Advances in Mathematical Sciences and Applications
Vol. 27, No. 1 (2018), pp. 263–275

GAKKOTOSHO
TOKYO JAPAN

GENERALIZED INTEGRAL TYPE CONTRACTION AND COMMON FIXED POINT THEOREMS USING AN AUXILIARY FUNCTION

Vishal Gupta
Department of Mathematics,
Maharishi Markandeshwar(Deemed to be University)
Mullana, Haryana, India.
(E-mail: vishal.gmn@gmail.com)

Naveen Mani *
Department of Mathematics,
Sandip University,
Nashik, Maharashtra, India.
(E-mail: naveenmani81@gmail.com)

and

Arslan H. Ansari
Department of Mathematics,
Karaj Branch, Islamic Azad University, Karaj, Iran.
(E-mail: analsisamirmath2@gmail.com)

Abstract. In this paper, with the help of new auxiliary function and without assuming the continuity and commutativity property of self maps, some common fixed theorems for generalized integral type contraction in complete metric spaces are proved. As application of our results, some corollaries are given. Also, some examples are given to justify the importance and existence of our findings in current research.

*Corresponding author
Communicated by Editors; Received August 18, 2017.
2010 AMS Subject Classification: 47H10, 54H25.
Keywords: Common fixed point, continuity, commutativity, auxiliary function, integral contraction.

1 Introduction

Banach [10], in 1922, full fledged an extraordinary fixed point result, which is one of the most significant finding of analysis. This invention is one of the commonly applied and useful results in the history of fixed point theory. Banach gave a simple structure of the proof which requires the completeness of metric spaces, a self map and a contraction to get fixed point.

Jungck [4] introduced the concept of commuting map as an inventive tool to obtain common fixed points of mappings. It was the turning point in the fixed point arena to get unique fixed point for pair of maps.

Branciari [1] in 2002, led down one of the fruitful generalization of Banach contraction mappings and proved a unique fixed point result satisfying integral type contractive condition. Many authors (see [3, 5, 6, 8, 9, 11, 12, 13, 14, 15]) have extended and generalized this result with different approach in different directions.

Ansari [2] in 2014-15, defined the notion of C -class function as a generalization of Banach contraction principle.

Definition 1.1. [2] We say $\phi : [0,+\infty) \to [0,+\infty)$ ultra distance function, if it is continuous and $\phi(0) \geq 0$, and $\phi(t) > 0, t > 0$.

Remark 1.2. We let Φ_u denote the class of the ultra distance functions.

Definition 1.3. [2] A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called *C-class* function if it is continuous and for all $r, t \in [0,\infty)$

1. $F(r,t) \leq r$;

2. $F(r,t) = r$ implies that either $r = 0$ or $t = 0$.

For brevity, we denote $\mathcal{C}$ as the family of C class functions.
It is also clear that, $F(0,0) = 0$. Some examples of C-class functions are given in [2].

Let us denote $\Psi_1 = \{\gamma | \gamma : [0,\infty) \to [0,\infty)\}$, which is Lebesgue-integrable, summable on each compact subset of R^+, and are such that for each $\epsilon > 0$, $\int_0^\epsilon \gamma(t)dt > 0$.

Mocanu and Popa [7] gave the following lemma.

Lemma 1.1. [7] *Let $\left(a_p\right)_{p\in N}$ be a non-negative sequence with $\lim_{p\to\infty} a_p = k$. If $\gamma \in \Psi_1$, then*

$$\lim_{p\to\infty} \int_0^{a_p} \gamma(t)dt = 0 \quad \textit{if and only if} \quad \lim_{p\to\infty} a_p = 0.$$

In next section 2, the main objective is to derive some results to get common fixed point for two self-maps satisfying generalized integral type contractive condition with C-class function. Here, the results are proved without using the continuity and commutative property of maps S and T. In section 3, as application, various consequence results of our findings are discussed. After that, In section 4, examples and remark are given to illustrate the fact that our findings are new, useable and applicable for future research.

