|^2)\,dx + \frac{1}{2}\int_\Gamma (|\nabla_\Gamma(\varepsilon\xi_\Gamma)|^2 - |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2)\,d\Gamma \\ &\quad + \frac{1}{2}\int_\Omega \left[\left(\nu f_\delta(\nabla\xi) + \frac{1}{\nu}\alpha(\zeta)\right)^2 - \left(\nu f_\delta(\nabla\theta) + \frac{1}{\nu}\alpha(\eta)\right)^2\right]dx \\ &= \Phi_\varepsilon^\delta(\zeta, w) - \Phi_\varepsilon^\delta(\eta, v), \quad \text{for any } z = [\zeta, w] = [\zeta, \xi, \xi_\Gamma] \in D(\Phi_\varepsilon^\delta) = \mathscr{V}_\varepsilon, \end{aligned}$$

and therefore

$$\begin{gathered} u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in D(\partial\Phi_\varepsilon^\delta) \text{ with } v = [\theta, \theta_\Gamma] \in V_\varepsilon, \\ \text{and } u^* = [\eta^*, v^*] = [\eta^*, \theta^*, \theta_\Gamma^*] \in \partial\Phi_\varepsilon^\delta(u) = \partial\Phi_\varepsilon^\delta(\eta, v) = \partial\Phi_\varepsilon^\delta(\eta, \theta, \theta_\Gamma) \text{ in } L^2(\Omega) \times H, \end{gathered}$$

by using Remark 1 (Fact 1)–(Fact 3), (3.10)–(3.12), and Young's inequality.

Thus, we conclude Key-Lemma 3. □

Key-Lemma 4 (Mosco-convergence for the convex energies)**.** *Let $\varepsilon_0 \geq 0$ be a fixed constant, and let us assume that the sequences $\{\varepsilon_n\}_{n=1}^\infty \subset [0,\infty)$ and $\{\delta_n\}_{n=1}^\infty \subset [0,1]$ satisfy $\varepsilon_n \to \varepsilon_0$ and $\delta_n \to 0$. Then, for the sequence of convex functions $\{\Phi_n\}_{n=1}^\infty := \{\Phi_{\varepsilon_n}^{\delta_n}\}_{n=1}^\infty$, it holds that:*

$$\Phi_n \to \Phi_{\varepsilon_0} \text{ on } \mathscr{H}\text{, in the sense of Mosco, as } n \to \infty.$$

Proof. First, we show the lower-bound condition (M1) in Definition 1. Let $\check{u} = [\check{\eta}, \check{\theta}, \check{\theta}_\Gamma] \in \mathscr{H}$ and $\{\check{u}_n = [\check{\eta}_n, \check{\theta}_n, \check{\theta}_{\Gamma,n}]\}_{n=1}^\infty \subset \mathscr{H}$ be such that:

$$\check{u}_n \to \check{u} \text{ weakly in } \mathscr{H}, \text{ as } n \to \infty.$$

Then, we may say $\underline{\lim}_{n\to\infty} \Phi_n(\check{u}_n) < \infty$ since another case is trivial. So, by taking a subsequence (not relabeled), we can reduce the situation to the case when $\underline{\lim}_{n\to\infty} \Phi_n(\check{u}_n) = \lim_{n\to\infty} \Phi_n(\check{u}_n) < \infty$. In this case, we may suppose that:

$$\nabla\check{\theta}_n \to \nabla\check{\theta} \text{ weakly in } L^2(\Omega)^N \text{ as } n \to \infty,$$

by taking more one subsequence if necessary. Here, if $\varepsilon_0 = 0$, then having in mind:

$$\begin{cases} \bullet \text{ the relationship } f_\delta \geq |\cdot|, \\ \bullet \text{ the weakly lower semi-continuities of the norms } |\cdot|_{L^2(\Omega)^N}, \text{ and } |\cdot|_{L^2(\Gamma)^N}, \\ \bullet \text{ the weakly lower semi-continuity of the convex function:} \\ [\eta, \omega] \in L^2(\Omega) \times L^2(\Omega)^N \mapsto \dfrac{1}{2}\displaystyle\int_\Omega \left(\nu|\omega| + \frac{1}{\nu}\alpha(\eta)\right)^2 dx \in [0,\infty), \end{cases}$$

we can show the condition (M1) as follows:

$$\begin{aligned} \underline{\lim}_{n\to\infty} \Phi_n(\check{u}_n) \geq{}& \frac{1}{2}\underline{\lim}_{n\to\infty} \int_\Omega |\nabla\check{\eta}_n|^2\,dx + \frac{1}{2}\underline{\lim}_{n\to\infty}\int_\Omega \left(\nu f_{\delta_n}(\nabla\check{\theta}_n) + \frac{1}{\nu}\alpha(\check{\eta}_n)\right)^2 dx \\ &+ \frac{1}{2}\underline{\lim}_{n\to\infty}\int_\Gamma |\nabla_\Gamma(\varepsilon_n\check{\theta}_{\Gamma,n})|^2\,d\Gamma \geq \Phi_{\varepsilon_0}(\check{u}). \end{aligned} \tag{3.13}$$

Meanwhile, if $\varepsilon_0 > 0$, then since the boundedness of $\{\Phi_n(\check{u}_n)\}_{n=1}^\infty$ implies that:

$$\begin{gathered} \{\check{\theta}_{\Gamma,n}\}_{n=1}^\infty \subset H^1(\Gamma), \text{ and } \nabla_\Gamma(\varepsilon_n\check{\theta}_{\Gamma,n}) \to \nabla_\Gamma(\varepsilon_0\check{\theta}_\Gamma) \text{ weakly in } L^2(\Omega)^N, \text{ as } n \to \infty, \\ \text{for some subsequence (not relabeled),} \end{gathered}$$

we can deduce the condition (M1) just as in (3.13).

Next, we show the optimality condition (M2) in Definition 1. Let us fix any $\hat{u} = [\hat{\eta}, \hat{\theta}, \hat{\theta}_\Gamma] \in \mathscr{V}_{\varepsilon_0}$. Besides, let us take a sequence $\{\hat{\varphi}_i\}_{i=1}^\infty \subset H^1(\Omega)$, such that:

$$|\hat{\varphi}_i - \hat{\eta}|_{H^1(\Omega)} \leq 2^{-i}, \text{ for any } i \in \mathbb{N}, \tag{3.14}$$

and let us take a sequence $\{\hat{\psi}_i\}_{i=1}^\infty \subset H^1(\Omega)$ in the following way:

$$\begin{cases} \bullet \text{ If } \varepsilon_0 > 0, \text{ then } \{\hat{\psi}_i\}_{i=1}^\infty = \{\hat{\theta}\}; \\ \bullet \text{ If } \varepsilon_0 = 0, \text{ then } \{\hat{\psi}_i\}_{i=1}^\infty \subset C^1(\overline{\Omega}) \text{ satisfies } |\hat{\psi}_i - \hat{\theta}|_{H^1(\Omega)} \leq 2^{-i}, \text{ for any } i \in \mathbb{N}. \end{cases} \tag{3.15}$$

Here, taking a subsequence if necessary, we can further impose that:

$$\begin{cases} \hat{\varphi}_i \to \hat{\eta} \text{ in the pointwise sense, a.e. in } \Omega, \\ \hat{\psi}_i \to \hat{\theta} \text{ in the pointwise sense, a.e. in } \Omega, \\ \hat{\psi}_{i|\Gamma} \to \hat{\theta}_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma), \text{ and in the pointwise sense, a.e. on } \Gamma, \\ \{\hat{\psi}_{i|\Gamma}\}_{i=1}^\infty \subset H^1(\Gamma), \text{ and } \varepsilon_0 \hat{\psi}_{i|\Gamma} \to \varepsilon_0 \hat{\theta}_\Gamma \text{ in } H^1(\Gamma), \end{cases} \text{as } i \to \infty. \tag{3.16}$$

By (3.16) and Lebesgue's dominated convergence theorem (cf. [14, Theorem 10]), we can configure a sequence $\{n_i\}_{i=0}^\infty \subset \mathbb{N}$ such that $1 =: n_0 < n_1 < n_2 < \cdots < n_i \uparrow \infty$, as $i \to \infty$, and for any $i \in \mathbb{N} \cup \{0\}$,

$$\sup_{n \geq n_i} \bigl| f_{\delta_n}(\nabla \hat{\psi}_i) - |\nabla \hat{\psi}_i| \bigr|_{L^2(\Omega)} < 2^{-i}, \text{ and } \sup_{n \geq n_i} |\varepsilon_n^2 - \varepsilon_0^2| |\nabla_\Gamma \hat{\psi}_{i|\Gamma}|^2_{L^2(\Gamma)^N} < 2^{-i}. \tag{3.17}$$

Based on these, let us define:

$$\hat{u}_n = [\hat{\eta}_n, \hat{\theta}_n, \hat{\theta}_{\Gamma,n}] := \begin{cases} [\hat{\varphi}_i, \hat{\psi}_i, \hat{\psi}_{i|\Gamma}], \text{ if } n_i \leq n < n_{i+1}, \text{ for } i \in \mathbb{N}, \\ [\hat{\varphi}_1, \hat{\psi}_1, \hat{\psi}_{1|\Gamma}], \text{ if } 1 \leq n < n_1. \end{cases} \tag{3.18}$$

Taking into account (3.1) and (3.14)–(3.18), we compute that:

$$\begin{aligned}
&\bigl|\Phi_n(\hat{u}_n) - \Phi_{\varepsilon_0}(\hat{u})\bigr| \\
&\leq \frac{1}{2} \int_\Omega \bigl| |\nabla \hat{\eta}_n|^2 - |\nabla \hat{\eta}|^2 \bigr| \, dx + \frac{1}{2} \int_\Gamma \bigl| |\nabla_\Gamma (\varepsilon_n \hat{\theta}_{\Gamma,n})|^2 - |\nabla_\Gamma(\varepsilon_0 \hat{\theta}_\Gamma)|^2 \bigr| \, d\Gamma \\
&\qquad + \frac{1}{2} \int_\Omega \left| \left(\nu f_{\delta_n}(\nabla \hat{\theta}_n) + \frac{1}{\nu} \alpha(\hat{\eta}_n) \right)^2 - \left(\nu |\nabla \hat{\theta}| + \frac{1}{\nu} \alpha(\hat{\eta}) \right)^2 \right| dx \\
&\leq \frac{1}{2} \bigl| |\nabla \hat{\eta}_n| + |\nabla \hat{\eta}| \bigr|_{L^2(\Omega)} \bigl| \nabla(\hat{\eta}_n - \hat{\eta}) \bigr|_{L^2(\Omega)^N} + \frac{1}{2} |\varepsilon_n^2 - \varepsilon_0^2| |\nabla_\Gamma \hat{\theta}_{\Gamma,n}|^2_{L^2(\Gamma)^N} \\
&\qquad + \frac{1}{2} \bigl| |\nabla_\Gamma(\varepsilon_0 \hat{\theta}_{\Gamma,n})| + |\nabla_\Gamma(\varepsilon_0 \hat{\theta}_\Gamma)| \bigr|_{L^2(\Gamma)} \bigl| \nabla_\Gamma(\varepsilon_0(\hat{\theta}_{\Gamma,n} - \hat{\theta}_\Gamma)) \bigr|_{L^2(\Gamma)^N} \\
&\quad + \frac{1}{2} \left| \nu \bigl(f_{\delta_n}(\nabla \hat{\theta}_n) + |\nabla \hat{\theta}| \bigr) + \frac{1}{\nu} \bigl(\alpha(\hat{\eta}_n) + \alpha(\hat{\eta}) \bigr) \right|_{L^2(\Omega)} \\
&\qquad \times \left[\nu \left(\bigl| f_{\delta_n}(\nabla \hat{\theta}_n) - |\nabla \hat{\theta}_n| \bigr|_{L^2(\Omega)} + \bigl| \nabla(\hat{\theta}_n - \hat{\theta}) \bigr|_{L^2(\Omega)^N} \right) + \frac{1}{\nu} \bigl| \alpha(\hat{\eta}_n) - \alpha(\hat{\eta}) \bigr|_{L^2(\Omega)} \right] \\
&\leq \frac{1}{2} \bigl| |\nabla \hat{\eta}_n| + |\nabla \hat{\eta}| \bigr|_{L^2(\Omega)} \bigl| \nabla(\hat{\eta}_n - \hat{\eta}) \bigr|_{L^2(\Omega)^N} + 2^{-i} \\
&\qquad + \frac{1}{2} \bigl| |\nabla_\Gamma(\varepsilon_0 \hat{\theta}_{\Gamma,n})| + |\nabla_\Gamma(\varepsilon_0 \hat{\theta}_\Gamma)| \bigr|_{L^2(\Gamma)} \bigl| \nabla_\Gamma(\varepsilon_0(\hat{\theta}_{\Gamma,n} - \hat{\theta}_\Gamma)) \bigr|_{L^2(\Gamma)^N} \\
&\quad + \frac{1}{2} \left(\nu \bigl| |\nabla \hat{\theta}_n| + \delta_n + |\nabla \hat{\theta}| \bigr|_{L^2(\Omega)} + \frac{1}{\nu} \bigl| \alpha(\hat{\eta}_n) + \alpha(\hat{\eta}) \bigr|_{L^2(\Omega)} \right) \\
&\qquad \times \left(\nu \bigl| \nabla(\hat{\theta}_n - \hat{\theta}) \bigr|_{L^2(\Omega)^N} + \frac{1}{\nu} |\alpha'|_{L^\infty(\mathbb{R})} |\hat{\eta}_n - \hat{\eta}|_{L^2(\Omega)} + \nu 2^{-i} \right), \\
&\qquad\quad \text{for any } i \in \mathbb{N} \cup \{0\} \text{ and any } n \geq n_i,
\end{aligned}$$

and therefore

$$\Phi_n(\hat{u}_n) \to \Phi_{\varepsilon_0}(\hat{u}) \text{ as } n \to \infty.$$

This implies that the sequence $\{\hat{u}_n = [\hat{\eta}_n, \hat{\theta}_n, \hat{\theta}_{\Gamma,n}]\}_{n=1}^{\infty} \subset H^1(\Omega)^2 \times H^{\frac{1}{2}}(\Gamma)$ is the required sequence to verify the optimality condition. □

Key-Lemma 5 (Representation of $\partial_v \Phi_\varepsilon$). *For any $\varepsilon \geq 0$, the following two items are equivalent.*

(O) *$u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in D(\partial_v \Phi_\varepsilon)$ and $v^* = [\theta^*, \theta_\Gamma^*] \in \partial_v \Phi_\varepsilon(\eta, v) = \partial_v \Phi_\varepsilon(\eta, \theta, \theta_\Gamma)$ in H, with $v = [\theta, \theta_\Gamma] \in V_\varepsilon$.*

(I) *$u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in \mathscr{V}_\varepsilon$ with $v = [\theta, \theta_\Gamma] \in V_\varepsilon$, and there exists $\omega^* \in L^\infty(\Omega)^N$, such that:*

$$\left\{ \begin{array}{l} \bullet\ \omega^* \in \mathrm{Sgn}(\nabla\theta), \text{ a.e. in } \Omega, \\ \bullet\ \alpha(\eta)\omega^* + \nu^2 \nabla\theta \in \boldsymbol{L}^2_{\mathrm{div}}(\Omega), \\ \bullet\ -\varDelta_\Gamma(\varepsilon^2 \theta_\Gamma) + \big[(\alpha(\eta)\omega^* + \nu^2 \nabla\theta)_{|_\Gamma} \cdot n_\Gamma\big] \in L^2(\Gamma), \end{array} \right. \tag{3.19}$$

and

$$\left\{ \begin{array}{l} \bullet\ \theta^* = -\mathrm{div}\big(\alpha(\eta)\omega^* + \nu^2\nabla\theta\big) \text{ in } L^2(\Omega), \\ \bullet\ \theta^*_\Gamma = -\varDelta_\Gamma(\varepsilon^2\theta_\Gamma) + \big[(\alpha(\eta)\omega^* + \nu^2 \nabla\theta)_{|_\Gamma} \cdot n_\Gamma\big] \text{ in } L^2(\Gamma). \end{array} \right. \tag{3.20}$$

Proof. Let us fix any $\varepsilon \geq 0$, and let us define a set-valued map $\mathcal{A}_\varepsilon : D(\mathcal{A}_\varepsilon) \subset \mathscr{H} \to 2^H$, by putting:

$$D(\mathcal{A}_\varepsilon) := \left\{ u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in \mathscr{V}_\varepsilon \;\middle|\; \begin{array}{l} \text{there exists } \omega^* \in L^\infty(\Omega)^N, \\ \text{such that (3.19) holds} \end{array} \right\}, \tag{3.21}$$

and

$$\begin{aligned} u = [\eta, v] = [\eta, \theta, \theta_\Gamma] \in D(\mathcal{A}_\varepsilon) \subset \mathscr{H} \mapsto \mathcal{A}_\varepsilon u = \mathcal{A}_\varepsilon[\eta, v] = \mathcal{A}_\varepsilon[\eta, \theta, \theta_\Gamma] \\ := \left\{ v^* = [\theta^*, \theta^*_\Gamma] \in H \;\middle|\; \begin{array}{l} \text{(3.20) holds, for some } \omega^* \in L^\infty(\Omega)^N, \\ \text{satisfying (3.19)} \end{array} \right\}. \end{aligned} \tag{3.22}$$

Then, the assertion of Key-Lemma 5 can be rephrased as follows:

$$\partial_v \Phi_\varepsilon(\eta, \cdot) = \mathcal{A}_\varepsilon(\eta, \cdot) \text{ in } H^2, \text{ for any } \varepsilon \geq 0 \text{ and any } \eta \in H^1(\Omega). \tag{3.23}$$

The above equality (3.23) can be shown via the following two Claims.

Claim 1. $\mathcal{A}_\varepsilon(\eta, \cdot)$ is a monotone such that $\mathcal{A}_\varepsilon(\eta, \cdot) \subset \partial_v \Phi_\varepsilon(\eta, \cdot)$ in H^2, for any $\eta \in H^1(\Omega)$.

Let us take $\eta \in H^1(\Omega)$, $v = [\theta, \theta_\Gamma] \in D(\mathcal{A}_\varepsilon(\eta, \cdot))$, and $v^* = [\theta^*, \theta^*_\Gamma] \in \mathcal{A}_\varepsilon(\eta, v) = \mathcal{A}_\varepsilon(\eta, \theta, \theta_\Gamma)$ in H. Then, from Remark 1 (Fact 1)–(Fact 3), (3.19)–(3.22), and [4, Key-Lemma 3], it is inferred that:

$$
\begin{aligned}
(v^*, w - v)_H &= (\theta^*, \xi - \theta)_{L^2(\Omega)} + (\theta^*_\Gamma, \xi_\Gamma - \theta_\Gamma)_{L^2(\Gamma)} \\
&= \int_\Omega (\alpha(\eta)\omega^* + \nu^2 \nabla\theta) \cdot \nabla(\xi - \theta)\, dx \\
&\qquad + \int_\Gamma \nabla_\Gamma(\varepsilon\theta_\Gamma) \cdot \nabla_\Gamma(\varepsilon(\xi_\Gamma - \theta_\Gamma))\, d\Gamma \\
&\leq \int_\Omega \alpha(\eta)(|\nabla\xi| - |\nabla\theta|)\, dx + \frac{\nu^2}{2}\int_\Omega (|\nabla\xi|^2 - |\nabla\theta|^2)\, dx \\
&\qquad + \frac{1}{2}\int_\Gamma (|\nabla_\Gamma(\varepsilon\xi_\Gamma)|^2 - |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2)\, d\Gamma \\
&= \Phi_\varepsilon(\eta, w) - \Phi_\varepsilon(\eta, v), \text{ for all } \eta \in H^1(\Omega), \text{ and } w = [\xi, \xi_\Gamma] \in V_\varepsilon.
\end{aligned}
$$

Thus, we have

$$
v = [\theta, \theta_\Gamma] \in D(\partial_v \Phi_\varepsilon(\eta, \cdot)) \text{ and } v^* = [\theta^*, \theta^*_\Gamma] \in \partial_v\Phi_\varepsilon(\eta, v) = \partial_v \Phi_\varepsilon(\eta, \theta, \theta_\Gamma) \text{ in } H,
$$

and we can say that:

$$
\mathcal{A}_\varepsilon(\eta, \cdot) \subset \partial_v \Phi_\varepsilon(\eta, \cdot) \text{ in } H^2, \text{ and } \mathcal{A}_\varepsilon(\eta, \cdot) \text{ is monotone graph on } H^2.
$$

Claim 2. $\mathcal{A}_\varepsilon(\eta, \cdot)$ is maximal in H^2.

Let us take $\eta \in H^1(\Omega)$ and $w = [\xi, \xi_\Gamma] \in H$. In the light of Claim 1 and Minty's theorem, it is sufficient to show $H \subset (\mathcal{A}_\varepsilon(\eta, \cdot) + \mathcal{I}_H)H$. Here, with Key-Lemma 2 in mind, we can apply Minty's theorem, and we can configure a class of functions $\{v^\delta = [\theta^\delta, \theta^\delta_\Gamma] \,|\, 0 < \delta \leq 1\} \subset V_\varepsilon$, by setting:

$$
v^\delta := (\mathcal{A}^\delta_\varepsilon(\eta) + \mathcal{I}_H)^{-1} w \text{ in } H, \text{ for any } 0 < \delta \leq 1,
$$

i.e.:

$$
w - v^\delta = \partial_v \Phi^\delta_\varepsilon(\eta, v^\delta) \text{ in } H, \text{ for any } 0 < \delta \leq 1. \tag{3.24}
$$

Also, we can see that:

$$
\begin{aligned}
&\int_\Omega (\alpha(\eta)\nabla f_\delta(\nabla\theta^\delta) + \nu^2\nabla\theta^\delta) \cdot \nabla\psi\, dx + \int_\Gamma \nabla_\Gamma(\varepsilon\theta^\delta_\Gamma) \cdot \nabla_\Gamma(\varepsilon\psi_\Gamma)\, d\Gamma \\
&= \int_\Omega (\xi - \theta^\delta)\psi\, dx + \int_\Gamma (\xi_\Gamma - \theta^\delta_\Gamma)\psi_\Gamma\, d\Gamma, \text{ for all } [\psi, \psi_\Gamma] \in V_\varepsilon \text{ and } 0 < \delta \leq 1.
\end{aligned} \tag{3.25}
$$

