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GAKKOTOSHO
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ABSTRACT APPROACH TO DEGENERATE PARABOLIC EQUATIONS WITH DYNAMIC BOUNDARY CONDITIONS

Takeshi Fukao
Department of Mathematics, Faculty of Education
Kyoto University of Education
1 Fujinomori, Fukakusa, Fushimi-ku, Kyoto 612-8522 Japan
(E-mail: `fukao@kyokyo-u.ac.jp`)

and

Taishi Motoda
Graduate School of Education, Kyoto University of Education
Kyoto University of Education
1 Fujinomori, Fukakusa, Fushimi-ku, Kyoto 612-8522 Japan
(E-mail: `motoda.math@gmail.com`)

Abstract. An initial boundary value problem of the nonlinear diffusion equation with dynamic boundary condition is treated. The existence problem of the initial-boundary value problem is discussed. The main idea of the proof is an abstract approach from evolution equations governed by subdifferentials. To apply this, the setting of suitable function spaces, more precisely the mean-zero function spaces, is important. In the case of dynamic boundary condition, the total mass, which is the sum of volumes in the bulk and on the boundary, is a point of emphasis. The existence of weak solutions is proved on this basis.

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1 Introduction

We consider the initial boundary value problem of the nonlinear diffusion equation (P), comprising

$$\frac{\partial u}{\partial t} - \Delta\xi = f, \quad \xi \in \beta(u) \quad \text{in } Q := (0,T) \times \Omega, \tag{1}$$

$$\xi|_\Gamma = \xi_\Gamma, \quad \frac{\partial u_\Gamma}{\partial t} + \partial_{\boldsymbol{\nu}}\xi - \Delta_\Gamma \xi_\Gamma = f_\Gamma, \quad \xi_\Gamma \in \beta(u_\Gamma) \quad \text{on } \Sigma := (0,T) \times \Gamma, \tag{2}$$

$$u(0,\cdot) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0,\cdot) = u_{0\Gamma} \quad \text{on } \Gamma, \tag{3}$$

where $0 < T < +\infty$, Ω is a bounded domain of $\mathbb{R}^d$ ($d = 2, 3$) with smooth boundary $\Gamma := \partial\Omega$, $\xi|_\Gamma$ stands for the trace of ξ to Γ, $\partial_{\boldsymbol{\nu}}$ is the outward normal derivative on Γ, Δ is the Laplacian, Δ_Γ is the Laplace–Beltrami operator (see, e.g., [15]), and $f : Q \to \mathbb{R}$, $f_\Gamma : \Sigma \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, and $u_{0\Gamma} : \Gamma \to \mathbb{R}$ are given data. Moreover, $\beta : \mathbb{R} \to 2^{\mathbb{R}}$, a maximal monotone graph, characterizes the first and second equations of (P) as the degenerate parabolic system. Indeed, by choosing various types of β given later, (P) will be various types of the degenerate parabolic system; e.g., (P) can be the Stefan problem, porous media equation, or fast diffusion equation (see, e.g., [10]). In particular, we allow β to be multivalued because we are also interested in the Hele-Shaw profile; more precisely, $\beta := \partial I_{[0,1]}$, the subdifferential of the indicator function $I_{[0,1]}$ on interval $[0, 1]$. In this paper, we treat a modified version of the Hele-Shaw profile.

In terms of the well-posedness of (P), an early result for the Stefan problem was given [11]. For this result, an abstract theory of the evolution equation in Hilbert space was applied. There are also treatments of (P) [5]; more precisely, there are two major approaches named the Hilbert space approach and L^1 approach. Results obtained using the Hilbert space approach have been presented [1, 2, 3] related to the Stefan problem with dynamic boundary condition, [12, 18] for a wider degenerate parabolic equation. Results obtained using the L^1 approach have been reported [4, 16, 17]. We refer to (2), which includes a time derivative, as the dynamic boundary condition. Asymptotic analysis of the Cahn–Hiliard equation has recently been performed [13, 14]. In this treatment, if we choose different values of β between (1) and (2), namely β and β_Γ, then we need a domination assumption [9, p.419, (A6)]. We improve this assumption in Section 4 of the present paper. In the cited studies, the important point is the setting of function spaces, where the total mass is zero. This property arises from the dynamic boundary condition (see also [10, 18, 23] for the setting of the Neumann boundary condition). In the present paper, to apply the pioneering idea of [11], we use the same setting [13, 14] to construct the duality mapping that plays the role of diffusion. One of the greatest difficulties of the problem is similar to the case of the Neumann boundary condition (see, e.g., [18, 22]).

The present paper proceeds as follows. Section 2 states the main theorem. We first prepare the notation used in this paper and set the suitable duality mapping and function spaces. We then introduce the definition of the weak solution of (P), and give the main theorem.

In Section 3, to apply the abstract theory of evolution equations governed by subdifferentials [7], we define the proper lower semicontinuous and convex functional. We consider the approximate problem using Moreau–Yosida regularization. We also give characterization lemma for the subdifferential. We then deduce uniform estimates of the approximate solutions. We finally prove the existence of weak solutions by passing to the limit.

In Section 4, we discuss improvements to the assumptions.

A detailed index of sections and subsections follows.

2 Main results

2.1 Notation

We use the spaces $H := L^2(\Omega)$, $H_\Gamma := L^2(\Gamma)$, $V := H^1(\Omega)$, $V_\Gamma := H^1(\Gamma)$ with respective standard norms $|\cdot|_H$, $|\cdot|_{H_\Gamma}$, $|\cdot|_V$, $|\cdot|_{V_\Gamma}$ and inner products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_{H_\Gamma}$, $(\cdot,\cdot)_V$, $(\cdot,\cdot)_{V_\Gamma}$. Moreover, we set $\boldsymbol{H} := H \times H_\Gamma$ and

$$\boldsymbol{V} := \bigl\{ \boldsymbol{z} := (z, z_\Gamma) \in V \times V_\Gamma \ : \ z|_\Gamma = z_\Gamma \text{ a.e. on } \Gamma \bigr\}.$$

$\boldsymbol{H}$ and $\boldsymbol{V}$ are then Hilbert spaces with inner products

$$\begin{aligned} (\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{H}} &:= (u, z)_H + (u_\Gamma, z_\Gamma)_{H_\Gamma} \quad \text{for all } \boldsymbol{u} := (u, u_\Gamma), \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{H}, \\ (\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{V}} &:= (u, z)_V + (u_\Gamma, z_\Gamma)_{V_\Gamma} \quad \text{for all } \boldsymbol{u} := (u, u_\Gamma), \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{V}. \end{aligned}$$

Note that $\boldsymbol{z} \in \boldsymbol{V}$ implies that the second component z_Γ of $\boldsymbol{z}$ is equal to the trace of the first component z of $\boldsymbol{z}$ on Γ, and $\boldsymbol{z} \in \boldsymbol{H}$ implies that $z \in H$ and $z_\Gamma \in H_\Gamma$ are independent. Throughout this paper, we use the bold letter $\boldsymbol{u}$ to represent the pair corresponding to the letter; i.e., $\boldsymbol{u} := (u, u_\Gamma)$.