2 Main results

Theorem 2.1. *Let S and T be two self-maps of X, and d be a metric on X such that (X,d) is complete. If for each $u,v\in X$*

$$\int_0^{d(Su,Tv)}\gamma(t)dt \leq F\Bigg(\frac{1}{\alpha+\beta}\left[\alpha\int_0^{N(u,v)}\gamma(t)dt+\beta\int_0^{d(u,v)}\gamma(t)dt\right], \phi\Bigg(\frac{1}{\alpha+\beta}\left[\alpha\int_0^{N(u,v)}\gamma(t)dt+\beta\int_0^{d(u,v)}\gamma(t)dt\right]\Bigg)\Bigg), \tag{1}$$

where $\alpha,\beta>0, F\in\mathcal{C},\phi\in\Phi_u,\ \gamma\in\Psi_1$ and

$$N(u,v)=\frac{d(v,Tv)\left[1+d(u,Su)\right]}{\left[1+d(u,v)\right]}, \tag{2}$$

then S and T have a unique common fixed point.

Proof. Choose $u_0\in X$ such that $Su_0=u_1$ and $Tu_1=u_2$. Continuing like this, we can construct sequences $\{u_p\}$ and $\{v_p\}$ in X, such as

$$v_{2p}=u_{2p+1}=Su_{2p} \ \text{ and } \ v_{2p+1}=u_{2p+2}=Tu_{2p+1}, \quad \text{where } \ p=0,1,2,\cdots. \tag{3}$$

We split the proof in several steps:
Step-1: Claim that sequence $\{u_p\}$ is Cauchy sequence.
Step-1.1: First suppose that $v_{2p-1}=v_{2p}$, for some $p\in N$.
Then from (1)

$$\begin{aligned}\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt &= \int_0^{d(Su_{2p},Tu_{2p+1})}\gamma(t)dt\\ &\leq F\Bigg(\frac{1}{\alpha+\beta}\left[\alpha\int_0^{N(u_{2p},u_{2p+1})}\gamma(t)dt+\beta\int_0^{d(u_{2p},u_{2p+1})}\gamma(t)dt\right],\\ &\phi\Bigg(\frac{1}{\alpha+\beta}\left[\alpha\int_0^{N(u_{2p},u_{2p+1})}\gamma(t)dt+\beta\int_0^{d(u_{2p},u_{2p+1})}\gamma(t)dt\right]\Bigg)\Bigg).\end{aligned} \tag{4}$$

From (2),

$$N(u_{2p},u_{2p+1})=\frac{d(u_{2p+1},Tu_{2p+1})\left[1+d(u_{2p},Su_{2p})\right]}{\left[1+d(u_{2p},u_{2p+1})\right]}=d(u_{2p+1},u_{2p+2}).$$

Hence from (4),

$$\begin{aligned}\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt \le F\Bigg(&\frac{1}{\alpha+\beta}\Bigg[\alpha\int_0^{d(u_{2p+1},u_{2p+2})}\gamma(t)dt+\beta\int_0^{d(u_{2p},u_{2p+1})}\gamma(t)dt\Bigg],\\ &\phi\Bigg(\frac{1}{\alpha+\beta}\Bigg[\alpha\int_0^{d(u_{2p+1},u_{2p+2})}\gamma(t)dt+\beta\int_0^{d(u_{2p},u_{2p+1})}\gamma(t)dt\Bigg]\Bigg)\Bigg)\\ &= F\Bigg(\frac{1}{\alpha+\beta}\Bigg[\alpha\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt+\beta\int_0^{d(v_{2p-1},v_{2p})}\gamma(t)dt\Bigg],\\ &\phi\Bigg(\frac{1}{\alpha+\beta}\Bigg[\alpha\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt+\beta\int_0^{d(v_{2p-1},v_{2p})}\gamma(t)dt\Bigg]\Bigg)\Bigg). \qquad (5)\end{aligned}$$

By the definition of C -class function, we get

$$\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt \le \frac{1}{\alpha+\beta}\Bigg[\alpha\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt+\beta\int_0^{d(v_{2p-1},v_{2p})}\gamma(t)dt\Bigg], \qquad (6)$$

Since $v_{2p-1}=v_{2p}$, thus we get

$$\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt \le \frac{\alpha}{\alpha+\beta}\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt.$$

This is possible only if $v_{2p}=v_{2p+1}$. Thus for all $q\ge 2p$, $v_q=v_{2p-1}$, and hence we obtain $u_q=u_{2p}$. Therefore, $\{u_p\}$ is a Cauchy sequence.