In the variational form (3.25), let us put $[\psi, \psi_\Gamma] = [\theta^\delta, \theta^\delta_\Gamma] \in V_\varepsilon$. Then, by using (A3) and Young's inequality, we deduce that:

$$
\begin{aligned}
\frac{1}{2}|v^\delta|^2_H + \nu^2|\nabla\theta^\delta|^2_{L^2(\Omega)^N} + |\nabla_\Gamma(\varepsilon\theta^\delta_\Gamma)|^2_{L^2(\Gamma)^N} &\leq \frac{1}{2}|w|^2_H + \delta\int_\Omega \alpha(\eta)\, dx \\
&\leq \frac{1}{2}|w|^2_H + (\mathcal{L}^N(\Omega)^{\frac{1}{2}}|\eta|_{L^2(\Omega)} + \delta_\alpha\mathcal{L}^N(\Omega)), \text{ for any } 0 < \delta \leq 1.
\end{aligned} \tag{3.26}
$$

(3.26) implies that $\{v^\delta \,|\, 0 < \delta \le 1\}$ is bounded in V_ε, and is compact in H. Also, as is easily checked,

$$|\nabla f_\delta(\nabla \theta_\delta)| = \left| \frac{\nabla \theta^\delta}{\sqrt{\delta^2 + |\nabla \theta^\delta|^2}} \right| \le 1, \text{ a.e. in } \Omega, \text{ for any } 0 < \delta \le 1. \tag{3.27}$$

Therefore, by (A3) and the estimates (3.26) and (3.27), we can find a sequence $\{\delta_n\}_{n=1}^\infty \subset (0,1]$, a pair of functions $v = [\theta, \theta_\Gamma] \in V_\varepsilon$, and a function $\omega^* \in L^\infty(\Omega)^N$, such that $\delta_n \downarrow 0$ as $n \to \infty$,

$$\begin{aligned} v_n = [\theta_n, \theta_{\Gamma,n}] := v^{\delta_n} = [\theta^{\delta_n}, \theta_\Gamma^{\delta_n}] \to v = [\theta, \theta_\Gamma] \text{ in } H,& \\ \text{and weakly in } V_\varepsilon, \text{ as } n \to \infty,& \end{aligned} \tag{3.28}$$

and

$$\nabla f_{\delta_n}(\nabla \theta_n) \to \omega^* \text{ weakly-}* \text{ in } L^\infty(\Omega)^N, \text{ as } n \to \infty. \tag{3.29}$$

Now, with (3.28) and (3.29) in mind, let us take any function $\psi_0 \in H_0^1(\Omega)$ and take $[\psi_0, 0]$ as the pair of test functions $[\psi, \psi_\Gamma]$ in (3.25). Then, putting $\delta = \delta_n$ with $n \in \mathbb{N}$, and letting $n \to \infty$ in (3.25) yields that:

$$\int_\Omega (\alpha(\eta)\omega^* + \nu^2 \nabla \theta) \cdot \nabla \psi_0 \, dx = (\xi - \theta, \psi_0)_{L^2(\Omega)}.$$

It implies that:

$$-\operatorname{div}(\alpha(\eta)\omega^* + \nu^2 \nabla \theta) = \xi - \theta \in L^2(\Omega) \text{ in } \mathcal{D}'(\Omega). \tag{3.30}$$

As well as, putting $\delta = \delta_n$, letting $n \to \infty$ in (3.25), and applying Remark 1 (Fact 1)–(Fact 3), and (3.28)–(3.30), we infer that:

$$\begin{aligned} (\xi_\Gamma - \theta_\Gamma, \psi_\Gamma)_{L^2(\Gamma)} &= \int_\Omega (\alpha(\eta)\omega^* + \nu^2 \nabla \theta) \cdot \nabla \psi \, dx - (\xi - \theta, \psi)_{L^2(\Omega)} \\ &\quad + \int_\Gamma \nabla_\Gamma(\varepsilon \theta_\Gamma) \cdot \nabla_\Gamma(\varepsilon \psi_\Gamma) \, d\Gamma \\ &= \left\langle \left[(\alpha(\eta)\omega^* + \nu^2 \nabla \theta)_{|\Gamma} \cdot n_\Gamma \right], \psi_{|\Gamma} \right\rangle_{H^{\frac{1}{2}}(\Gamma)} \\ &\quad + \left\langle -\Delta_\Gamma(\varepsilon \theta_\Gamma), \varepsilon \psi_\Gamma \right\rangle_{H^1(\Gamma)}, \text{ for any } [\psi, \psi_\Gamma] \in V_\varepsilon. \end{aligned}$$

It is seen that:

$$-\Delta_\Gamma(\varepsilon^2 \theta_\Gamma) + \left[(\alpha(\eta)\omega^* + \nu^2 \nabla \theta)_{|\Gamma} \cdot n_\Gamma \right] = \xi_\Gamma - \theta_\Gamma \in L^2(\Gamma) \text{ in } H^{-1}(\Gamma). \tag{3.31}$$

Finally, by Key-Lemma 4, (3.24) and (3.28), we can apply Remark 4 (Fact 4) to see that:

$$w - v \in \partial_v \Phi_\varepsilon(\eta, v) \text{ in } H,$$

and

$$\Phi_\varepsilon^{\delta_n}(\eta, v_n) \to \Phi_\varepsilon(\eta, v), \text{ as } n \to \infty. \tag{3.32}$$

Also, taking into account (3.1), (3.28), and lower semi-continuities of the norms $|\cdot|_{L^2(\Omega)^N}$, $|\cdot|_{L^2(\Gamma)^N}$, and the convex function:

$$\omega \in L^2(\Omega)^N \mapsto \int_\Omega \alpha(\eta)|\omega|\,dx \in [0,\infty),$$

we can see that:

$$\begin{cases} \displaystyle\varliminf_{n\to\infty} \int_\Omega \alpha(\eta) f_{\delta_n}(\nabla\theta_n)\,dx \geq \int_\Omega \alpha(\eta)|\nabla\theta|\,dx, \\ \displaystyle\varliminf_{n\to\infty} \left(\frac{\nu^2}{2}\int_\Omega |\nabla\theta_n|^2\,dx\right) \geq \frac{\nu^2}{2}\int_\Omega |\nabla\theta|^2\,dx, \\ \displaystyle\varliminf_{n\to\infty} \left(\frac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_{\Gamma,n})|^2\,d\Gamma\right) \geq \frac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2\,d\Gamma. \end{cases} \tag{3.33}$$

From (3.32) and (3.33), it follows that:

$$\begin{aligned} &\frac{\nu^2}{2}\int_\Omega |\nabla\theta|^2\,dx \leq \frac{\nu^2}{2}\varliminf_{n\to\infty}\int_\Omega |\nabla\theta_n|^2\,dx \leq \frac{\nu^2}{2}\varlimsup_{n\to\infty}\int_\Omega |\nabla\theta_n|\,dx \\ &\leq \lim_{n\to\infty} \Phi_\varepsilon^{\delta_n}(\eta, v_n) - \frac{1}{2}\int_\Omega |\nabla\eta|^2\,dx - \frac{1}{2\nu^2}\int_\Omega (\alpha(\eta))^2\,dx \\ &\qquad - \varliminf_{n\to\infty}\int_\Omega \alpha(\eta) f_{\delta_n}(\nabla\theta_n)\,dx - \frac{1}{2}\varliminf_{n\to\infty}\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_{\Gamma,n})|^2\,d\Gamma \\ &\leq \Phi_\varepsilon(\eta, v) - \frac{1}{2}\int_\Omega |\nabla\eta|^2\,dx - \frac{1}{2\nu^2}\int_\Omega (\alpha(\eta))^2\,dx \\ &\qquad - \int_\Omega \alpha(\eta)|\nabla\theta|\,dx - \frac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2\,d\Gamma = \frac{\nu^2}{2}\int_\Omega |\nabla\theta|^2\,dx. \end{aligned} \tag{3.34}$$

Having in mind (Fact 0), (3.28), (3.34), and the uniform convexity of L^2-based topologies, we deduced that:

$$\theta_n \to \theta \text{ in } H^1(\Omega)\ (\nabla\theta_n \to \nabla\theta \text{ in } L^2(\Omega)^N), \text{ as } n \to \infty. \tag{3.35}$$

In view of Remark 2, Remark 4 (Fact 4), (3.29), (3.35), and [3, Proposition 2.16], it is inferred that:

$$\omega^* \in \mathrm{Sgn}(\nabla\theta), \text{ a.e. in } \Omega. \tag{3.36}$$

As a consequence of (3.30), (3.31), (3.36), we verify Claim 2.

Now, by using Claims 1, 2, and the maximality of $\mathcal{A}_\varepsilon(\eta,\cdot)$ in H^2, we can show the coincidence (3.23), and we conclude this Key-Lemma 5. □

4 Time-discretization

In this paper, the solution to $(\mathrm{KWC})_\varepsilon$ is to be obtained by means of the time-discretization methods. In view of this, we fix the constants $\varepsilon \geq 0$ and $0 < \delta \leq 1$, and assume (A0)–(A5). On this basis, we denote by $0 < \tau \leq 1$ the constant of time-step-size, and consider the following time-discretization scheme, denoted by $(\mathrm{AP})^\delta_\tau$.

$$(\mathrm{AP})^\delta_\tau\colon\ A_0(u^\delta_{\tau,i-1})\frac{u^\delta_{\tau,i}-u^\delta_{\tau,i-1}}{\tau}+\partial\Phi^\delta_\varepsilon(u^\delta_{\tau,i})+\mathcal{G}(u^\delta_{\tau,i})=0 \text{ in } \mathscr{H}, \text{ for } i=1,2,3,\cdots, \quad (4.1)$$

with the initial condition:

$$\eta^\delta_{\tau,0}=\eta_0 \text{ in } L^2(\Omega), \text{ and } v^\delta_{\tau,0}=[\theta^\delta_{\tau,0},\theta^\delta_{\Gamma,\tau,0}]=v_0=[\theta_0,\theta_{\Gamma,0}] \text{ in } H. \quad (4.2)$$

Definition 3 (Solution to $(\mathrm{AP})^\delta_\tau$)**.** A sequence of $\{u^\delta_{\tau,i}\}_{i=0}^\infty=\{[\eta^\delta_{\tau,i},\theta^\delta_{\tau,i},\theta^\delta_{\Gamma,\tau,i}]\}_{i=0}^\infty\subset\mathscr{H}$ is called a solution to $(\mathrm{AP})^\delta_\tau$, iff. $\{u^\delta_{\tau,i}\}_{i=0}^\infty\subset\mathscr{V}_\varepsilon$, and $\{u^\delta_{\tau,i}\}_{i=0}^\infty$ fulfills (4.1) and (4.2).

Now, the the objective of this Section is to prove the following Theorem.