Let $m : \boldsymbol{H} \to \mathbb{R}$ be the special mean function defined by

$$m(\boldsymbol{z}) := \frac{1}{|\Omega| + |\Gamma|} \left\{ \int_\Omega z dx + \int_\Gamma z_\Gamma d\Gamma \right\} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H},$$

where $|\Omega| := \int_\Omega 1dx, |\Gamma| := \int_\Gamma 1d\Gamma$. We then define $\boldsymbol{H}_0 := \{\boldsymbol{z} \in \boldsymbol{H} : m(\boldsymbol{z}) = 0\}$, $\boldsymbol{V}_0 := \boldsymbol{V} \cap \boldsymbol{H}_0$. Moreover, $\boldsymbol{V}^*, \boldsymbol{V}_0^*$ denote the dual spaces of $\boldsymbol{V}, \boldsymbol{V}_0$, respectively; the duality pairing between $\boldsymbol{V}_0^*$ and $\boldsymbol{V}_0$ is denoted $\langle \cdot, \cdot \rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0}$. We define the norm of $\boldsymbol{H}_0$ by $|\boldsymbol{z}|_{\boldsymbol{H}_0} := |\boldsymbol{z}|_{\boldsymbol{H}}$ for all $\boldsymbol{z} \in \boldsymbol{H}_0$. We now use the bilinear form $a(\cdot, \cdot) : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$, defined by

$$a(\boldsymbol{u}, \boldsymbol{z}) := \int_\Omega \nabla u \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma z_\Gamma d\Gamma \quad \text{for all } \boldsymbol{u}, \boldsymbol{z} \in \boldsymbol{V}.$$

Then, for all $\boldsymbol{z} \in \boldsymbol{V}_0$, $|\boldsymbol{z}|_{\boldsymbol{V}_0} := \sqrt{a(\boldsymbol{z}, \boldsymbol{z})}$ is the norm of $\boldsymbol{V}_0$. Also, for all $\boldsymbol{z} \in \boldsymbol{V}_0$, we let $\boldsymbol{F} : \boldsymbol{V}_0 \to \boldsymbol{V}_0^*$ be the duality mapping defined by

$$\langle \boldsymbol{F}\boldsymbol{z}, \tilde{\boldsymbol{z}} \rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} := a(\boldsymbol{z}, \tilde{\boldsymbol{z}}) \quad \text{for all } \tilde{\boldsymbol{z}} \in \boldsymbol{V}_0.$$

Then from the Poincaré–Wirtinger inequality, there exists a positive constant c_{P} such that

$$|z|_{\boldsymbol{V}}^2 \le c_{\mathrm{P}} \left\{ a(\boldsymbol{z}, \boldsymbol{z}) + \left| m(\boldsymbol{z}) \right|^2 \right\} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}. \tag{4}$$

Moreover, we define the inner product of $\boldsymbol{V}_0^*$ by

$$(\boldsymbol{z}^*, \tilde{\boldsymbol{z}}^*)_{\boldsymbol{V}_0^*} := \langle \boldsymbol{z}^*, \boldsymbol{F}^{-1} \tilde{\boldsymbol{z}}^* \rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} \quad \text{for all } \boldsymbol{z}^*, \tilde{\boldsymbol{z}}^* \in \boldsymbol{V}_0^*.$$

We have $\boldsymbol{V}_0 \hookrightarrow\hookrightarrow \boldsymbol{H}_0 \hookrightarrow\hookrightarrow \boldsymbol{V}_0^*$, where "$\hookrightarrow\hookrightarrow$" stands for compact embedding (see [9, Lemmas A and B]). One of the essential ideas of the present paper is the setting of the function space $\boldsymbol{V}_0$ and the duality mapping $\boldsymbol{F}$ that plays the role of diffusion, as in [11].

2.2 Definition of the solution and main theorem

In this subsection, we define our solution for the initial-boundary value problem (1)–(3), named by (P), and then state the main theorem.

Definition 2.1. *The quadruplet $(u, u_\Gamma, \xi, \xi_\Gamma)$ is called a weak solution of* (P) *if*

$$u \in H^1(0,T;V^*) \cap L^\infty(0,T;H), \quad u_\Gamma \in H^1(0,T;V_\Gamma^*) \cap L^\infty(0,T;H_\Gamma), \tag{5}$$

$$\xi \in L^2(0,T;V), \quad \xi_\Gamma \in L^2(0,T;V_\Gamma),$$

$$\xi \in \beta(u) \quad \text{a.e. in } Q,$$

$$\xi_\Gamma \in \beta(u_\Gamma), \quad \xi|_\Gamma = \xi_\Gamma \quad \text{a.e. on } \Sigma \tag{6}$$

satisfying

$$\begin{aligned} &\langle u'(t), z \rangle_{V^*,V} + \langle u_\Gamma'(t), z_\Gamma \rangle_{V_\Gamma^*, V_\Gamma} + \int_\Omega \nabla \xi(t) \cdot \nabla z dx + \int_\Gamma \nabla_\Gamma \xi_\Gamma(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma \\ &= \int_\Omega f(t) z dx + \int_\Gamma f_\Gamma(t) z_\Gamma d\Gamma \quad \text{for all } \boldsymbol{z} := (z, z_\Gamma) \in \boldsymbol{V}, \end{aligned} \tag{7}$$

for a.a. $t \in (0,T)$, with

$$u(0, \cdot) = u_0 \quad \text{a.e. in } \Omega, \quad u_\Gamma(0, \cdot) = u_{0\Gamma} \quad \text{a.e. on } \Gamma.$$

We assume the following.

(A1) $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph, which is the subdifferential $\beta = \partial_{\mathbb{R}} \widehat{\beta}$ of some proper lower semicontinuous convex function $\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ satisfying $\widehat{\beta}(0) = 0$;

(A2) there exist positive constants c_1, c_2 such that $\widehat{\beta}(r) \geq c_1 r^2 - c_2$ for all $r \in \mathbb{R}$;

(A3) $\boldsymbol{f} := (f, f_\Gamma) \in L^2(0, T; \boldsymbol{H}_0)$;

(A4) $\boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in \boldsymbol{H}_0, \widehat{\beta}(u_0) \in L^1(\Omega)$ and $\widehat{\beta}(u_{0\Gamma}) \in L^1(\Gamma)$.

In particular, (A1) yields $0 \in \beta(0)$. These assumptions (A1)–(A4) are standard comparing with the literature [10, 11, 12, 13, 14, 22]. Additionally, in the present paper, β is modified to a singleton and is similar to a segment far from the origin in the following sense.

(A5) There exist constants c_0, $M_0 > 0$, and $c_0' \geq 0$ such that

$$\beta(r) = \begin{cases} c_0 r + c_0' & \text{if } r \geq M_0, \\ c_0 r - c_0' & \text{if } r \leq -M_0, \end{cases} \tag{8}$$

this implies $D(\beta) = \mathbb{R}$.

Remark 2.1. Condition (A5) is a technical yet essential assumption. If we can expect that the components of u and u_Γ of the solution are bounded below by $-M_0$ and above by M_0, then this modification (8) is negligible because β no longer takes these values. In the case that we want to treat a maximal monotone graph whose domain is a proper subset of $\mathbb{R}$ (e.g., $\partial I_{[0,1]}$), an example of modification is

$$\beta(r) = \begin{cases} r - c_0' & \text{if } r < 0, \\ [-c_0', 0] & \text{if } r = 0, \\ 0 & \text{if } 0 < r < 1, \\ [0, c_0' + 1] & \text{if } r = 1, \\ r + c_0' & \text{if } r > 1. \end{cases} \quad \widehat{\beta}(r) = \begin{cases} \dfrac{1}{2} r^2 - c_0' r & \text{if } r < 0, \\ 0 & \text{if } 0 \leq r \leq 1, \\ \dfrac{1}{2}(r-1)^2 + (c_0' + 1)(r - 1) & \text{if } r > 1. \end{cases}$$

This assumption is used to obtain the uniform boundedness of the total mass. (cf. [18, 22]).