Step-1.2: Secondly assume that $v_p\ne v_{p+1}$ for all integers p.

From (6), we obatin

$$\left(1-\frac{\alpha}{\alpha+\beta}\right)\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt \le \frac{\beta}{\alpha+\beta}\int_0^{d(v_{2p-1},v_{2p})}\gamma(t)dt,$$

implies that

$$\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt \le \int_0^{d(v_{2p-1},v_{2p})}\gamma(t)dt,$$

similarly,

$$\int_0^{d(v_{2p-1},v_{2p})}\gamma(t)dt \le \int_0^{d(v_{2p-2},v_{2p-1})}\gamma(t)dt,$$

Thus, we get a sequence $\left\{\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt\right\}$ of numbers, which is monotone, decreasing and lower bounded. Therefore, there exists a $r\ge 0$ such that

$$\lim_{p\to\infty}\int_0^{d(v_{2p},v_{2p+1})}\gamma(t)dt = r. \qquad (7)$$

On taking $\lim_{p\to\infty}$ in (5), we get

$$r \leq F(r, \phi(r)),$$

this implies using Definition 1.1 and Definition 1.3, either $r = 0$ or $\phi(r) = 0$. Hence

$$\lim_{p\to\infty} \int_0^{d(v_{2p}, v_{2p+1})} \gamma(t)dt = 0.$$

On using Lemma 1.1, we get

$$\lim_{p\to\infty} d(v_{2p}, v_{2p+1}) = 0.$$

Consequently,

$$\lim_{p\to\infty} d(u_{2p+1}, u_{2p+2}) = 0, \quad \forall \ \ p = 0, 1, 2, \cdots . \tag{8}$$

On contradictory, assume that $\{u_{2p}\}$ is not a Cauchy sequence. Then for an $\epsilon > 0$, we can find two sub-sequences of positive integers q_i and p_i, where $p_i > q_i > i$ for all $i > 0$ such that

$$d(u_{2q_i}, u_{2p_i}) \geq \epsilon \quad \text{and} \quad d(u_{2q_i}, u_{2p_{i-2}}) < \epsilon. \tag{9}$$

By using (9), we get

$$\epsilon \leq d(u_{2q_i}, u_{2p_i}) \leq d(u_{2q_i}, u_{2p_{i-2}}) + d(u_{2p_{i-2}}, u_{2p_{i-1}}) + d(u_{2p_{i-1}}, u_{2p_i}).$$

On taking $\lim_{i\to\infty}$ in above inequality, we get

$$\lim_{i\to\infty} d(u_{2q_i}, u_{2p_i}) = \epsilon. \tag{10}$$

Consider,

$$d(u_{2p_i}, u_{2q_{i-1}}) \leq d(u_{2p_i}, u_{2q_i}) + d(u_{2q_i}, u_{2q_{i-1}}),$$

Again on taking $\lim_{i\to\infty}$ in above inequality, we get

$$\lim_{i\to\infty} d(u_{2p_i}, u_{2q_{i-1}}) = \epsilon.$$

Similarly, we have

$$\lim_{i\to\infty} d(u_{2p_i}, u_{2q_{i+1}}) = \epsilon. \tag{11}$$

Consider,

$$d(u_{2p_i}, u_{2q_i}) \leq d(u_{2p_i}, u_{2p_{i+1}}) + d(u_{2p_{i+1}}, u_{2q_i}) = d(u_{2p_i}, u_{2p_{i+1}}) + d(Su_{2p_i}, Tu_{2q_{i+1}}).$$

taking $\lim_{i\to\infty}$, we obtain

$$\epsilon \leq \lim_{i\to\infty} d(Su_{2p_i}, Tu_{2q_{i+1}}).$$

As γ is a Lebesgue-integrable function, therefore

$$\int_0^{\epsilon} \gamma(t)dt \leq \lim_{i\to\infty} \int_0^{d(Su_{2p_i}, Tu_{2q_{i+1}})} \gamma(t)dt. \tag{12}$$