Theorem 1 (Solvability and energy estimate for $(\mathrm{AP})^\delta_\tau$)**.** *There exists a small positive constant $0<\tau^*<1$, such that for any $0<\tau<\tau^*$, the time-discretization scheme $(AP)^\delta_\tau$ admits a unique solution $\{u^\delta_{\tau,i}\}_{i=0}^\infty=\{[\eta^\delta_{\tau,i},\theta^\delta_{\tau,i},\theta^\delta_{\Gamma,\tau,i}]\}_{i=0}^\infty\subset\mathscr{V}_\varepsilon$, such that:*

$$0\le\eta^\delta_{\tau,i}\le 1,\ m_0\le\theta^\delta_{\tau,i}\le M_0 \ \text{a.e. in } \Omega,\ m_0\le\theta^\delta_{\Gamma,\tau,i}\le M_0,\ \text{a.e. on } \Gamma, \quad (4.3)$$

and

$$\frac{1}{2\tau}\left|A_0(u^\delta_{\tau,i-1})^{\frac12}(u^\delta_{\tau,i}-u^\delta_{\tau,i-1})\right|^2_{\mathscr{H}}+\mathscr{F}^\delta_\varepsilon(u^\delta_{\tau,i})\le\mathscr{F}^\delta_\varepsilon(u^\delta_{\tau,i-1}),\ \text{for } i=1,2,3\ldots, \quad (4.4)$$

where $\mathscr{F}^\delta_\varepsilon$ is a relaxed free-energy defined as:

$$\begin{aligned}u=[\eta,\theta,\theta_\Gamma]\in\mathscr{V}_\varepsilon\mapsto\mathscr{F}^\delta_\varepsilon(u)&=\mathscr{F}^\delta_\varepsilon(\eta,\theta,\theta_\Gamma):=\Phi^\delta_\varepsilon(u)+\int_\Omega\hat{G}(\eta)\,dx\\ &=\frac12\int_\Omega|\nabla\eta|^2\,dx+\int_\Omega\hat{g}(\eta)\,dx\\ &\quad+\int_\Omega\alpha(\eta)f_\delta(\nabla\theta)\,dx+\frac{\nu^2}{2}\int_\Omega|\nabla\theta|^2\,dx+\frac12\int_\Gamma|\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2\,d\Gamma\in[0,\infty).\end{aligned} \quad (4.5)$$

Remark 11. As is shown in Key-Lemma 3, the subdifferential $\partial\Phi^\delta_\varepsilon$ coincides with $\partial_\eta\Phi_\varepsilon\times\partial_v\Phi^\delta_\varepsilon$ in $\mathscr{H}^2$, so that the equality as in (4.1) is valid. Additionally, the scheme $(\mathrm{AP})^\delta_\tau$ can be reformulated to the following system:

$$\begin{aligned}&\frac1\tau(\eta^\delta_{\tau,i}-\eta^\delta_{\tau,i-1})-\mathit{\Delta}_N\eta^\delta_{\tau,i}+g(\eta^\delta_{\tau,i})+\alpha'(\eta^\delta_{\tau,i})f_\delta(\nabla\theta^\delta_{\tau,i})=0 \ \text{ in } L^2(\Omega),\\ &\tilde{A}_0(\eta^\delta_{\tau,i-1})\frac{v^\delta_{\tau,i}-v^\delta_{\tau,i-1}}{\tau}+\partial_v\Phi^\delta_\varepsilon(\eta^\delta_{\tau,i},v^\delta_{\tau,i})=0 \ \text{ in } H,\end{aligned}$$

with the initial condition (4.2).

For the proof of Theorem 1, we prepare some Lemmas.

Lemma 1. *Let us assume that* $\zeta_k \in D(\varDelta_N)$, $\zeta_{0,k} \in H^1(\Omega)$, $k = 1, 2$, $\theta \in H^1(\Omega)$, *and*

$$\frac{1}{\tau}(\zeta_1 - \zeta_{0,1}) - \varDelta_N \zeta_1 + g(\zeta_1) + \alpha'(\zeta_1) f_\delta(\nabla\theta) \leq 0, \ \ a.e. \ in \ \Omega, \tag{4.6}$$

$$\frac{1}{\tau}(\zeta_2 - \zeta_{0,2}) - \varDelta_N \zeta_2 + g(\zeta_2) + \alpha'(\zeta_2) f_\delta(\nabla\theta) \geq 0, \ \ a.e \ in \ \Omega. \tag{4.7}$$

Then, there exists a small positive constant $\tau_0 \in (0, 1)$ *such that:*

$$|[\zeta_1 - \zeta_2]^+|_{H^1(\Omega)}^2 \leq \frac{1}{\tau} |[\zeta_{0,1} - \zeta_{0,2}]^+|_{L^2(\Omega)}^2, \ \ for \ any \ \tau \in (0, \tau_0). \tag{4.8}$$

Proof. Let us take the difference between (4.6) and (4.7), and multiply the both sides by $[\zeta_1 - \zeta_2]^+$. Then, by (A1), (A3), Green's formula, and Young's inequality, we can observe that:

$$\begin{aligned}
&\frac{1}{\tau}|[\zeta_1 - \zeta_2]^+|_{L^2(\Omega)}^2 + |\nabla[\zeta_1 - \zeta_2]^+|_{L^2(\Omega)^N}^2 \\
&= \frac{1}{\tau}(\zeta_{0,1} - \zeta_{0,2}, [\zeta_1 - \zeta_2]^+)_{L^2(\Omega)} - (g(\zeta_1) - g(\zeta_2), [\zeta_1 - \zeta_2]^+)_{L^2(\Omega)} \\
&\qquad - \int_\Omega (\alpha'(\zeta_1) - \alpha'(\zeta_2))[\zeta_1 - \zeta_2]^+ f_\delta(\nabla\theta)\, dx \\
&\leq \left(\frac{1}{2\tau} + |g'|_{L^\infty(\mathbb{R})} \right) |[\zeta_1 - \zeta_2]^+|_{L^2(\Omega)}^2 + \frac{1}{2\tau} |[\zeta_{0,1} - \zeta_{0,2}]^+|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.9}$$

Here, putting

$$\tau_0 := \frac{1}{1 + 2|g'|_{L^\infty(\mathbb{R})}} \in (0, 1),$$

the inequality (4.8) is obtained as a consequence of (4.9). $\square$

Lemma 2. *Let us fix* $\eta \in H^1(\Omega)$, $w_{0,k} = [\xi_{0,k}, \xi_{\Gamma,0,k}] \in V_\varepsilon$, $k = 1, 2$, *and assume that:*

$$z = [\eta, w_k] = [\eta, \xi_k, \xi_{\Gamma,k}] \in D(\partial_w \Phi_\varepsilon^\delta) \ with \ w_k = [\xi_k, \xi_{\Gamma,k}] \in V_\varepsilon,$$
$$and \ \ w_k^* = [\xi_k^*, \xi_{\Gamma,k}^*] \in \partial_w \Phi_\varepsilon^\delta(\eta, w_k) = \partial_w \Phi_\varepsilon(\eta, \xi_k, \xi_{\Gamma,k}) \ in \ H, \ k = 1, 2,$$

$$\tilde{A}_0(\eta)(w_1 - w_{0,1}) + w_1^* \leq 0 \ \left(= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \ in \ H, \tag{4.10}$$

and

$$\tilde{A}_0(\eta)(w_2 - w_{0,2}) + w_2^* \geq 0 \ \left(= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \ in \ H. \tag{4.11}$$

Then, it holds that:

$$\left| \tilde{A}_0(\eta)[w_1 - w_2]^+ \right|_H^2 \leq \left| \tilde{A}_0(\eta)[w_{0,1} - w_{0,2}]^+ \right|_H^2.$$

Proof. This lemma is concluded by taking the difference between (4.10) and (4.11), by multiplying the both sides by $[w_1 - w_2]^+$, and by applying an inequality of the so-called T-monotonicity:

$$(w_1^* - w_2^*, [w_1 - w_2]^+)_H \geq 0,$$

which is verified as follows.

$$
\begin{aligned}
&(w_1^* - w_2^*, [w_1 - w_2]^+)_H = (w_1^*, [w_1 - w_2]^+)_H + (w_2^*, -[w_1 - w_2]^+)_H \\
&= (w_1^*, w_1 - (w_1 \wedge w_2))_H + (w_2^*, w_2 - (w_1 \vee w_2))_H \\
&\geq \Phi_\varepsilon^\delta(\eta, w_1) - \Phi_\varepsilon^\delta(\eta, w_1 \wedge w_2) + \Phi_\varepsilon^\delta(\eta, w_2) - \Phi_\varepsilon^\delta(\eta, w_1 \vee w_2) \\
&= \Phi_\varepsilon^\delta(\eta, w_1) + \Phi_\varepsilon^\delta(\eta, w_2) \\
&\quad - \frac{1}{2} \int_{\{\xi_1 \leq \xi_2\}} \left(\nu f_\delta(\nabla \xi_1) + \frac{1}{\nu} \alpha(\eta) \right)^2 dx - \frac{1}{2} \int_{\{\xi_{\Gamma,1} \leq \xi_{\Gamma,2}\}} |\nabla_\Gamma (\varepsilon \xi_{\Gamma,1})|^2 \, d\Gamma \\
&\quad - \frac{1}{2} \int_{\{\xi_1 > \xi_2\}} \left(\nu f_\delta(\nabla \xi_2) + \frac{1}{\nu} \alpha(\eta) \right)^2 dx - \frac{1}{2} \int_{\{\xi_{\Gamma,1} > \xi_{\Gamma,2}\}} |\nabla_\Gamma (\varepsilon \xi_{\Gamma,2})|^2 \, d\Gamma \\
&\quad - \frac{1}{2} \int_{\{\xi_1 \leq \xi_2\}} \left(\nu f_\delta(\nabla \xi_2) + \frac{1}{\nu} \alpha(\eta) \right)^2 dx - \frac{1}{2} \int_{\{\xi_{\Gamma,1} \leq \xi_{\Gamma,2}\}} |\nabla_\Gamma (\varepsilon \xi_{\Gamma,2})|^2 \, d\Gamma \\
&\quad - \frac{1}{2} \int_{\{\xi_1 > \xi_2\}} \left(\nu f_\delta(\nabla \xi_1) + \frac{1}{\nu} \alpha(\eta) \right)^2 dx - \frac{1}{2} \int_{\{\xi_{\Gamma,1} > \xi_{\Gamma,2}\}} |\nabla_\Gamma (\varepsilon \xi_{\Gamma,1})|^2 \, d\Gamma \\
&= (\Phi_\varepsilon^\delta(\eta, w_1) + \Phi_\varepsilon^\delta(\eta, w_2)) - (\Phi_\varepsilon^\delta(\eta, w_1) + \Phi_\varepsilon^\delta(\eta, w_2)) = 0.
\end{aligned}
$$

□

Lemma 3. *Let us fix any $\tilde{u}_0 = [\tilde{\eta}_0, \tilde{\theta}_0, \tilde{\theta}_{\Gamma,0}] \in \mathscr{H}$, and consider the following auxiliary equation:*

$$\frac{1}{\tau} A_0(\tilde{u}_0)(u - \tilde{u}_0) + \partial \Phi_\varepsilon^\delta(u) + \mathcal{G}(u) = 0 \text{ in } \mathscr{H}. \tag{4.12}$$

Then, there exists a small positive constant $\tau_1 \in (0, 1]$ such that under $\tau \in (0, \tau_1]$, the equation (4.12) *admits a unique solution $u = [\eta, \theta, \theta_\Gamma] \in \mathscr{V}_\varepsilon$, and*

$$\frac{1}{2\tau} \left| A_0(\tilde{u}_0)^{\frac{1}{2}} (u - \tilde{u}_0) \right|_{\mathscr{H}}^2 + \mathscr{F}_\varepsilon^\delta(u) \leq \mathscr{F}_\varepsilon^\delta(\tilde{u}_0). \tag{4.13}$$