We now give our main theorem.

Theorem 2.1. *Under assumptions* (A1)–(A5)*, there exists a unique weak solution to the problem* (P)*.*

The continuous dependence of the problem (P) is completely the same as that in [13, Theorem 2.2], and we therefore omit the proof of the uniqueness in this paper.

3 Approximate problem and uniform estimates

3.1 Abstract formulation

We apply the abstract theory of evolution equations [7] to prove the main theorem, following on from the essential idea of [11]. To do so, we define a convex functional $\varphi : \boldsymbol{V}_0^* \to [0, +\infty]$ by

$$\varphi(\boldsymbol{z}) := \begin{cases} \displaystyle\int_\Omega \widehat{\beta}(z)dx + \int_\Gamma \widehat{\beta}(z_\Gamma)d\Gamma & \text{if} \quad \boldsymbol{z} \in \boldsymbol{H}_0, \widehat{\beta}(z) \in L^1(\Omega), \widehat{\beta}(z_\Gamma) \in L^1(\Gamma), \\ +\infty & \text{otherwise.} \end{cases} \tag{9}$$

Note that the assumption of the growth condition (A2) plays an important role for the lower semicontinuity on $\boldsymbol{V}_0^*$ of φ.

Lemma 3.1. *The proper convex functional $\varphi : \boldsymbol{V}_0^* \to [0, +\infty]$ is lower semicontinuous on $\boldsymbol{V}_0^*$.*

Proof It is enough to show that the level set $[\varphi \le \lambda] := \{\boldsymbol{z} \in \boldsymbol{V}_0^* : \varphi(\boldsymbol{z}) \le \lambda\}$ is closed in $\boldsymbol{V}_0^*$ for all $\lambda \in \mathbb{R}$ (see, e.g., [6, p.70, Proposition 2.5]). We first take any $\{\boldsymbol{z}_n\}_{n\in\mathbb{N}} \subset [\varphi \le \lambda]$ with $\boldsymbol{z}_n \to \boldsymbol{z}$ in $\boldsymbol{H}_0$ as $n \to +\infty$. $\widehat{\beta}$ is now lower semicontinuous on $\mathbb{R}$. Therefore, by applying the Fatou lemma to subsequences $\{z_{n_k}\}_{k\in\mathbb{N}}$ and $\{z_{\Gamma,n_k}\}_{k\in\mathbb{N}}$, which respectively converge to z and z_Γ almost everywhere, we see that $\varphi(\boldsymbol{z}) \le \liminf_{k\to\infty} \varphi(\boldsymbol{z}_{n_k}) \le \lambda$; i.e., $[\varphi \le \lambda]$ is closed with respect to the topology of $\boldsymbol{H}_0$. Second, from the convexity of φ, we see that $[\varphi \le \lambda]$ is closed with respect to the weak topology of $\boldsymbol{H}_0$ (see, e.g., [6, p.72, Proposition 2.10]). We finally take any $\{\boldsymbol{z}_n\}_{n\in\mathbb{N}} \subset [\varphi \le \lambda]$ with $\boldsymbol{z}_n \to \boldsymbol{z}$ in $\boldsymbol{V}_0^*$ as $n \to +\infty$. In this case, from the assumption of growth condition (A2), we can take a bounded subsequence $\{\boldsymbol{z}_{n_k}\}_{k\in\mathbb{N}}$ in $\boldsymbol{H}_0$ such that $\boldsymbol{z}_{n_k} \to \boldsymbol{z}$ weakly in $\boldsymbol{H}_0$ as $k \to +\infty$. We thus conclude that $\boldsymbol{z} \in [\varphi \le \lambda]$. $\square$

We now define the projection $\boldsymbol{P} : \boldsymbol{H} \to \boldsymbol{H}_0$ by

$$\boldsymbol{P}\boldsymbol{z} := \boldsymbol{z} - m(\boldsymbol{z})\mathbf{1} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{H},$$

where $\mathbf{1} := (1, 1)$.

3.2 Approximate problem for (P)

We next consider an approximate problem to show the existence of a weak solution to (P). For each $\lambda > 0$, we define the Moreau–Yosida regularization $\widehat{\beta}_\lambda$ of $\widehat{\beta} : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{\beta}_\lambda(r) := \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\lambda}|r - s|^2 + \widehat{\beta}(s) \right\} = \frac{1}{2\lambda}\left|r - J_\lambda(r)\right|^2 + \widehat{\beta}\bigl(J_\lambda(r)\bigr) \tag{10}$$

for all $r \in \mathbb{R}$, where the resolvent operator $J_\lambda : \mathbb{R} \to \mathbb{R}$ of β is given by $J_\lambda(r) := (I + \lambda\beta)^{-1}r$. We also define

$$\varphi_\lambda(\boldsymbol{z}) := \begin{cases} \displaystyle\int_\Omega \widehat{\beta}_\lambda(z)dx + \int_\Gamma \widehat{\beta}_\lambda(z_\Gamma)d\Gamma & \text{if } \boldsymbol{z} \in \boldsymbol{H}_0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, for each $\lambda > 0$, φ_λ is a proper lower semicontinuous convex function on $\boldsymbol{V}_0^*$. We now give the representation of subdifferential operator $\partial_{\boldsymbol{V}_0^*}\varphi_\lambda$ by the next lemma.

Lemma 3.2. *For any $\boldsymbol{z} \in \boldsymbol{H}_0$, the following equivalence holds: $\boldsymbol{z}^* \in \partial_{\boldsymbol{V}_0^*}\varphi_\lambda(\boldsymbol{z})$ in $\boldsymbol{V}_0^*$ if and only if $\boldsymbol{\beta}_\lambda(\boldsymbol{z}) := (\beta_\lambda(z), \beta_\lambda(z_\Gamma)) \in \boldsymbol{V}$ and*

$$\boldsymbol{z}^* = \boldsymbol{F}\boldsymbol{P}\boldsymbol{\beta}_\lambda(\boldsymbol{z}) \quad \text{in } \boldsymbol{V}_0^*.$$

That is to say, $\partial_{\boldsymbol{V}_0^*}\varphi_\lambda(\boldsymbol{z})$ is a singleton.