Also, from (1)

$$\int_0^{d(Su_{2p_i}, Tu_{2q_{i+1}})} \gamma(t)dt \leq F\Bigg(\frac{1}{\alpha+\beta}\Big[\alpha \int_0^{N(u_{2p_i}, u_{2q_{i+1}})} \gamma(t)dt + \beta \int_0^{d(u_{2p_i}, u_{2q_{i+1}})} \gamma(t)dt\Big], \phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \int_0^{N(u_{2p_i}, u_{2q_{i+1}})} \gamma(t)dt + \beta \int_0^{d(u_{2p_i}, u_{2q_{i+1}})} \gamma(t)dt\Big]\Big)\Bigg) \tag{13}$$

From (2) and using (8), we have

$$\begin{aligned} \lim_{i\to\infty} N(u_{2p_i}, u_{2q_{i+1}}) &= \lim_{i\to\infty} \frac{d\left(u_{2q_{i+1}}, Tu_{2q_{i+1}}\right)[1 + d(u_{2p_i}, Su_{2p_i})]}{[1 + d(u_{2p_i}, u_{2q_{i+1}})]} \\ &= \lim_{i\to\infty} \frac{d\left(u_{2q_{i+1}}, u_{2q_{i+2}}\right)[1 + d(u_{2p_i}, u_{2p_{i+1}})]}{[1 + d(u_{2p_i}, u_{2q_{i+1}})]} = 0. \end{aligned} \tag{14}$$

Taking limit as $i \to \infty$ in (13) and using (14), we get

$$\lim_{i\to\infty} \int_0^{d(Su_{2p_i}, Tu_{2q_{i+1}})} \gamma(t)dt \leq F\Bigg(\int_0^{\epsilon} \gamma(t)dt, \phi\Big(\int_0^{\epsilon} \gamma(t)dt\Big)\Bigg).$$

Hence from (12),

$$\int_0^{\epsilon} \gamma(t)dt \leq F\Bigg(\int_0^{\epsilon} \gamma(t)dt, \phi\Big(\int_0^{\epsilon} \gamma(t)dt\Big)\Bigg).$$

Using Definition 1.1 and Definition 1.3, this is possible only if $\int_0^{\epsilon} \gamma(t)dt = 0$ or $\phi(\int_0^{\epsilon} \gamma(t)dt) = 0$ implies that $\int_0^{\epsilon} \gamma(t)dt = 0$. This is a contradiction. Hence $\{u_{2p}\}$ is a Cauchy sequence. Therefore, there exists $a \in X$ such that

$$\lim_{p\to\infty} u_{2p} = a. \tag{15}$$

From (3), we obtain

$$\lim_{p\to\infty} Su_{2p} = a; \qquad \lim_{p\to\infty} Tu_{2p+1} = a. \tag{16}$$

Step- 2: Claim that a is a common fixed point of S and T.
From (1),

$$\int_0^{d(Su_{2p}, Ta)} \gamma(t)dt \leq F\Bigg(\frac{1}{\alpha+\beta}\Big[\alpha \int_0^{N(u_{2p}, a)} \gamma(t)dt + \beta \int_0^{d(u_{2p}, a)} \gamma(t)dt\Big], \phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha \int_0^{N(u_{2p}, a)} \gamma(t)dt + \beta \int_0^{d(u_{2p}, a)} \gamma(t)dt\Big]\Big)\Bigg), \tag{17}$$

where,

$$N(u_{2p},a)=\frac{d(a,Ta)\,[1+d(u_{2p},Su_{2p})]}{[1+d(u_{2p},a)]}=d(a,Ta). \tag{18}$$