Proof. First, for the proof of existence, let us define a functional $\tilde{\mathscr{F}}_\varepsilon^\delta : \mathscr{H} \to (-\infty, \infty]$, by letting:

$$
\begin{aligned}
u = [\eta, \theta, \theta_\Gamma] \in \mathscr{H} \mapsto \tilde{\mathscr{F}}_\varepsilon^\delta(u) &= \tilde{\mathscr{F}}_\varepsilon^\delta(\eta, \theta, \theta_\Gamma) \\
&:= \frac{1}{2\tau} \left| A_0(\tilde{u}_0)^{\frac{1}{2}} (u - \tilde{u}_0) \right|_{\mathscr{H}}^2 + \Phi_\varepsilon^\delta(u) + \int_\Omega \hat{G}(\eta) \, dx,
\end{aligned}
$$

and let us set:

$$\tilde{\tau}_0 := \frac{\delta_\alpha \nu^2}{\delta_\alpha \nu^2 + 4|\alpha'|_{L^\infty(\mathbb{R})}^2} \in (0, 1).$$

Then, in the light of (A2)–(A4), it is easily checked that $\tilde{\mathscr{F}}_\varepsilon^\delta$ is a proper and l.s.c. functional on $\mathscr{H}$, and

$$
\begin{aligned}
\tilde{\mathscr{F}}_\varepsilon^\delta(u) \geq \frac{(1 \wedge \delta_\alpha)}{2} |u - \tilde{u}_0|_{\mathscr{H}}^2 &+ \frac{(1 \wedge \nu^2 \wedge \varepsilon^2)}{2} |\nabla u|_{L^2(\Omega)^N \times L^2(\Omega)^N \times L^2(\Gamma)^N} \\
&- \frac{4}{\nu^2} \left(|\alpha'|_{L^\infty(\mathbb{R})}^2 |\tilde{\eta}_0|_{L^2(\Omega)}^2 + \delta_\alpha^2 \mathcal{L}^N(\Omega) \right), \text{ whenever } 0 < \tau < \tilde{\tau}_0,
\end{aligned}
$$

via the following computations:

$$\Phi_\varepsilon^\delta(u) \geq \frac{1}{2}|\nabla \eta|^2_{L^2(\Omega)^N} + \frac{\nu^2}{2}|\nabla \theta|^2_{L^2(\Omega)^N} + \frac{1}{2}|\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2_{L^2(\Gamma)^N},$$

and

$$\begin{aligned}
\int_\Omega \hat{G}(\eta)\,dx &\geq -\frac{1}{\nu^2}\int_\Omega (\alpha(\eta))^2\,dx = -\frac{1}{\nu^2}\int_\Omega 2\big(|\alpha(\eta)-\alpha(\tilde{\eta}_0)|^2 + |\alpha(\tilde{\eta}_0)|^2\big)\,dx \\
&\geq -\frac{2}{\nu^2}\int_\Omega \big(|\alpha'|^2_{L^\infty(\mathbb{R})}|\eta-\tilde{\eta}_0|^2 + |\alpha(\tilde{\eta}_0)|^2\big)\,dx \\
&\geq -\frac{2|\alpha'|^2_{L^\infty(\mathbb{R})}}{\delta_\alpha \nu^2}\big|A_0(\tilde{u}_0)^{\frac{1}{2}}(u-\tilde{u}_0)\big|^2_{\mathscr{H}} - \frac{2}{\nu^2}\int_\Omega (2|\alpha'|^2_{L^\infty(\mathbb{R})}|\tilde{\eta}_0|^2_{L^2(\Omega)} + 2\delta_\alpha^2)\,dx \\
&\geq -\frac{4}{\nu^2}\big(|\alpha'|^2_{L^\infty(\mathbb{R})}|\tilde{\eta}_0|^2_{L^2(\Omega)} + \delta_\alpha^2 \mathcal{L}^N(\Omega)\big)
\end{aligned}$$

In addition, the equation (4.12) coincides with the stationary equation for $\mathscr{F}_\varepsilon^\delta$, and hence, when $\tau \in (0, \tilde{\tau}_0]$, the solution to (4.12) is obtained, by means of the direct method of calculation of variations (cf. [1, Theorem 3.2.1]).

Next, for the proof of uniqueness, we suppose that there are two solutions $u_k \in \mathscr{V}_\varepsilon$, $k = 1, 2$, to the equation (4.12). Besides, let us take the difference between equations (4.12) corresponding to u_k, $k = 1, 2$. Then, multiplying the both sides of the results by $u_1 - u_2$, and using (A1) and (A3), we arrive at:

$$\begin{aligned}
&\frac{1}{\tau}(\delta_\alpha - \tau|G'|_{L^\infty(\mathbb{R})})|u_1 - u_2|^2_{\mathscr{H}} \\
&\quad\leq \frac{1}{\tau}\big|A_0(\tilde{u}_0)^{\frac{1}{2}}(u_1-u_2)\big|^2_{\mathscr{H}} + (\mathcal{G}(u_1) - \mathcal{G}(u_2), u_1 - u_2)_{\mathscr{H}} = 0.
\end{aligned}$$

Hence, the uniqueness for (4.12) holds, under the following sufficient condition:

$$0 < \tau \leq \tilde{\tau}_1 := \frac{\delta_\alpha}{2(1 + |G'|_{L^\infty(\mathbb{R})})}.$$

Finally, to verify (4.13), let us multiply the both sides of (4.12) by $u - u_0$. Then, by (2.4), we observe that:

$$\begin{aligned}
&\frac{1}{\tau}\big|A_0(\tilde{u}_0)^{\frac{1}{2}}(u-\tilde{u}_0)\big|^2_{\mathscr{H}} + \Phi_\varepsilon^\delta(u) - \Phi_\varepsilon^\delta(\tilde{u}_0) \\
&\quad\leq \int_\Omega G(\eta)(\tilde{\eta}_0 - \eta)\,dx \\
&\quad\leq \int_\Omega \left(\hat{G}(\tilde{\eta}_0) - \hat{G}(\eta) + \frac{|G'|_{L^\infty(\mathbb{R})}}{2}|\tilde{\eta}_0 - \eta|^2\right)\,dx \\
&\quad\leq \int_\Omega \hat{G}(\tilde{\eta}_0)\,dx - \int_\Omega \hat{G}(\eta)\,dx + \frac{|G'|_{L^\infty(\mathbb{R})}}{2\delta_\alpha}\big|A_0(\tilde{u}_0)^{\frac{1}{2}}(u-\tilde{u}_0)\big|^2_{\mathscr{H}}.
\end{aligned} \tag{4.14}$$

So, putting

$$\tilde{\tau}_2 := \frac{2\delta_\alpha}{1 + |G'|_{L^\infty(\mathbb{R})}} \in (0, 1),$$

the inequality (4.13) is inferred from (4.14), under the sufficient condition $0 < \tau < \tilde{\tau}_2$.

Now, we conclude that $\tau_1 := \tilde{\tau}_0 \wedge \tilde{\tau}_1 \wedge \tilde{\tau}_2$ is the required constant to realize (4.12) and (4.13). $\square$

Proof of Theorem 1. Let us set τ_1, given in Lemma 3 as the required constant in this theorem, and let us fix any time-step-size $\tau \in (0, \tau_*)$. Then, since the value of constant τ_* is independent of the time-index $i \in \mathbb{N} \cup \{0\}$, the solution $\{u^\delta_{\tau,i}\}_{i=0}^\infty = \{[\eta^\delta_{\tau,i}, \theta^\delta_{\tau,i}, \theta^\delta_{\Gamma,\tau,i}]\}_{i=1}^\delta \subset \mathscr{V}_\varepsilon$ to the time-discretization scheme $(\mathrm{AP})^\delta_\tau$ is obtained by applying Lemma 3 to the equation (4.1), inductively, and moreover, the energy inequality (4.4) is obtained as a straightforward sequence of (4.13), for every $i \in \mathbb{N}$.

Next, we verify (4.3). To this end, we fix any $i \in \mathbb{N}$, and suppose that:

$$\begin{aligned} &0 \leq \eta^\delta_{\tau,i-1} \leq 1, \ m_0 \leq \theta^\delta_{\tau,i-1} \leq M_0 \text{ a.e. in } \Omega, \text{ and} \\ &m_0 \leq \theta^\delta_{\Gamma,\tau,i-1} \leq M_0, \text{ a.e. on } \Gamma, \text{ for } i = 1, 2, 3, \cdots. \end{aligned} \tag{4.15}$$

Also, let us invoke Remark 11, and confirm that:

$$\frac{1}{\tau}(\eta^\delta_{\tau,i} - \eta^\delta_{\tau,i-1}) - \mathit{\Delta}_N \eta^\delta_{\tau,i} + g(\eta^\delta_{\tau,i}) + \alpha'(\eta^\delta_{\tau,i}) f_\delta(\nabla \theta^\delta_{\tau,i}) = 0 \text{ in } L^2(\Omega),$$

and

$$\frac{1}{\tau}\tilde{A}_0(\eta^\delta_{\tau,i-1}) \begin{bmatrix} \theta^\delta_{\tau,i} - \theta^\delta_{\tau,i-1} \\ \theta^\delta_{\Gamma,\tau,i} - \theta^\delta_{\Gamma,\tau,i-1} \end{bmatrix} + \partial_v \Phi^\delta_\varepsilon(\eta^\delta_{\tau,i}, \theta^\delta_{\tau,i}, \theta^\delta_{\Gamma,\tau,i}) \ni 0 \text{ in } H.$$

Additionally, owing to (A1), (4.15), the constant functions $0(\in D(\mathit{\Delta}_N))$ and $1(\in D(\mathit{\Delta}_N))$ satisfy that:

$$\frac{1}{\tau}(0 - \eta^\delta_{\tau,i}) - \mathit{\Delta}_N 0 + g(0) + \alpha'(0)|\nabla \theta^\delta_{\tau,i}| \leq 0, \text{ a.e. in } \Omega,$$

and

$$\frac{1}{\tau}(1 - \eta^\delta_{\tau,i-1}) - \mathit{\Delta}_N 1 + g(1) + \alpha'(1)|\nabla \theta^\delta_{\tau,i}| \geq 0, \text{ a.e. in } \Omega,$$

and the pairs of constants $[m_0, m_0](\in H)$, $[M_0, M_0](\in H)$, and $[0, 0](\in H)$ satisfy that:

$$\begin{cases} [m_0, m_0] \in D(\partial_v \Phi^\delta_\varepsilon), \\ [0, 0] \in \partial_v \Phi^\delta_\varepsilon(\eta^\delta_{\tau,i}, m_0, m_0) \text{ in } H, \end{cases} \qquad \begin{cases} [M_0, M_0] \in D(\partial_v \Phi^\delta_\varepsilon), \\ [0, 0] \in \partial_v \Phi^\delta_\varepsilon(\eta^\delta_{\tau,i}, M_0, M_0) \text{ in } H, \end{cases}$$

$$\frac{1}{\tau}\tilde{A}_0(\eta^\delta_{\tau,i-1}) \begin{bmatrix} m_0 - \theta^\delta_{\tau,i-1} \\ m_0 - \theta^\delta_{\Gamma,\tau,i-1} \end{bmatrix} \left(+ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ in } H, \tag{4.16}$$

and

$$\frac{1}{\tau}\tilde{A}_0(\eta^\delta_{\tau,i-1}) \begin{bmatrix} M_0 - \theta^\delta_{\tau,i-1} \\ M_0 - \theta^\delta_{\Gamma,\tau,i-1} \end{bmatrix} \left(+ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ in } H. \tag{4.17}$$