Proof For each fixed $\boldsymbol{z} \in D(\varphi_\lambda) = \boldsymbol{H}_0$, we set $\boldsymbol{z}^* \in \partial_{\boldsymbol{V}_0^*}\varphi_\lambda(\boldsymbol{z}) \in \boldsymbol{V}_0^*$. We see from the definition of the subdifferential that

$$(\boldsymbol{z}^*, \tilde{\boldsymbol{z}} - \boldsymbol{z})_{\boldsymbol{V}_0^*} \le \varphi_\lambda(\tilde{\boldsymbol{z}}) - \varphi_\lambda(\boldsymbol{z}) \quad \text{for all } \tilde{\boldsymbol{z}} \in \boldsymbol{V}_0^*.$$

Now, for each $\delta \in (0,1]$ and $\bar{\boldsymbol{z}} \in \boldsymbol{H}_0$, taking $\tilde{\boldsymbol{z}} := \boldsymbol{z} + \delta\bar{\boldsymbol{z}} \in \boldsymbol{H}_0$ in the above, we have

$$(\boldsymbol{z}^*, \delta\bar{\boldsymbol{z}})_{\boldsymbol{V}_0^*} \le \int_\Omega \widehat{\beta}_\lambda(z + \delta\bar{z})dx - \int_\Omega \widehat{\beta}_\lambda(z)dx + \int_\Gamma \widehat{\beta}_\lambda(z_\Gamma + \delta\bar{z}_\Gamma)d\Gamma - \int_\Gamma \widehat{\beta}_\lambda(z_\Gamma)d\Gamma. \tag{11}$$

Now, according to the intermediate value theorem, there exist $\xi : \Omega \to \mathbb{R}$ between z and $z + \delta\bar{z}$ a.e. in Ω and $\xi_\Gamma : \Gamma \to \mathbb{R}$ between z_Γ and $z_\Gamma + \delta\bar{z}_\Gamma$ a.e. on Γ such that

$$\frac{\widehat{\beta}_\lambda(z + \delta\bar{z}) - \widehat{\beta}_\lambda(z)}{\delta} = \beta_\lambda(\xi)\bar{z} \quad \text{a.e. in } \Omega,$$
$$\frac{\widehat{\beta}_\lambda(z_\Gamma + \delta\bar{z}_\Gamma) - \widehat{\beta}_\lambda(z_\Gamma)}{\delta} = \beta_\lambda(\xi_\Gamma)\bar{z}_\Gamma \quad \text{a.e. on } \Gamma.$$

We thus deduce that

$$\begin{aligned}\left|\frac{\widehat{\beta}_\lambda(z + \delta\bar{z}) - \widehat{\beta}_\lambda(z)}{\delta}\right| &= \bigl|\beta_\lambda(\xi) - \beta_\lambda(0)\bigr||\bar{z}| \\ &\le \frac{1}{\lambda}|\xi - 0||\bar{z}| \\ &\le \frac{1}{\lambda}\bigl(|z| + \delta|\bar{z}|\bigr)|\bar{z}|\end{aligned}$$

a.e. in Ω, where the Lipschitz continuity of β_λ with the Lipschitz constant $1/\lambda$ is used. Now, letting δ tend to zero, we obtain $\xi \to z$ a.e. in Ω, $\beta_\lambda(\xi) \to \beta_\lambda(z)$ a.e. in Ω. From the Lebesgue dominated convergence theorem, it follows that

$$\lim_{\delta \to 0} \int_\Omega \frac{\widehat{\beta}_\lambda(z + \delta\bar{z}) - \widehat{\beta}_\lambda(z)}{\delta}dx = \int_\Omega \beta_\lambda(z)\bar{z}dx = \bigl(\beta_\lambda(z), \bar{z}\bigr)_H.$$

Similarly,

$$\lim_{n \to \infty} \int_\Gamma \frac{\widehat{\beta}_\lambda(z_\Gamma + \delta\bar{z}_\Gamma) - \widehat{\beta}_\lambda(z_\Gamma)}{\delta}d\Gamma = \bigl(\beta_\lambda(z_\Gamma), \bar{z}_\Gamma\bigr)_{H_\Gamma}.$$

Thus, through dividing by $\delta > 0$ in (11) and letting δ tend to zero, we infer that

$$\begin{aligned}(\boldsymbol{z}^*, \bar{\boldsymbol{z}})_{\boldsymbol{V}_0^*} &\le \big(\beta_\lambda(z), \bar{z}\big)_H + \big(\beta_\lambda(z_\Gamma), \bar{z}_\Gamma\big)_{H_\Gamma} \\ &= \big(\boldsymbol{\beta}_\lambda(\boldsymbol{z}), \bar{\boldsymbol{z}}\big)_{\boldsymbol{H}} \\ &= \big(\boldsymbol{P}\boldsymbol{\beta}_\lambda(\boldsymbol{z}), \bar{\boldsymbol{z}}\big)_{\boldsymbol{H}_0} \quad \text{for all } \bar{\boldsymbol{z}} \in \boldsymbol{H}_0.\end{aligned}$$

Next, taking $\tilde{\boldsymbol{z}} := \boldsymbol{z} - \delta\bar{\boldsymbol{z}}$, we see that $(\boldsymbol{z}^*, \bar{\boldsymbol{z}})_{\boldsymbol{V}_0^*} \ge (\boldsymbol{P}\boldsymbol{\beta}_\lambda(\boldsymbol{z}), \bar{\boldsymbol{z}})_{\boldsymbol{H}_0}$ for all $\bar{\boldsymbol{z}} \in \boldsymbol{H}_0$. That is to say, we have $(\boldsymbol{z}^*, \bar{\boldsymbol{z}})_{\boldsymbol{V}_0^*} = (\boldsymbol{P}\boldsymbol{\beta}_\lambda(\boldsymbol{z}), \bar{\boldsymbol{z}})_{\boldsymbol{H}_0}$ for all $\bar{\boldsymbol{z}} \in \boldsymbol{H}_0$. This implies that $\boldsymbol{F}^{-1}\boldsymbol{z}^* = \boldsymbol{P}\boldsymbol{\beta}_\lambda(\boldsymbol{z})$ in $\boldsymbol{H}_0$, that is, in $\boldsymbol{V}_0$ by comparison. We therefore get $\boldsymbol{\beta}_\lambda(\boldsymbol{z}) \in \boldsymbol{V}$ and $\boldsymbol{z}^* = \boldsymbol{F}\boldsymbol{P}\boldsymbol{\beta}_\lambda(\boldsymbol{z})$ in $\boldsymbol{V}_0^*$. Meanwhile, if $\boldsymbol{\beta}_\lambda(\boldsymbol{z}) \in \boldsymbol{V}$, then $\boldsymbol{P}\boldsymbol{\beta}_\lambda(\boldsymbol{z}) \in \boldsymbol{V}_0$ and

$$\begin{aligned}\big(\boldsymbol{F}\boldsymbol{P}\boldsymbol{\beta}_\lambda(\boldsymbol{z}), \tilde{\boldsymbol{z}} - \boldsymbol{z}\big)_{\boldsymbol{V}_0^*} &= \big\langle \tilde{\boldsymbol{z}} - \boldsymbol{z}, \boldsymbol{P}\boldsymbol{\beta}_\lambda(\boldsymbol{z})\big\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} \\ &= \big(\tilde{\boldsymbol{z}} - \boldsymbol{z}, \boldsymbol{\beta}_\lambda(\boldsymbol{z})\big)_{\boldsymbol{H}} \\ &\le \varphi_\lambda(\tilde{\boldsymbol{z}}) - \varphi_\lambda(\boldsymbol{z}) \quad \text{for all } \tilde{\boldsymbol{z}} \in \boldsymbol{H}_0, \end{aligned} \tag{12}$$

because $\tilde{\boldsymbol{z}} - \boldsymbol{z} \in \boldsymbol{H}_0$. If $\tilde{\boldsymbol{z}} \in \boldsymbol{V}_0^* \setminus \boldsymbol{H}_0$, then $\varphi_\lambda(\tilde{\boldsymbol{z}}) = +\infty$, and (12) thus holds for all $\tilde{\boldsymbol{z}} \in \boldsymbol{V}_0^*$. This gives us $\boldsymbol{z}^* \in \partial_{\boldsymbol{V}_0^*}\varphi_\lambda(\boldsymbol{z})$ in $\boldsymbol{V}_0^*$. □