On taking $\lim_{p\to\infty}$ in (17), and using (18), we get

$$\int_0^{d(a,Ta)}\gamma(t)dt\le F\Bigg(\frac{\alpha}{\alpha+\beta}\int_0^{d(a,Ta)}\gamma(t)dt,\phi\Big(\frac{\alpha}{\alpha+\beta}\int_0^{d(a,Ta)}\gamma(t)dt\Big)\Bigg),$$

this is possible only (using Definition 1.1 and Definition 1.3) if $\int_0^{d(a,Ta)}\gamma(t)dt=0$. Hence by using Lemma 1.1, we get $d(a,Ta)=0$. Thus $Ta=a$.

Next claim that every fixed point of T is also a fixed point of S. i.e. $Sa=a$.

Again using (1),

$$\begin{aligned}\int_0^{d(Sa,u_{2p+1})}\gamma(t)dt&=\int_0^{d(Sa,Tu_{2p})}\gamma(t)dt\\ &\le F\Bigg(\frac{1}{\alpha+\beta}\Big[\alpha\int_0^{N(a,u_{2p})}\gamma(t)dt+\beta\int_0^{d(a,u_{2p})}\gamma(t)dt\Big],\\ &\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\int_0^{N(a,u_{2p})}\gamma(t)dt+\beta\int_0^{d(a,u_{2p})}\gamma(t)dt\Big]\Big)\Bigg).\end{aligned} \tag{19}$$

where from (2),

$$N(a,a)=\frac{d(a,Ta)\,[1+d(a,Sa)]}{[1+d(a,a)]}=0. \tag{20}$$

Taking $\lim_{p\to\infty}$ in (19) and using (20), we have

$$\int_0^{d(Sa,a)}\gamma(t)dt\le F(0,\phi(0))=0.$$

This implies that $d(Sa,a)=0$.
Thus, S and T have common fixed points.
Step- 3: Prove that fixed points of maps are unique.
Suppose not, therefore there exists another point $b\neq a$ such that $Sb=Tb=b$.
Consider,

$$\begin{aligned}\int_0^{d(b,a)}\gamma(t)dt&=\int_0^{d(Sb,Ta)}\gamma(t)dt\\ &\le F\Bigg(\frac{1}{\alpha+\beta}\Big[\alpha\int_0^{N(b,a)}\gamma(t)dt+\beta\int_0^{d(b,a)}\gamma(t)dt\Big],\\ &\phi\Big(\frac{1}{\alpha+\beta}\Big[\alpha\int_0^{N(b,a)}\gamma(t)dt+\beta\int_0^{d(b,a)}\gamma(t)dt\Big]\Big)\Bigg),\end{aligned} \tag{21}$$

where,

$$N(b,a) = \frac{d(a,Ta)\,[1+d(b,Sb)]}{[1+d(b,a)]} = 0.$$

Hence, from (21)

$$\int_0^{d(b,a)} \gamma(t)dt \leq F\bigg(\frac{\beta}{\alpha+\beta}\int_0^{d(b,a)} \gamma(t)dt, \phi\Big(\frac{\beta}{\alpha+\beta}\int_0^{d(b,a)} \gamma(t)dt\Big)\bigg).$$

Using the fact that F and ϕ are continuous, and $\frac{\beta}{\alpha+\beta} < 1$ for all $\alpha, \beta > 0$, we obtain

$$\int_0^{d(b,a)} \gamma(t)dt < F\bigg(\int_0^{d(b,a)} \gamma(t)dt, \phi\Big(\int_0^{d(b,a)} \gamma(t)dt\Big)\bigg).$$

Using Definition 1.1 and Definition 1.3, this is possible only if $\int_0^{d(b,a)} \gamma(t)dt = 0$ or $\phi(\int_0^{d(b,a)} \gamma(t)dt) = 0$ implies that $\int_0^{d(b,a)} \gamma(t)dt = 0$. Using Lemma 1.1, we get $d(a,b) = 0$. This is a contradiction to our assumption and hence fixed points are unique.
This completes the proof of main result. □

Next we derive a result for selfmaps without using both completeness of metric spaces and continuity of maps S and T.