Now, applying Lemma 1 to the case when:

$$\begin{cases} \zeta_1 = 0, & \zeta_{0,1} = \eta^\delta_{\tau,i-1}, \\ \zeta_2 = \eta^\delta_{\tau,i}, & \zeta_{0,2} = \eta^\delta_{\tau,i-1}, \end{cases} \quad \left(\text{resp. } \begin{cases} \zeta_1 = \eta^\delta_{\tau,i}, & \zeta_{0,1} = \eta^\delta_{\tau,i-1}, \\ \zeta_2 = 1, & \zeta_{0,2} = \eta^\delta_{\tau,i-1}, \end{cases} \right),$$

it is deduced that:

$$\begin{aligned} &|[-\eta^\delta_{\tau,i}]^+|_{L^2(\Omega)} \leq 0 \ \ (\text{ resp. } |[\eta^\delta_{\tau,i}-1]^+|_{L^2(\Omega)} \leq 0), \\ &\text{i.e. } \ 0 \leq \eta^\delta_{\tau,i} \leq 1, \text{ a.e. in } \Omega, \text{ for } i = 1,2,3,\dots. \end{aligned} \tag{4.18}$$

As well as, having in mind (A4), (4.16) and (4.17), we can apply Lemma 2 to the case when:

$$\begin{cases} w_1 = \begin{bmatrix} m_0 \\ m_0 \end{bmatrix}, & w_{0,1} = \begin{bmatrix} \theta^\delta_{\tau,i-1} \\ \theta^\delta_{\Gamma,\tau,i-1} \end{bmatrix}, \\ w_2 = \begin{bmatrix} \theta^\delta_{\tau,i} \\ \theta^\delta_{\Gamma,\tau,i} \end{bmatrix}, & w_{0,2} = \begin{bmatrix} \theta^\delta_{\tau,i-1} \\ \theta^\delta_{\Gamma,\tau,i-1} \end{bmatrix}, \end{cases}$$

$$\left(\text{resp. } \begin{cases} w_1 = \begin{bmatrix} \theta^\delta_{\tau,i} \\ \theta^\delta_{\Gamma,\tau,i} \end{bmatrix}, & w_{0,1} = \begin{bmatrix} \theta^\delta_{\tau,i-1} \\ \theta^\delta_{\Gamma,\tau,i-1} \end{bmatrix}, \\ w_2 = \begin{bmatrix} M_0 \\ M_0 \end{bmatrix}, & w_{0,2} = \begin{bmatrix} \theta^\delta_{\tau,i-1} \\ \theta^\delta_{\Gamma,\tau,i-1} \end{bmatrix}, \end{cases} \right),$$

one can see that:

$$\left| \begin{bmatrix} \delta_\alpha[m_0 - \theta^\delta_{\tau,i}]^+ \\ [m_0 - \theta^\delta_{\Gamma,\tau,i}]^+ \end{bmatrix} \right|_H^2 \leq \left| \tilde{A}_0(\eta^\delta_{\tau,i}) \begin{bmatrix} [m_0 - \theta^\delta_{\tau,i}]^+ \\ [m_0 - \theta^\delta_{\Gamma,\tau,i}]^+ \end{bmatrix} \right|_H^2 \leq 0,$$

$$\left(\text{resp. } \left| \begin{bmatrix} \delta_\alpha[\theta^\delta_{\tau,i} - M_0]^+ \\ [\theta^\delta_{\Gamma,\tau,i} - M_0]^+ \end{bmatrix} \right|_H^2 \leq \left| \tilde{A}_0(\eta^\delta_{\tau,i}) \begin{bmatrix} [\theta^\delta_{\tau,i} - M_0]^+ \\ [\theta^\delta_{\Gamma,\tau,i} - M_0]^+ \end{bmatrix} \right|_H^2 \leq 0 \right),$$

i.e.:

$$m_0 \leq \theta^\delta_{\tau,i} \leq M_0, \text{ a.e. in } \Omega \ \text{ and } \ m_0 \leq \theta^\delta_{\Gamma,\tau,i} \leq M_0, \text{ a.e on } \Gamma, \text{ for } i = 1,2,3,\dots. \tag{4.19}$$

By (4.18) and (4.19), we verify (4.3), and conclude this theorem. □

5 Proofs of Main Theorems

This section is devoted to the proof of Main Theorems.

5.1 Proof of Main Theorem 1

First, we prove the item (A). Let us fix any $u_0 = [\eta_0, \theta_0, \theta_{\Gamma,0}] \in \mathscr{D}_\varepsilon$. Let $0 < \tau_* < 1$ be the constant given in Theorem 1, and for every $0 < \delta \leq 1$ and $0 < \tau < \tau_*$, let $u^\delta_{\tau,i} = [\eta^\delta_{\tau,i}, \theta^\delta_{\tau,i}, \theta^\delta_{\Gamma,\tau,i}]$ be the solution to $(\mathrm{AP})^\delta_\tau$, subject to the initial condition (4.2).

Besides, we let:

$$t_i := i\tau, \text{ for } i = 0,1,2,\dots,$$

and we define the following time-interpolations:

$$\begin{aligned}
&\overline{u}_\tau^\delta(t) = [\overline{\eta}_\tau^\delta(t), \overline{\theta}_\tau^\delta(t), \overline{\theta}_{\Gamma,\tau}^\delta(t)] := u_{\tau,i}^\delta = [\eta_{\tau,i}^\delta, \theta_{\tau,i}^\delta, \theta_{\Gamma,\tau,i}^\delta], && \text{if } t_{i-1} < t \le t_i, \\
&\underline{u}_\tau^\delta(t) = [\underline{\eta}_\tau^\delta(t), \underline{\theta}_\tau^\delta(t), \underline{\theta}_{\Gamma,\tau}^\delta(t)] := u_{\tau,i-1}^\delta = [\eta_{\tau,i-1}^\delta, \theta_{\tau,i-1}^\delta, \theta_{\Gamma,\tau,i-1}^\delta], && \text{if } t_{i-1} \le t < t_i, \\
&\widehat{u}_\tau^\delta(t) = [\widehat{\eta}_\tau^\delta(t), \widehat{\theta}_\tau^\delta(t), \widehat{\theta}_{\Gamma,\tau}^\delta(t)] := \frac{t - t_{i-1}}{\tau}\overline{u}_\tau^\delta(t) + \frac{t_i - t}{\tau}\underline{u}_\tau^\delta(t), && \text{if } t_{i-1} \le t < t_i,
\end{aligned}$$

with some $i \in \mathbb{N}$, for all $t \ge 0$.

Then, from (4.3) in Theorem 1, we can see that:

$$\begin{cases}
0 \le \overline{\eta}_\tau^\delta \le 1,\ 0 \le \underline{\eta}_\tau^\delta \le 1, \text{ and } 0 \le \widehat{\eta}_\tau^\delta \le 1, \text{ a.e. in } Q, \\
m_0 \le \overline{\theta}_\tau^\delta \le M_0,\ m_0 \le \underline{\theta}_\tau^\delta \le M_0, \text{ and } m_0 \le \widehat{\theta}_\tau^\delta \le M_0, \text{ a.e. in } Q, \\
m_0 \le \overline{\theta}_{\Gamma,\tau}^\delta \le M_0,\ m_0 \le \underline{\theta}_{\Gamma,\tau}^\delta \le M_0, \text{ and } m_0 \le \widehat{\theta}_{\Gamma,\tau}^\delta \le M_0, \text{ a.e. on } \Sigma.
\end{cases} \tag{5.1}$$

Also, putting $n_t := \min\{\tilde{n} \in \mathbb{N} \,|\, \tilde{n}\tau \ge t\}$, for $t \in [0,T]$, we infer from (4.4) that:

$$\begin{aligned}
&\frac{1}{2}\int_0^t |\partial_t \widehat{\eta}_\tau^\delta(\sigma)|_{L^2(\Omega)}^2\, d\sigma + \frac{1}{2}\int_0^t \Big|\sqrt{\alpha_0(\overline{\eta}_\tau^\delta(\sigma))}\partial_t \widehat{\theta}_\tau^\delta(\sigma)\Big|_{L^2(\Omega)}^2\, d\sigma \\
&\quad + \frac{1}{2}\int_0^t |\partial_t \widehat{\theta}_{\Gamma,\tau}^\delta(\sigma)|_{L^2(\Gamma)}^2\, d\sigma + \mathscr{F}_\varepsilon^\delta(\overline{\eta}_\tau^\delta(t), \overline{\theta}_\tau^\delta(t), \overline{\theta}_{\Gamma,\tau}^\delta(t)) \\
&\le \frac{1}{2\tau}\sum_{i=1}^{n_t} \Big|A_0(u_{\tau,i-1}^\delta)^{\frac{1}{2}}(u_{\tau,i}^\delta - u_{\tau,i-1}^\delta)\Big|_{\mathscr{H}}^2 + \mathscr{F}_\varepsilon^\delta(u_{\tau,n_t}^\delta) \\
&\le \mathscr{F}_\varepsilon^\delta(u_0), \text{ for all } t \in [0,T],
\end{aligned} \tag{5.2}$$

and therefore

$$\begin{aligned}
&\frac{1}{2}\int_0^T \Big|A_0(\overline{u}_\tau^\delta)^{\frac{1}{2}}\partial_t \widehat{u}(t)\Big|_{\mathscr{H}}^2\, dt + \sup_{t\in[0,T]} \mathscr{F}_\varepsilon^\delta(\overline{u}_\tau^\delta(t)) \\
&\quad \le 2 \sup_{0\le\delta\le 1} \mathscr{F}_\varepsilon^\delta(u_0) \le 2\mathscr{F}_\varepsilon(u_0) + 2\delta|\alpha|_{C([0,1])}\mathcal{L}^N(\Omega).
\end{aligned} \tag{5.3}$$

As is checked from (5.2) and (5.3):

$$\begin{cases}
\bullet\ \{\overline{u}_\tau^\delta \mid 0 < \delta \le 1,\ 0 < \tau < \tau_*\} \text{ is bounded in } L^\infty(0,T;\mathscr{V}_\varepsilon), \\
\bullet\ \{\underline{u}_\tau^\delta \mid 0 < \delta \le 1,\ 0 < \tau < \tau_*\} \text{ is bounded in } L^\infty(0,T;\mathscr{V}_\varepsilon), \\
\bullet\ \{\widehat{u}_\tau^\delta \mid 0 < \delta \le 1,\ 0 < \tau < \tau_*\} \text{ is bounded in } W^{1,2}(0,T;\mathscr{H}) \text{ and in } L^\infty(0,T;\mathscr{V}_\varepsilon).
\end{cases} \tag{5.4}$$