For each $\lambda \in (0,1]$, applying the abstract theory of Brézis (see [7]), we see that for each $\boldsymbol{f}$ and $\boldsymbol{u}_0$ satisfying (A3) and (A4), there exists a unique function $\boldsymbol{u}_\lambda \in H^1(0,T;\boldsymbol{V}_0^*) \cap L^\infty(0,T;D(\varphi_\lambda))$ such that

$$\begin{cases} \boldsymbol{u}_\lambda'(t) + \partial_{\boldsymbol{V}_0^*}\varphi_\lambda\big(\boldsymbol{u}_\lambda(t)\big) = \boldsymbol{f}(t) \quad \text{in } \boldsymbol{V}_0^* \quad \text{for a.a. } t \in (0,T), \\ \boldsymbol{u}_\lambda(0) = \boldsymbol{u}_0 \quad \text{in } \boldsymbol{V}_0^*. \end{cases} \tag{13}$$

From Lemma 3.2, it follows that $\boldsymbol{\beta}_\lambda(\boldsymbol{u}_\lambda(t)) \in \boldsymbol{V}$ for a.a. $t \in (0,T)$ and

$$\begin{aligned}\boldsymbol{f}(t) - \boldsymbol{u}_\lambda'(t) &= \partial_{\boldsymbol{V}_0^*}\varphi_\lambda\big(\boldsymbol{u}_\lambda(t)\big) \\ &= \boldsymbol{F}\boldsymbol{P}\boldsymbol{\beta}_\lambda\big(\boldsymbol{u}_\lambda(t)\big) \quad \text{in } \boldsymbol{V}_0^*.\end{aligned}$$

This yields

$$\big\langle \boldsymbol{u}_\lambda'(t), \boldsymbol{z}\big\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} + a\big(\boldsymbol{\beta}_\lambda\big(\boldsymbol{u}_\lambda(t)\big), \boldsymbol{z}\big) = \big(\boldsymbol{f}(t), \boldsymbol{z}\big)_{\boldsymbol{H}_0} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}_0, \tag{14}$$

for a.a. $t \in (0,T)$.

3.3 Uniform estimates

In this subsection, we obtain the uniform estimates, independent of λ, to prove the suitable convergence.

Lemma 3.3. *There exists a positive constant M_1, independent of $\lambda \in (0,1]$, such that*

$$\frac{1}{2}\int_0^t \big|\boldsymbol{u}_\lambda'(s)\big|_{\boldsymbol{V}_0^*}^2 ds + \int_\Omega \widehat{\beta}_\lambda\big(u_\lambda(t)\big)dx + \int_\Gamma \widehat{\beta}_\lambda\big(u_{\Gamma,\lambda}(t)\big)d\Gamma \le M_1$$

for all $t \in [0,T]$.

Proof For a.a. $s \in (0,T)$, we have that

$$\big(\boldsymbol{u}_\lambda'(s), \partial_{\boldsymbol{V}_0^*}\varphi_\lambda\big(\boldsymbol{u}_\lambda(s)\big)\big)_{\boldsymbol{V}_0^*} = \frac{d}{ds}\varphi_\lambda\big(\boldsymbol{u}_\lambda(s)\big).$$

Hence, we deduce from (13) that

$$\begin{aligned} \big|\boldsymbol{u}_\lambda'(s)\big|_{\boldsymbol{V}_0^*}^2 &= \big(\boldsymbol{u}_\lambda'(s), \boldsymbol{f}(s) - \partial_{\boldsymbol{V}_0^*}\varphi_\lambda\big(\boldsymbol{u}_\lambda(s)\big)\big)_{\boldsymbol{V}_0^*} \\ &= \big\langle \boldsymbol{u}_\lambda'(s), \boldsymbol{F}^{-1}\boldsymbol{f}(s)\big\rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0} - \frac{d}{ds}\varphi_\lambda\big(\boldsymbol{u}_\lambda(s)\big). \end{aligned}$$

Now, integrating over $(0,t)$ with respect to s, we obtain

$$\int_0^t \big|\boldsymbol{u}_\lambda'(s)\big|_{\boldsymbol{V}_0^*}^2 ds + \varphi_\lambda\big(\boldsymbol{u}_\lambda(t)\big) \le \varphi(\boldsymbol{u}_0) + \int_0^t \big|\boldsymbol{u}_\lambda'(s)\big|_{\boldsymbol{V}_0^*}\big|\boldsymbol{F}^{-1}\boldsymbol{f}(s)\big|_{\boldsymbol{V}_0} ds$$

for all $t \in [0,T]$. Then, using the Young inequality and taking

$$M_1 := \big|\widehat{\beta}(u_0)\big|_{L^1(\Omega)} + \big|\widehat{\beta}(u_{0\Gamma})\big|_{L^1(\Gamma)} + \frac{1}{2}|\boldsymbol{f}|_{L^2(0,T;\boldsymbol{H}_0)}^2,$$

we get the conclusion. □

Lemma 3.4. *There exist a value $\bar{\lambda} \in (0,1]$ and a positive constant M_2 independent of $\lambda \in (0,\bar{\lambda}]$, such that*

$$\big|\boldsymbol{u}_\lambda(t)\big|_{\boldsymbol{H}_0}^2 \le M_2 \tag{15}$$

for all $t \in [0,T]$ and $\lambda \in (0,\bar{\lambda}]$.

Proof By virtue of (10) with (A2), we have that

$$\widehat{\beta}_\lambda\big(u_\lambda(t)\big) \ge \frac{1}{2\lambda}\big|u_\lambda(t) - J_\lambda\big(u_\lambda(t)\big)\big|^2 + c_1\big|J_\lambda\big(u_\lambda(t)\big)\big|^2 - c_2,$$

for all $t \in [0,T]$. We now set $\bar{\lambda} := \min\{1, 1/(2c_1)\}$. Then, for each $\lambda \in (0,\bar{\lambda}]$, we have $\lambda \le \bar{\lambda} \le 1/(2c_1)$; i.e., $1/(2\lambda) \ge c_1$. It follows from Lemma 3.3 that

$$\begin{aligned} M_1 &\ge \int_\Omega \widehat{\beta}_\lambda\big(u_\lambda(t)\big) dx \\ &\ge c_1 \int_\Omega \Big\{\big|u_\lambda(t) - J_\lambda\big(u_\lambda(t)\big)\big|^2 + \big|J_\lambda\big(u_\lambda(t)\big)\big|^2\Big\} dx - c_2|\Omega| \\ &\ge \frac{c_1}{2}\int_\Omega \big|u_\lambda(t)\big|^2 dx - c_2|\Omega|. \end{aligned}$$

This yields

$$\big|u_\lambda(t)\big|_H^2 \le \frac{2}{c_1}\big(M_1 + c_2|\Omega|\big) \quad \text{for all } t \in [0,T].$$

Similarly,

$$\big|u_{\Gamma,\lambda}(t)\big|_{H_\Gamma}^2 \le \frac{2}{c_1}\big(M_1 + c_2|\Gamma|\big) \quad \text{for all } t \in [0,T].$$

Thus, setting $M_2 := (2/c_1)(2M_1 + c_2(|\Omega| + |\Gamma|))$, we obtain (15). □

Lemma 3.5. *There exist positive constants M_3 and M_4, independent of $\lambda \in (0,1]$, such that*

$$\int_0^t \bigl|\boldsymbol{P}\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(s)\bigr)\bigr|_{\boldsymbol{V}_0}^2 ds \le M_3, \quad \bigl|m\bigl(\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(t)\bigr)\bigr)\bigr| \le M_4$$

for all $t \in [0,T]$.