Theorem 2.2. *Let S and T be two self-maps of X, and d be a metric on X such that (X,d) is a metric spaces. If for each $u, v \in X$*

$$\int_0^{d(Su,Tv)} \gamma(t)dt \leq F\bigg(\int_0^{N(u,v)} \gamma(t)dt, \phi\Big(\int_0^{N(u,v)} \gamma(t)dt\Big)\bigg), \tag{22}$$

where $N(u,v)$ is given by eq (2), $F \in \mathcal{C}, \phi \in \Phi_u$ and $\gamma \in \Psi_1$, then S and T have a unique common fixed point.

Proof. Let us define a sequence $\{u_p\}$ in X s.t.

$$u_{2p+1} = Su_{2p} \ \text{ and } \ u_{2p+2} = Tu_{2p+1}, \ \forall \ \ p = 0, 1, 2, \cdots . \tag{23}$$

Now we assume that for no $n \in N$

$$u_{2p+1} = u_{2p+2}. \tag{24}$$

Then from (22), we get

$$\begin{aligned} \int_0^{d(u_{2p+1},u_{2p+2})} \gamma(t)dt &= \int_0^{d(Su_{2p},Tu_{2p+1})} \gamma(t)dt \\ &\leq F\bigg(\int_0^{N(u_{2p},u_{2p+1})} \gamma(t)dt, \phi\Big(\int_0^{N(u_{2p},u_{2p+1})} \gamma(t)dt\Big)\bigg), \end{aligned} \tag{25}$$

where,

$$N\left(u_{2p},u_{2p+1}\right)=\frac{d(u_{2p+1},Tu_{2p+1})\left[1+d\left(u_{2p},Su_{2p}\right)\right]}{\left[1+d\left(u_{2p},u_{2p+1}\right)\right]}=d(u_{2p+1},u_{2p+2}). \tag{26}$$

Thus (25) implies that,

$$\int_0^{d(u_{2p+1},u_{2p+2})}\gamma(t)dt\le F\Bigg(\int_0^{d(u_{2p+1},u_{2p+2})}\gamma(t)dt,\phi\Big(\int_0^{d(u_{2p+1},u_{2p+2})}\gamma(t)dt\Big)\Bigg).$$

Using Definition 1.1 and Definition 1.3, this is possible only if $\int_0^{d(u_{2p+1},u_{2p+2})}\gamma(t)dt=0$, implies $d(u_{2p+1},u_{2p+2})=0$. Thus our assumption (24) is wrong and so $u_{2p+1}=u_{2p+2}$ for some $p\in N$. Let $p=k$, then we get $u_{2k+1}=u_{2k+2}$. If $a=u_{2k+1}$, then from (23), we obtain $Ta=a$.
Similarly, if we consider

$$\begin{aligned}\int_0^{d(Sa,a)}\gamma(t)dt=\int_0^{d(Sa,Ta)}\gamma(t)dt&\le F\Bigg(\int_0^{N(a,a)}\gamma(t)dt,\phi\Big(\int_0^{N(a,a)}\gamma(t)dt\Big)\Bigg)\\&=F(0,0)=0,\end{aligned} \tag{27}$$

Thus $\int_0^{d(Sa,a)}\gamma(t)dt=0$ implies $d(Sa,a)=0$. Thus we get $Sa=a=Ta$
For uniqueness, assume there exists another point $b\neq a$ s.t $Sb=Tb=b$.
Consider from (22),

$$\int_0^{d(b,a)}\gamma(t)dt=\int_0^{d(Sb,Ta)}\gamma(t)dt\le F\Bigg(\int_0^{N(b,a)}\gamma(t)dt,\phi\Big(\int_0^{N(b,a)}\gamma(t)dt\Big)\Bigg), \tag{28}$$

where, $N(b,a)=\frac{d(a,Ta)[1+d(b,Sb)]}{[1+d(b,a)]}=0$. Hence from (28), $\int_0^{d(b,a)}\le F(0,0)=0$.
This completes proof of our Theorem 2.2. □

3 Applications

In this section, we give several corollaries, as applications of our main result, in the underlying spaces.