By virtue of (5.1)–(5.4), we can apply the theories of compactness of Aubin's type [23, Corollary 4], Arzerá–Ascoli [24, Theorem 1.3.1], and Alaoglu–Bourbaki–Kakutani [24, Theorem 1.2.5], and can find sequences $\{\delta_n\}_{n=1}^\infty \subset (0,1)$, $\{\tau_n\}_{n=1}^\infty \subset (0,\tau_*)$, and a triplet $u = [\eta,\theta,\theta_\Gamma] \in L^2(0,T;\mathscr{H})$ of functions, such that $\delta_n \to 0$, $\tau_n \to 0$, as $n \to \infty$,

$$u = [\eta,\theta,\theta_\Gamma] \in W^{1,2}(0,T;\mathscr{H}) \cap L^\infty(0,T;\mathscr{V}_\varepsilon),$$

$$0 \le \eta \le 1,\ m_0 \le \theta \le M_0, \text{ a.e. in } Q, \text{ and } m_0 \le \theta_\Gamma \le M_0, \text{ a.e. on } \Sigma,$$

$$\begin{aligned}\widehat{u}_n = [\widehat{\eta}_n, \widehat{\theta}_n, \widehat{\theta}_{\Gamma,n}] := \widehat{u}^{\delta_n}_{\tau_n} = [\widehat{\eta}^{\delta_n}_{\tau_n}, \widehat{\theta}^{\delta_n}_{\tau_n}, \widehat{\theta}^{\delta_n}_{\Gamma,\tau_n}] \to u = [\eta, \theta, \theta_\Gamma] \text{ in } C([0,T]; \mathscr{H}),\\ \text{weakly in } W^{1,2}(0,T;\mathscr{H}), \text{ and weakly-}* \text{ in } L^\infty(0,T;\mathscr{V}_\varepsilon), \text{ as } n \to \infty,\end{aligned} \tag{5.5}$$

$$u(0) = u_n(0) = u_0 \text{ in } \mathscr{H}, \text{ for } n = 1,2,3,\ldots, \tag{5.6}$$

and therefore

$$\begin{aligned}\overline{u}_n = [\overline{\eta}_n, \overline{\theta}_n, \overline{\theta}_{\Gamma,n}] := \overline{u}^{\delta_n}_{\tau_n} = [\overline{\eta}^{\delta_n}_{\tau_n}, \overline{\theta}^{\delta_n}_{\tau_n}, \overline{\theta}^{\delta_n}_{\Gamma,\tau_n}] \to u = [\eta, \theta, \theta_\Gamma]\\ \text{in } L^\infty(0,T;\mathscr{H}), \text{ and weakly-}* \text{ in } L^\infty(0,T;\mathscr{V}_\varepsilon), \text{ as } n \to \infty,\\ \underline{u}_n = [\underline{\eta}_n, \underline{\theta}_n, \underline{\theta}_{\Gamma,n}] := \underline{u}^{\delta_n}_{\tau_n} = [\underline{\eta}^{\delta_n}_{\tau_n}, \underline{\theta}^{\delta_n}_{\tau_n}, \underline{\theta}^{\delta_n}_{\Gamma,\tau_n}] \to u = [\eta, \theta, \theta_\Gamma]\\ \text{in } L^\infty(0,T;\mathscr{H}), \text{ and weakly-}* \text{ in } L^\infty(0,T;\mathscr{V}_\varepsilon), \text{ as } n \to \infty.\end{aligned} \tag{5.7}$$

Here, from (5.1) and (5.7), it follows that:

$$\begin{cases} \overline{\eta}_n \to \eta \text{ weakly-}* \text{ in } L^\infty(Q), \text{ and in the pointwise sense, a.e. in } Q,\\ \overline{\theta}_n \to \theta \text{ weakly-}* \text{ in } L^\infty(Q), \text{ and in the pointwise sense, a.e. in } Q,\\ \overline{\theta}_{\Gamma,n} \to \theta_\Gamma \text{ weakly-}* \text{ in } L^\infty(\Sigma), \text{ and in the pointwise sense, a.e. on } \Sigma, \end{cases} \tag{5.8}$$

by taking a subsequence if necessary. Invoking (A2), (5.5), (5.7), and (5.8), we can apply the dominated convergence theorem (cf. [14, Theorem 10]), and can obtain the following convergences:

$$\begin{cases} A_0(\overline{u}_n)\partial_t \widehat{u}_n = [\partial_t \widehat{\eta}_n, \alpha_0(\overline{\eta}_n)\partial_t \widehat{\theta}_n, \partial_t \widehat{\theta}_{\Gamma,n}] \to A_0(u)\partial_t u = [\partial_t \eta, \alpha_0(\eta)\partial_t \theta, \partial_t \theta_\Gamma]\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{weakly in } L^2(0,T;\mathscr{H}),\\ \mathcal{G}(\overline{u}_n) \to \mathcal{G}(u) \text{ in } L^2(0,T;\mathscr{H}), \text{ as } n \to \infty. \end{cases} \tag{5.9}$$

Furthermore, having in mind (4.1), (5.7), (5.9), Key-Lemma 4, and [4, Lemma 4.1], we can see that:

$$-A_0(\overline{u})\partial_t \widehat{u} - \mathcal{G}(\overline{u}) \in \partial \hat{\Phi}^T_\varepsilon(\overline{u}) \text{ in } L^2(0,T;\mathscr{H}), \text{ for any } \varepsilon \geq 0, \tag{5.10}$$

and

$$\hat{\Phi}^{T,n}_\varepsilon \to \hat{\Phi}^T_\varepsilon \text{ on } L^2(0,T;\mathscr{H}), \text{ in the sense of Mosco, as } n \to \infty, \tag{5.11}$$

where for every $\varepsilon \in [0,\infty)$,

$$\begin{cases} \tilde{u} \in L^2(0,T;\mathscr{H}) \mapsto \hat{\Phi}^T_\varepsilon(\tilde{u}) := \displaystyle\int_0^T \Phi_\varepsilon(\tilde{u}(t))\,dt \in [0,\infty],\\ \tilde{u} \in L^2(0,T;\mathscr{H}) \mapsto \hat{\Phi}^{T,n}_\varepsilon(\tilde{u}) := \displaystyle\int_0^T \Phi^{\delta_n}_\varepsilon(\tilde{u}(t))\,dt \in [0,\infty], \ n = 1,2,3,\ldots. \end{cases}$$

By (5.6), (5.10), and [3, Proposition 2.16], we can observe that $u = [\eta, \theta, \theta_\Gamma]$ is a solution to the Cauchy problem (2.5).

Next, we consider the constant case of α_0 to verify the uniqueness. In this case, the operator $A_0 = A_0(u)$ becomes just a positive diagonal matrix $\overline{A}_0$, i.e.:

$$\overline{A}_0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3},$$

and referring to [5, Proposition 5.9 in Chapter 1], the Cauchy problem (2.5) can be reduced to:

$$\begin{cases} u'(t)+\partial(\Phi_\varepsilon\circ(\overline{A}_0)^{-1})(u(t))+(\overline{A}_0)^{-1}\mathcal{G}(u(t)) \ni 0 \text{ in } \mathscr{H}, \text{ a.e. } t\in(0,T), \\ u(0) = u_0 \text{ in } \mathscr{H}. \end{cases}$$

Since $(\overline{A}_0)^{-1}\mathcal{G} : \mathscr{H} \to \mathscr{H}$ is a Lipschitz operator, we can apply the general theory of nonlinear evolution equation [3, Proposition 3.12], and can obtain the uniqueness of the solution.

Finally, the item (B) is verified as a straightforward consequence of Remarks 3 and 8, Key-Lemmas 1 and 5.

Thus, we can conclude Main Theorem 1. □

5.2 Proof of Main Theorem 2

First, we show the item (C). Then, under (0.6), (3.1), (4.5), and (A4), we can observe from (5.2) and (5.3) that:

$$\begin{aligned}
&\frac{1}{2}|\partial_t\eta_\varepsilon|^2_{L^2(0,T;L^2(\Omega))} + \frac{\delta_\alpha}{2}|\partial_t\theta_\varepsilon|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}|\partial_t\theta_{\Gamma,\varepsilon}|^2_{L^2(0,T;L^2(\Gamma))} \\
&\quad + \frac{1}{2}|\nabla\eta_\varepsilon|^2_{L^2(0,T;L^2(\Omega)^N)} + \frac{\nu^2}{2}|\nabla\theta_\varepsilon|^2_{L^2(0,T;L^2(\Omega)^N)} + \frac{1}{2}|\nabla_\Gamma(\varepsilon\theta_{\Gamma,\varepsilon})|^2_{L^2(0,T;L^2(\Gamma)^N)} \\
&\le \varliminf_{\delta\downarrow 0}\left(\frac{1}{2}\left|A_0(u_\varepsilon^\delta)^{\frac{1}{2}}\partial_t u_\varepsilon^\delta\right|^2_{L^2(0,T;\mathscr{H})}\right) + \varliminf_{\delta\downarrow 0}\mathscr{F}_\varepsilon^\delta(u_\varepsilon^\delta) \\
&\le \varlimsup_{\delta\downarrow 0}\left(\frac{1}{2}\left|A_0(u_\varepsilon^\delta)^{\frac{1}{2}}\partial_t u_\varepsilon^\delta\right|^2_{L^2(0,T;\mathscr{H})} + \mathscr{F}_\varepsilon^\delta(u_\varepsilon^\delta)\right) \\
&\le \lim_{\delta\downarrow 0}\mathscr{F}_\varepsilon^\delta(u_{0,\varepsilon}) = \mathscr{F}_\varepsilon(u_{0,\varepsilon}), \text{ for any } \varepsilon \ge 0.
\end{aligned} \tag{5.12}$$

Besides, let us set $J_n := [\frac{\varepsilon_0}{n+1}, \frac{1}{n} + \varepsilon_0]$, for $n = 1, 2, 3, \ldots$. Then, by (0.6), (2.6), (A3), and (S0), we have:

$$
\begin{aligned}
0 \leq \mathscr{F}_\varepsilon(u_{0,\varepsilon}) &= \frac{1}{2}\int_\Omega |\nabla \eta_{0,\varepsilon}|^2\,dx + \int_\Omega \hat{g}(\eta_{0,\varepsilon})\,dx \\
&\quad + \int_\Omega \left(\alpha(\eta_{0,\varepsilon})|\nabla\theta_{0,\varepsilon}| + \frac{\nu^2}{2}|\nabla\theta_{0,\varepsilon}|^2\right)dx + \frac{1}{2}\int_\Gamma |\nabla_\Gamma(\varepsilon\theta_\Gamma)|^2\,d\Gamma \\
&\leq \frac{1}{2}|\eta_{0,\varepsilon}|^2_{H^1(\Omega)} + \frac{\nu^2}{2}|\theta_{0,\varepsilon}|^2_{H^1(\Omega)} + \frac{1}{2}|\varepsilon\theta_{\Gamma,0,\varepsilon}|^2_{H^1(\Gamma)} \\
&\quad + |\hat{g}(\eta_{0,\varepsilon})|_{L^1(\Omega)} + |\alpha(\eta_{0,\varepsilon})|_{L^\infty(\Omega)}|\nabla\theta_{0,\varepsilon}|_{L^1(\Omega)^N} \\
&\leq \bigl(1+\nu^2\bigr)\bigl(|\eta_{0,\varepsilon}|^2_{H^1(\Omega)} + |\theta_{0,\varepsilon}|^2_{H^1(\Omega)} + |\varepsilon\theta_{\Gamma,0,\varepsilon}|^2_{H^1(\Gamma)}\bigr) \\
&\quad + |\hat{g}|_{L^\infty(0,1)}\mathcal{L}^N(\Omega) + \frac{1}{2}|\theta_{0,\varepsilon}|^2_{H^1(\Omega)} + \frac{1}{2}|\alpha|^2_{C([0,1])}\mathcal{L}^N(\Omega) \\
&\leq 2(1+\nu^2)\sup_{\varepsilon\in J_1}\bigl|[\eta_{0,\varepsilon}, \theta_{0,\varepsilon}, \varepsilon\theta_{\Gamma,0,\varepsilon}]\bigr|^2_{H^1(\Omega)^2\times H^1(\Gamma)} + (1 + |\alpha|_{C([0,1])} + |\hat{g}|_{L^\infty(0,1)})^2\mathcal{L}^N(\Omega) \\
&=: R_0 < \infty, \text{ for all } \varepsilon \in J_1 = [\tfrac{\varepsilon_0}{2}, 1+\varepsilon_0].
\end{aligned}
\tag{5.13}
$$