Proof For all $t \in [0,T]$, from (13) with Lemma 3.3, we deduce that

$$\begin{aligned}\int_0^t \bigl|\partial_{\boldsymbol{V}_0^*}\varphi_\lambda\bigl(\boldsymbol{u}_\lambda(s)\bigr)\bigr|_{\boldsymbol{V}_0^*}^2 ds &= \int_0^t \bigl|\boldsymbol{f}(s) - \boldsymbol{u}'_\lambda(s)\bigr|_{\boldsymbol{V}_0^*}^2 ds \\ &\le 2\int_0^T \bigl|\boldsymbol{f}(s)\bigr|_{\boldsymbol{V}_0^*}^2 ds + 2\int_0^T \bigl|\boldsymbol{u}'_\lambda(s)\bigr|_{\boldsymbol{V}_0^*}^2 ds \\ &\le 2|\boldsymbol{f}|_{L^2(0,T;\boldsymbol{H}_0)}^2 + 4M_1.\end{aligned}$$

Now, by setting $M_3 := 2|\boldsymbol{f}|_{L^2(0,T;\boldsymbol{H}_0)}^2 + 4M_1$ we infer from Lemma 3.2 that

$$\begin{aligned}\int_0^t \bigl|\boldsymbol{P}\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(s)\bigr)\bigr|_{\boldsymbol{V}_0}^2 ds &= \int_0^t \bigl|\boldsymbol{F}\boldsymbol{P}\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(s)\bigr)\bigr|_{\boldsymbol{V}_0^*}^2 ds \\ &= \int_0^t \bigl|\partial_{\boldsymbol{V}_0^*}\varphi_\lambda\bigl(\boldsymbol{u}_\lambda(s)\bigr)\bigr|_{\boldsymbol{V}_0^*}^2 ds \\ &\le M_3\end{aligned}$$

for all $t \in [0,T]$. To prove the second estimate, we set

$$\begin{aligned}\Omega_1 = \Omega_1(u_\lambda) &:= \bigl\{x \in \Omega; \bigl|u_\lambda(x)\bigr| \le M_0\bigr\}, \quad \Omega_2 = \Omega_2(u_\lambda) := \bigl\{x \in \Omega; \bigl|u_\lambda(x)\bigr| > M_0\bigr\}, \\ \Gamma_1 = \Gamma_1(u_{\Gamma,\lambda}) &:= \bigl\{x \in \Gamma; \bigl|u_{\Gamma,\lambda}(x)\bigr| \le M_0\bigr\}, \quad \Gamma_2 = \Gamma_2(u_{\Gamma,\lambda}) := \bigl\{x \in \Gamma; \bigl|u_{\Gamma,\lambda}(x)\bigr| > M_0\bigr\},\end{aligned}$$

and positive constant $c^* := \max\{\beta(M_0), -\beta(-M_0)\}$. For all $t \in [0,T]$, we infer that

$$\begin{aligned}&\bigl|m\bigl(\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(t)\bigr)\bigr)\bigr| \\ &\quad = \frac{1}{|\Omega| + |\Gamma|}\left|\int_\Omega \beta_\lambda\bigl(u_\lambda(t)\bigr)dx + \int_\Gamma \beta_\lambda\bigl(u_{\Gamma,\lambda}(t)\bigr)d\Gamma\right| \\ &\quad \le \frac{1}{|\Omega| + |\Gamma|}\left\{c^*\bigl(|\Omega| + |\Gamma|\bigr) + c_0\left|\int_{\Omega_2} u_\lambda(t)dx + \int_{\Gamma_2} u_{\Gamma,\lambda}(t)d\Gamma\right| + c'_0\bigl(|\Omega| + |\Gamma|\bigr)\right\}.\end{aligned}$$

Now, from total mass conservation we have

$$\begin{aligned}\left|\int_{\Omega_2} u_\lambda(t)dx + \int_{\Gamma_2} u_{\Gamma,\lambda}(t)d\Gamma\right| &= \left|-\left(\int_{\Omega_1} u_\lambda(t)dx + \int_{\Gamma_1} u_{\Gamma,\lambda}(t)d\Gamma\right)\right| \\ &\le M_0\bigl(|\Omega| + |\Gamma|\bigr).\end{aligned}$$

We thus have

$$\bigl|m\bigl(\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(t)\bigr)\bigr)\bigr| \le c^* + c_0 M_0 + c'_0 =: M_4$$

for all $t \in [0,T]$. □

Lemma 3.6. *There exists a positive constant M_5, independent of $\lambda \in (0,1]$, such that*

$$\int_0^t \bigl|\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(s)\bigr)\bigr|_{\boldsymbol{V}}^2 ds \le M_5$$

for all $t \in [0,T]$.

Proof We consider that $a(\boldsymbol{z},\boldsymbol{z}) = |\boldsymbol{P}\boldsymbol{z}|_{\boldsymbol{V}_0}^2$ for all $\boldsymbol{z} \in \boldsymbol{V}$. From (4) with Lemma 3.5, we have

$$\begin{aligned}\int_0^t \bigl|\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(s)\bigr)\bigr|_{\boldsymbol{V}}^2 ds &\le c_{\mathrm{P}} \left\{ \int_0^t \bigl|\boldsymbol{P}\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(s)\bigr)\bigr|_{\boldsymbol{V}_0}^2 ds + \int_0^t \bigl|m\bigl(\boldsymbol{\beta}_\lambda\bigl(\boldsymbol{u}_\lambda(s)\bigr)\bigr)\bigr|^2 ds \right\} \\ &\le c_{\mathrm{P}}(M_3 + M_4^2 T) =: M_5\end{aligned}$$

for all $t \in [0,T]$ □

3.4 Passage to the limit

In this subsection, we obtain the weak solution of (P) from the passage to the limit for the approximate problem.