If we take $F(r,t)=\frac{r}{1+t}$ in Theorem 2.1, we obtain following result.

Corollary 3.1. *Let (X,d) be a complete metric space, and S and T be two self maps of X such that for each $u,v\in X$*

$$\int_0^{d(Su,Tv)}\gamma(t)dt\le\frac{\frac{1}{\alpha+\beta}\left[\alpha\int_0^{N(u,v)}\gamma(t)dt+\beta\int_0^{d(u,v)}\gamma(t)dt\right]}{\left[1+\phi\big(\frac{1}{\alpha+\beta}\left[\alpha\int_0^{N(u,v)}\gamma(t)dt+\beta\int_0^{d(u,v)}\gamma(t)dt\right]\big)\right]},$$

where $N(u,v)$ is given by eq(2), $\alpha,\beta>0$, $\phi\in\Phi_u$ and $\gamma\in\Psi_1$, then S and T have a unique common fixed point.

If we take $F(r,t)=r-\frac{t}{k+t}$ in Theorem 2.1, we obtain a new result as follows.

Corollary 3.2. *Let S and T be two self-maps of X, and d be a metric on X such that (X,d) is complete. If for each $u,v\in X$*

$$\int_0^{d(Su,Tv)}\gamma(t)dt \le \frac{1}{\alpha+\beta}\big[\alpha\int_0^{N(u,v)}\gamma(t)dt+\beta\int_0^{d(u,v)}\gamma(t)dt\big] - \frac{\phi\big(\frac{1}{\alpha+\beta}\big[\alpha\int_0^{N(u,v)}\gamma(t)dt+\beta\int_0^{d(u,v)}\gamma(t)dt\big]\big)}{\big[k+\phi\big(\frac{1}{\alpha+\beta}\big[\alpha\int_0^{N(u,v)}\gamma(t)dt+\beta\int_0^{d(u,v)}\gamma(t)dt\big]\big)},$$

where $N(u,v)$ is given by eq(2), $\alpha,\beta>0$, $\phi\in\Phi_u$ and $\gamma\in\Psi_1$, then S and T have a unique common fixed point.

If we take $F(r,t)=\frac{r}{(1+r)^2}$ in Theorem 2.1, we obtain the following result.

Corollary 3.3. *Let S and T be two self maps of X, and d be a metric on X such that (X,d) is complete. If for each $u,v\in X$*

$$\int_0^{d(Su,Tv)}\gamma(t)dt \le \frac{\frac{1}{\alpha+\beta}\big[\alpha\int_0^{N(u,v)}\gamma(t)dt+\beta\int_0^{d(u,v)}\gamma(t)dt\big]}{\Big[1+\frac{1}{\alpha+\beta}\big[\alpha\int_0^{N(u,v)}\gamma(t)dt+\beta\int_0^{d(u,v)}\gamma(t)dt\big]\Big]^2},$$

where $N(u,v)$ is given by eq(2), $\alpha,\beta>0$ and $\gamma\in\Psi_1$, then S and T have a unique common fixed point.

If we take $F(r,t)=r\ \theta(r)$ in Theorem 2.2, we derive following result.

Corollary 3.4. *Let S and T be two self maps of X, and d be a metric on X such that (X,d) is complete. If for each $u,v\in X$*

$$\int_0^{d(Su,Tv)}\gamma(t)dt \le \int_0^{N(u,v)}\gamma(t)dt\ \theta\Big(\int_0^{N(u,v)}\gamma(t)dt\Big),$$

where $N(u,v)$ is given by eq(2), $\theta:[0,\infty)\to[0,1)$ be a function and $\gamma\in\Psi_1$, then S and T have a unique common fixed point.

If we take $F(r,t)=r-\theta(r)$ in Theorem 2.2, we find following result.