On account of (5.12), (5.13), and Remark 10, we can see that:

$$
\{u_\varepsilon\}_{\varepsilon\in J_1} = \{[\eta_\varepsilon, \theta_\varepsilon, \theta_{\Gamma,\varepsilon}]\}_{\varepsilon\in J_1} \text{ is bounded in } W^{1,2}(0,T;\mathscr{H}) \cap L^\infty(0,T;\mathscr{V}_{\varepsilon_0}).
$$

Therefore, applying general theories of compactness, e.g. Aubin's type [23, Corollary 4], Arzerá–Ascoli [24, Theorem 1.3.1], and Alaoglu–Bourbaki–Kakutani [24, Theorem 1.2.5], we find a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset J_1$, and a limiting triplet $u = [\eta, \theta, \theta_\Gamma] \in W^{1,2}(0,T;\mathscr{H}) \cap L^\infty(0,T;\mathscr{V}_{\varepsilon_0})$, such that:

$$
\varepsilon_n \to \varepsilon_0, \text{ as } n \to \infty, \tag{5.14}
$$

and the sequence $\{u_n\}_{n=1}^\infty = \{[\eta_n, \theta_n, \theta_{\Gamma,n}]\}_{n=1}^\infty := \{u_{\varepsilon_n}\}_{n=1}^\infty = \{[\eta_{\varepsilon_n}, \theta_{\varepsilon_n}, \theta_{\Gamma,\varepsilon_n}]\}_{n=1}^\infty$ satisfies:

$$
\begin{aligned}
&u_n = [\eta_n, \theta_n, \theta_{\Gamma,n}] \to u = [\eta, \theta, \theta_\Gamma] \text{ in } C([0,T];\mathscr{H}), \\
&\qquad \text{weakly in } W^{1,2}(0,T;\mathscr{H}), \text{ and weakly-}* \text{ in } L^\infty(0,T;\mathscr{V}_{\varepsilon_0}), \text{ as } n \to \infty,
\end{aligned}
\tag{5.15}
$$

and in particular,

$$
\begin{aligned}
u_n(0) &= [\eta_n(0), \theta_n(0), \theta_{\Gamma,n}(0)] = [\eta_{0,\varepsilon_n}, \theta_{0,\varepsilon_n}, \theta_{\Gamma,0,\varepsilon_n}] \to [\eta_{0,\varepsilon_0}, \theta_{0,\varepsilon_0}, \theta_{\Gamma,0,\varepsilon_0}] \\
&\qquad = [\eta(0), \theta(0), \theta_\Gamma(0)] = u(0) \text{ in } \mathscr{H}, \text{ and weakly in } \mathscr{V}_{\varepsilon_0}, \text{ as } n \to \infty.
\end{aligned}
\tag{5.16}
$$

Thus, $\varlimsup_{\varepsilon\to\varepsilon_0} \mathfrak{S}_\varepsilon(u_{0,\varepsilon}) \supset \{u\} \neq \emptyset$.

Additionally, by (5.12) and (5.13), it is inferred that:

$$
\begin{aligned}
&|u_\varepsilon|^2_{W^{1,2}(0,T;\mathscr{H})} + |[\nabla\eta, \nabla\theta]|^2_{L^2(0,T;(L^2(\Omega)^N)^2)} \\
&\leq |u_\varepsilon|^2_{L^2(0,T;\mathscr{H})} + |\partial_t u_\varepsilon|^2_{L^2(0,T;\mathscr{H})} + |[\nabla\eta, \nabla\theta]|^2_{L^2(0,T;(L^2(\Omega)^N)^2)} + |\nabla_\Gamma(\varepsilon\theta_{\Gamma,\varepsilon})|^2_{L^2(0,T;L^2(\Gamma)^N)} \\
&\leq 2T|u_{0,\varepsilon}|^2_{\mathscr{H}} + (1+2T)|\partial_t u_\varepsilon|^2_{L^2(0,T;\mathscr{H})} \\
&\qquad + |\nabla\eta_\varepsilon|^2_{L^2(0,T;L^2(\Omega)^N)} + |\nabla\theta_\varepsilon|^2_{L^2(0,T;L^2(\Omega)^N)} + |\nabla(\varepsilon\theta_{\Gamma,\varepsilon})|^2_{L^2(0,T;L^2(\Gamma)^N)} \\
&\leq 2T|u_{0,\varepsilon}|^2_{\mathscr{V}_{\varepsilon_0}} + \frac{4(1+T)}{1\wedge\delta_\alpha\wedge\nu^2}\mathscr{F}_\varepsilon(u_{0,\varepsilon}) \\
&\leq 2T\sup_{\varepsilon\in J_1}|u_{0,\varepsilon}|^2_{\mathscr{V}_{\varepsilon_0}} + \frac{4(1+T^2)}{1\wedge\delta_\alpha\wedge\nu^2}R_0 =: R_1 < \infty,
\end{aligned}
$$

$$
|\theta_{\Gamma,\varepsilon}|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} = |\theta_{\varepsilon|_\Gamma}|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \leq C_\Gamma^2|\theta_\varepsilon|^2_{L^2(0,T;H^1(\Omega))} \leq C_\Gamma^2 R_1 =: R_2,
$$

and

$$
|u_\varepsilon|^2_{W^{1,2}(0,T;\mathscr{H})} + |u_\varepsilon|^2_{L^2(0,T;H^1(\Omega)^2\times H^{\frac{1}{2}}(\Gamma))} \leq 2(R_1+R_2) =: R_3, \text{ for all } \varepsilon \in J_1,
$$

where C_Γ is the operator norm of the trace operator $\mathrm{tr}_\Gamma \in \mathcal{L}(H^1(\Omega); H^{\frac{1}{2}}(\Gamma))$. Therefore, it will be estimated that:

$$
\begin{aligned}
&\overline{\lim_{\varepsilon\to\varepsilon_0}}\,\mathfrak{S}_\varepsilon(u_{0,\varepsilon}) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{\varepsilon\in J_n}\mathfrak{S}_\varepsilon(u_{0,\varepsilon})}^{\mathscr{H}} \subset \overline{\bigcup_{\varepsilon\in J}\mathfrak{S}_\varepsilon(u_{0,\varepsilon})}^{\mathscr{H}} \\
&\quad\subset \left\{ \tilde{u} \in L^2(0,T;\mathscr{H}) \;\middle|\; \begin{array}{l} \tilde{u} \in W^{1,2}(0,T;\mathscr{H}) \cap L^2(0,T;H^1(\Omega)^2 \times H^{\frac{1}{2}}(\Gamma)), \\ \text{and } |\tilde{u}|^2_{W^{1,2}(0,T;\mathscr{H})} + |\tilde{u}|^2_{L^2(0,T;H^1(\Omega)^2\times H^{\frac{1}{2}}(\Gamma))} \leq R_3 \end{array} \right\}.
\end{aligned} \tag{5.17}
$$

The compactness of $\overline{\lim_{\varepsilon\to\varepsilon_0}}\,\mathfrak{S}_\varepsilon(u_{0,\varepsilon})$ is verified as a consequence of (5.17) and the compactness theory of Aubin's type [23, Corollary 4].

Next, we show the item (D). Let us take any $u = [\eta, \theta, \theta_\Gamma] \in \overline{\lim_{\varepsilon\to\varepsilon_0}}\,\mathfrak{S}_\varepsilon(u_{0,\varepsilon})$ to show $u \in \mathfrak{S}_{\varepsilon_0}(u_{0,\varepsilon})$. Then, in the light of (5.12) and (5.13), we may suppose the existence of sequences $\{\varepsilon_n\} \subset J_1$, and $\{u_n = [\eta_n, \theta_n, \theta_{\Gamma,n}] \in \mathfrak{S}_{\varepsilon_0}(u_{0,\varepsilon_n})\}_{n=1}^\infty$, satisfying (5.14)–(5.16). Meanwhile, by Definition 2 (S0), we have:

$$
\begin{gathered}
0 \leq \eta_n \leq 1,\ m_0 \leq \theta_n \leq M_0, \text{ a.e. in } Q, \text{ and } m_0 \leq \theta_{\Gamma,n}(t,y) \leq M_0, \text{ a.e. on } \Sigma, \\
\text{for } n = 1,2,3,\ldots,
\end{gathered} \tag{5.18}
$$

and by (5.15) and (5.18), we can derive:

$$
0 \leq \eta \leq 1,\ m_0 \leq \theta \leq M_0, \text{ a.e. in } Q, \text{ and } m_0 \leq \theta_\Gamma \leq M_0, \text{ a.e. on } \Sigma, \tag{5.19}
$$

and

$$
\left\{
\begin{array}{l}
\eta_n \to \eta \text{ weakly-}* \text{ in } L^\infty(Q), \text{ and in the pointwise sense, a.e. in } Q, \\
\theta_n \to \theta \text{ weakly-}* \text{ in } L^\infty(Q), \text{ and in the pointwise sense, a.e. in } Q, \\
\theta_{\Gamma,n} \to \theta_\Gamma \text{ weakly-}* \text{ in } L^\infty(\Sigma), \text{ and in the pointwise sense, a.e. on } \Sigma,
\end{array}
\right. \tag{5.20}
$$

for some subsequence (not relabeled). (5.15), (5.20), and the dominated convergence theorem (cf. [14, Theorem 10]) allow us to infer that:

$$\begin{cases} A_0(u_n)\partial_t u_n = [\partial_u \eta_n, \alpha_0(\eta_n)\partial_t \theta_n, \partial_t \theta_{\Gamma,n}] \to A_0(u)\partial_t u = [\partial_t \eta, \alpha_0(\eta)\partial_t \theta, \partial_t \theta_\Gamma] \\ \hfill \text{weakly in } L^2(0,T;\mathscr{H}), \\ \mathcal{G}(u_n) \to \mathcal{G}(u) \text{ in } L^2(0,T;\mathscr{H}), \text{ as } n \to \infty. \end{cases} \tag{5.21}$$

Furthermore, from Key-Lemma 4 and [4, Lemma 4.1], it follows that:

$$\hat{\Phi}_{\varepsilon_n}^T \to \hat{\Phi}_{\varepsilon_0}^T \text{ on } L^2(0,T;\mathscr{H}), \text{ in the sense of Mosco, as } n \to \infty, \tag{5.22}$$

where

$$\begin{cases} \tilde{u} \in L^2(0,T;\mathscr{H}) \mapsto \hat{\Phi}_{\varepsilon_0}^T(\tilde{u}) := \displaystyle\int_0^T \Phi_{\varepsilon_0}(\tilde{u}(t))\,dt \in [0,\infty], \\ \tilde{u} \in L^2(0,T;\mathscr{H}) \mapsto \hat{\Phi}_n^T(\tilde{u}) := \displaystyle\int_0^T \Phi_{\varepsilon_n}(\tilde{u}(t))\,dt \in [0,\infty], n = 1,2,3,\dots. \end{cases}$$

Taking into account (5.15), (5.21), (5.22), and Remark 4 (Fact 4), it is inferred that:

$$[-A_0(u)\partial_t u - \mathcal{G}(u), u] \in \partial \hat{\Phi}_{\varepsilon_0}^T \text{ in } L^2(0,T;\mathscr{H})^2, \tag{5.23}$$

and moreover, applying [3, Proposition 2.16] to (5.23), and invoking (5.16) and (5.19), it is deduced that $u = [\eta, \theta, \theta_\Gamma] \in \mathfrak{S}_{\varepsilon_0}(u_{0,\varepsilon_0})$.

Thus, the item (D) is verified, and the proof of Main Theorem 2 is complete. □

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