Proof of Theorem 2.1. On the basis of the previous estimates in Lemmas 3.3, 3.4, and 3.6, there exist a subsequence $\{\lambda_k\}_{k\in\mathbb{N}}$ with $\lambda_k \to 0$ as $k \to +\infty$ and limit functions $\boldsymbol{u} \in H^1(0,T;\boldsymbol{V}_0^*) \cap L^\infty(0,T;\boldsymbol{H}_0)$ and $\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{V})$ such that

$$\boldsymbol{u}_{\lambda_k} \to \boldsymbol{u} \quad \text{weakly star in } H^1(0,T;\boldsymbol{V}_0^*) \cap L^\infty(0,T;\boldsymbol{H}_0), \tag{16}$$

$$\boldsymbol{\beta}_{\lambda_k}(\boldsymbol{u}_{\lambda_k}) \to \boldsymbol{\xi} \quad \text{weakly in } L^2(0,T;\boldsymbol{V}) \tag{17}$$

as $k \to +\infty$. Now, from (16) and the Ascoli–Arzela theorem (see, e.g., [24]), we see that there exists a subsequence (not relabeled) such that

$$\boldsymbol{u}_{\lambda_k} \to \boldsymbol{u} \quad \text{in } C\bigl([0,T];\boldsymbol{V}_0^*\bigr) \tag{18}$$

as $k \to +\infty$; i.e., $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in $\boldsymbol{V}_0^*$. Now, from (14) and by letting $k \to +\infty$, we obtain

$$\bigl\langle \boldsymbol{u}'(t), \boldsymbol{z} \bigr\rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0} + a\bigl(\boldsymbol{\xi}(t), \boldsymbol{z}\bigr) = \bigl(\boldsymbol{f}(t), \boldsymbol{z}\bigr)_{\boldsymbol{H}_0} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}_0, \tag{19}$$

for a.a. $t \in (0,T)$. To prove the main theorem, we show that $\xi(t) \in \beta(u(t))$ a.e. in Ω and $\xi_\Gamma(t) \in \beta(u_\Gamma(t))$ a.e. on Γ. We now define two operators $\boldsymbol{B}, \boldsymbol{B}_{\lambda_k} : L^2(0,T;\boldsymbol{H}) \to L^2(0,T;\boldsymbol{H})$ by

$$\begin{aligned}\boldsymbol{B}\boldsymbol{\eta} &:= \bigl\{ \boldsymbol{\xi} := (\xi, \xi_\Gamma) \in L^2(0,T;\boldsymbol{H}) \ : \ \xi \in \beta(\eta) \text{ a.e. in } Q,\ \xi_\Gamma \in \beta(\eta_\Gamma) \text{ a.e. on } \Sigma \bigr\}, \\ \boldsymbol{B}_{\lambda_k}\boldsymbol{\eta} &:= \bigl(\beta_{\lambda_k}(\eta), \beta_{\lambda_k}(\eta_\Gamma)\bigr) \quad \text{for all } \boldsymbol{\eta} \in L^2(0,T;\boldsymbol{H}).\end{aligned}$$

Then, from the maximal monotonicity of β, we see that $\boldsymbol{B}$ and $\boldsymbol{B}_{\lambda_k}$ are maximal monotone operators on $L^2(0,T;\boldsymbol{H})$. Now, from (16) and (17) we already have

$$\boldsymbol{u}_{\lambda_k} \to \boldsymbol{u} \quad \text{weakly star in } L^\infty(0,T;\boldsymbol{H}),$$
$$\boldsymbol{B}_{\lambda_k}\boldsymbol{u}_{\lambda_k} = \boldsymbol{\beta}_{\lambda_k}(\boldsymbol{u}_{\lambda_k}) \to \boldsymbol{\xi} \quad \text{weakly in } L^2(0,T;\boldsymbol{V})$$

as $k \to +\infty$. Moreover, we deduce from (18) that

$$\begin{aligned}\int_0^T \big(\boldsymbol{B}_{\lambda_k}\boldsymbol{u}_{\lambda_k}(s), \boldsymbol{u}_{\lambda_k}(s)\big)_{\boldsymbol{H}} ds &= \int_0^T \big\langle \boldsymbol{u}_{\lambda_k}(s), \boldsymbol{\beta}_{\lambda_k}\big(\boldsymbol{u}_{\lambda_k}(s)\big)\big\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} ds \\ &\to \int_0^T \big\langle \boldsymbol{u}(s), \boldsymbol{\xi}(s)\big\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} ds \\ &= \int_0^T \big(\boldsymbol{\xi}(s), \boldsymbol{u}(s)\big)_{\boldsymbol{H}} ds\end{aligned}$$

as $k \to +\infty$. Therefore, by applying [19, p.260, Lemma 7.1], we deduce that $\boldsymbol{\xi} \in \boldsymbol{B}\boldsymbol{u}$ in $L^2(0,T;\boldsymbol{H})$. We finally check (5) and (7). Firstly, it follows from [9, Remark 2] that the function $\boldsymbol{u}' \in L^2(0,T;\boldsymbol{V}_0^*)$ can be extended to $L^2(0,T;\boldsymbol{V}^*)$ by setting $\langle \boldsymbol{u}'(t), \boldsymbol{1}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = 0$; i.e.,

$$\big\langle \boldsymbol{u}'(t), \boldsymbol{z}\big\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} := \big\langle \boldsymbol{u}'(t), \boldsymbol{P}\boldsymbol{z}\big\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}.$$

We next see that $\boldsymbol{V}$ is a subspace of $V \times V_\Gamma$, and therefore, from the Hahn–Banach extension theorem, we can also extend $\boldsymbol{u}'(t)$ to $(V \times V_\Gamma)^*$; i.e., $u \in H^1(0,T;V^*) \cap L^\infty(0,T;H)$ and $u_\Gamma \in H^1(0,T;V_\Gamma^*) \cap L^\infty(0,T;H_\Gamma)$ with $\boldsymbol{u} = (u, u_\Gamma) \in H^1(0,T;\boldsymbol{V}^*) \cap L^\infty(0,T;\boldsymbol{H})$. Therefore, from (19) we obtain (7). $\square$

4 Improvement

In this section, we consider an improvement to the main theorem.

4.1 Improvement of the initial condition to a nonzero mean value

The essential idea of the proof of Theorem 2.1 is the setting of suitable function spaces, or more precisely, the mean-zero function space $\boldsymbol{H}_0$. This idea comes from the treatment of the Cahn–Hilliard system (see, e.g., [9, 20, 21]). Considering this idea, we can improve our assumption for the initial data to the general $\boldsymbol{H}$-function. In this subsection, we assume that

(A4)$'$ $\boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in \boldsymbol{H}, \widehat{\beta}(u_0) \in L^1(\Omega)$ and $\widehat{\beta}(u_{0\Gamma}) \in L^1(\Gamma)$.

We can then improve our main theorem as follows.

Theorem 4.1. *Under assumptions* (A1)–(A3) *and* (A4)$'$*, there exists a unique weak solution to the problem* (P).