Corollary 3.5. *Let S and T be two self maps of X, and d be a metric on X such that (X,d) is complete. If for each $u,v\in X$*

$$\int_0^{d(Su,Tv)}\gamma(t)dt \le \int_0^{N(u,v)}\gamma(t)dt-\theta\Big(\int_0^{N(u,v)}\gamma(t)dt\Big),$$

where $N(u,v)$ is given by eq(2), $\theta:[0,\infty)\to[0,\infty)$ is a continuous function such that $\theta(t)=0$ if and only if $t=0$ and $\gamma\in\Psi_1$, then S and T have a unique common fixed point.

If we assume that $F(r,t)=\theta(r)$ in Theorem 2.1, then we obtain a new result.

Corollary 3.6. *Let S and T be two self maps of X, and d be a metric on X such that (X,d) is complete. If for each $u,v \in X$*

$$\int_0^{d(Su,Tv)} \gamma(t)dt \leq \theta\Big(\frac{1}{\alpha+\beta}\big[\alpha\int_0^{N(u,v)} \gamma(t)dt + \beta\int_0^{d(u,v)} \gamma(t)dt\big]\Big)$$

where $N(u,v)$ is given by eq(2), $\alpha,\beta > 0$, $\theta : [0,\infty) \to [0,\infty)$ is a upper semi-continuous function such that $\theta(0) = 0$, and $\theta(t) < t$ for all $t > 0$ and $\gamma \in \Psi_1$, then S and T have a unique common fixed point.

If we take $F(r,t) = r - t$ in Theorem 2.2, then we get the following result.

Corollary 3.7. *Let S and T be two self maps of X, and d be a metric on X such that (X,d) is complete. If for each $u,v \in X$*

$$\int_0^{d(Su,Tv)} \gamma(t)dt \leq \int_0^{N(u,v)} \gamma(t)dt - \phi\Big(\int_0^{N(u,v)} \gamma(t)dt\Big),$$

where $N(u,v)$ is given by eq(2), $\phi \in \Phi_u$ and $\gamma \in \Psi_1$, then S and T have a unique common fixed point.

4 Remarks and Example

Remark 4.1. To the best of my knowledge, contractions given in Corollary 3.1, Corollary 3.2 and Corollary 3.3 are new. These results further can be utilized for future research.

Example 4.2. Let $X = [1,\infty)$ be a space endowed with usual metric $d(u,v) = |u-v|$. Clearly, (X,d) be a metric spaces.
Define maps $S,T : X \to X$ by

$$S(u) = \sqrt[2]{u}, \quad \text{and} \quad T(u) = \sqrt[3]{u} \quad \text{for all} \quad u \in X.$$

Let $\phi, \gamma : [0,+\infty) \to [0,+\infty)$ be defined as

$$\gamma(t) = 2t \quad \text{and} \quad \phi(t) = \frac{t}{10} \text{ for all } t \in R^+,$$

then $\phi \in \Phi_u$ and for each $\epsilon > 0$, γ is a Lebesgue-integrable function which is summable on each compact subset of R^+ such that

$$\int_0^{\epsilon} \gamma(t)dt = \epsilon^2 > 0.$$

Let us define a function $F : [0,\infty)^2 \to \mathbb{R}$ as $F(r,t) = r - t$, for all $r,t \in [0,\infty)$, then clearly, F is a *C-class* function.
If we fix the constants $\alpha = \frac{1}{2} > 0$ and $\beta = \frac{1}{2} > 0$, then step by step calculation, we found that all the assumptions of Theorem 2.1 are satisfied. Also $u = 1$ is the unique common fixed point of S and T in X.

Example 4.3. Consider the assumptions as in Example 4.2 and if we define $\gamma : [0, +\infty) \to [0, +\infty)$ as $\gamma(t) = 1$ for all $t \in R^+$, then for each $\epsilon > 0$, γ is a Lebesgue-integrable function which is summable on each compact subset of R^+ such that

$$\int_0^\epsilon \gamma(t)dt = \epsilon > 0.$$

Then we can see that all the conditions of Theorem 2.1 are satisfied and $u = 1$ is a unique common fixed point of S and T in X.

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