Proof. Let us consider a Cauchy problem of evolution equations:

$$\begin{cases} \boldsymbol{v}'(t) + \partial_{\boldsymbol{V}_0^*}\varphi_{m_0}\big(\boldsymbol{v}(t)\big) \ni \boldsymbol{f}(t) \quad \text{in } \boldsymbol{V}_0^* \quad \text{for a.a. } t \in (0,T), \\ \boldsymbol{v}(0) = P\boldsymbol{u}_0 \quad \text{in } \boldsymbol{H}_0, \end{cases} \tag{20}$$

where the convex functional $\varphi_{m_0} : \boldsymbol{V}_0^* \to [0,+\infty]$ is defined by

$$\varphi_{m_0}(\boldsymbol{z}) := \begin{cases} \displaystyle\int_\Omega \widehat{\beta}(z+m_0)dx + \int_\Gamma \widehat{\beta}(z_\Gamma + m_0)d\Gamma \\ \quad \text{if} \quad \boldsymbol{z} \in \boldsymbol{H}_0, \widehat{\beta}(z+m_0) \in L^1(\Omega), \widehat{\beta}(z_\Gamma + m_0) \in L^1(\Gamma), \\ +\infty \quad \text{otherwise}. \end{cases}$$

Then, this is also proper lower semicontinuous and convex on $\boldsymbol{V}_0^*$. Therefore, in the same way as for Theorem 2.1, we see that there exists a unique weak solution $(v, v_\Gamma, \xi, \xi_\Gamma)$ satisfying $\boldsymbol{v} \in H^1(0,T;\boldsymbol{V}_0^*) \cap L^\infty(0,T;\boldsymbol{H}_0)$ and $\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{V})$ with $\xi \in \beta(v+m_0)$ a.e. in Q and $\xi_\Gamma \in \beta(v_\Gamma + m_0)$ a.e. on Σ such that (20) holds; i.e., $\boldsymbol{v}$ satisfies

$$\begin{aligned} \big\langle v'(t), z \big\rangle_{V^*,V} + \big\langle v_\Gamma'(t), z_\Gamma \big\rangle_{V_\Gamma^*,V_\Gamma} + \int_\Omega \nabla\xi(t)\cdot\nabla z dx + \int_\Gamma \nabla_\Gamma \xi_\Gamma(t)\cdot\nabla_\Gamma z_\Gamma d\Gamma \\ = \int_\Omega f(t) z dx + \int_\Gamma f_\Gamma(t) z_\Gamma d\Gamma \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}, \end{aligned}$$

for a.a. $t \in (0,T)$. We now set $\boldsymbol{u} := \boldsymbol{v} + m_0\mathbf{1}$, and have $\boldsymbol{u}(0) = \boldsymbol{v}(0) + m_0\mathbf{1} = \boldsymbol{u}_0$ in $\boldsymbol{H}$ and $\boldsymbol{u}' = \boldsymbol{v}'$. Thus, $(u, u_\Gamma, \xi, \xi_\Gamma)$ is our weak solution to problem (P). $\square$

4.2 Nonlinear diffusions of different β and β_Γ

In this subsection, we treat the different maximal monotone graphs β and β_Γ in Ω and on Γ, respectively. This implies that we can consider different nonlinearities of the diffusion in the bulk and on the boundary. We assume that

(A1)* $\beta, \beta_\Gamma : \mathbb{R} \to 2^{\mathbb{R}}$ are maximal monotone graphs, which are the subdifferentials $\beta = \partial_{\mathbb{R}}\widehat{\beta}$ and $\beta_\Gamma = \partial_{\mathbb{R}}\widehat{\beta}_\Gamma$ of some proper lower semicontinuous convex functions $\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to [0,+\infty]$ satisfying $\widehat{\beta}(0) = 0$ and $\widehat{\beta}_\Gamma(0) = 0$;

(A2)* there exist positive constants c_1, c_2, c_3, and c_4 such that $\widehat{\beta}(r) \geq c_1 r^2 - c_2$ and $\widehat{\beta}_\Gamma(r) \geq c_3 r^2 - c_4$ for all $r \in \mathbb{R}$;

(A4)* $\boldsymbol{u}_0 := (u_0, u_{0\Gamma}) \in \boldsymbol{H}_0, \widehat{\beta}(u_0) \in L^1(\Omega)$ and $\widehat{\beta}_\Gamma(u_{0\Gamma}) \in L^1(\Gamma)$;

(A5)* there exist constants c_0, $M_0 > 0$, and $c_0', c_0'' \geq 0$ such that

$$\beta(r) = \begin{cases} c_0 r + c_0' & \text{if } r \geq M_0, \\ c_0 r - c_0' & \text{if } r \leq -M_0, \end{cases} \qquad \beta_\Gamma(r) = \begin{cases} c_0 r + c_0'' & \text{if } r \geq M_0, \\ c_0 r - c_0'' & \text{if } r \leq -M_0. \end{cases}$$

We then obtain the same result.

Theorem 4.2. *Under assumptions* (A1)*, (A2)*, (A3), (A4)*, *and* (A5)* *there exists a unique weak solution to the problem* (P) *with* (6) *replaced with*

$$\xi_\Gamma \in \beta_\Gamma(u_\Gamma), \quad \xi|_\Gamma = \xi_\Gamma \quad a.e.\ on\ \Sigma. \tag{21}$$

Proof. We define a convex functional

$$\varphi(\boldsymbol{z}) := \begin{cases} \displaystyle\int_\Omega \widehat{\beta}(z)dx + \int_\Gamma \widehat{\beta}_\Gamma(z_\Gamma)d\Gamma & \text{if} \quad \boldsymbol{z} \in \boldsymbol{H}_0, \widehat{\beta}(z) \in L^1(\Omega), \widehat{\beta}_\Gamma(z_\Gamma) \in L^1(\Gamma), \\ +\infty & \text{otherwise.} \end{cases}$$

(cf., (9)). From growth conditions in (A2)*, we obtain coercivities of β and β_Γ. We thus obtain surjectivities of β and β_Γ; i.e., $R(\beta) = R(\beta_\Gamma) = \mathbb{R}$. Indeed, to obtain the trace condition in (21), we can use the surjectivities of β and β_Γ; otherwise, the trace condition makes no sense if $R(\beta) \cap R(\beta_\Gamma) = \emptyset$. The entire proof of Theorem 4.2 is the same as that of Theorem 2.1, but with $\boldsymbol{\beta}$ replaced by (β, β_Γ). □

Remark 4.1. The characterization of the degenerate parabolic equation as the asymptotic limit of Cahn–Hilliard systems has recently been discussed [10, 13, 14]. In the case of a dynamic boundary condition, for example, the existence result [9] was used at the level of approximation. Comparing the proofs of Theorem 4.2 and [13, Theorem 2.1], we readily see that (A5) is too restrictive. Moreover, the growth condition in (A2) has been improved [14, Theorem 2.1] (see also the advantage of the Cahn–Hilliard approach [10, Section 6]). However, to treat different nonlinearities of β and β_Γ, we will assume the condition (see [9, p.419, (A6)]):

There exist positive constants ρ_1, ρ_2 such that

$$\left|\beta^\circ(r)\right| \le \rho_1 \left|\beta_\Gamma^\circ(r)\right| + \rho_2 \quad \text{for all } r \in \mathbb{R}. \tag{22}$$

Here, the minimal section β° of β is defined by $\beta^\circ(r) := \{r^* \in \beta(r) : |r^*| = \min_{s\in\beta(r)} |s|\}$ and the same definition applies to β_Γ°. Indeed, the dominated inequality (22) is the same as [8, 9], which gives us the same inequality at the level of the Yosida approximation [8, p.19, Lemma 4.4]. This dominated inequality provides suitable uniform estimates related to $\beta(u)$ and $\beta_\Gamma(u_\Gamma)$; more precisely, we can treat the estimate of $\beta_\Gamma(u_\Gamma)$ against $\beta(u_\Gamma)$ on the boundary (see [9, Lemmas 4.3 and 4.4]). However, if we apply the main theorem of the present paper, we do not need to assume such a dominated inequality because we do not treat directly the estimate of $\beta_\Gamma(u_\Gamma)$ against $\beta(u_\Gamma)$. This is one advantage of the abstract approach from abstract evolution equations to degenerate parabolic equations.

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