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ASYMPTOTIC ANALYSIS OF AN ε-STOKES PROBLEM CONNECTING STOKES AND PRESSURE-POISSON PROBLEMS

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Abstract. In this Note, we prepare an ε-Stokes problem connecting the Stokes problem and the corresponding pressure-Poisson equation using one parameter $\varepsilon > 0$. We prove that the solution to the ε-Stokes problem, convergences as ε tends to 0 or ∞ to the Stokes and pressure-Poisson problem, respectively.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 2, n \in \mathbb{N})$ with Lipschitz continuous boundary Γ and let $F \in L^2(\Omega)^n, u_b \in H^{1/2}(\Gamma)^n$ satisfy $\int_\Gamma u_b \cdot \nu = 0$, where ν is the unit outward normal vector for Γ. The weak form of the Stokes problem is: Find $u_S \in H^1(\Omega)^n$ and $p_S \in L^2(\Omega)/\mathbb{R}$ satisfying

$$\begin{cases} -\Delta u_S + \nabla p_S = F & \text{in } H^{-1}(\Omega)^n, \\ \operatorname{div} u_S = 0 & \text{in } L^2(\Omega), \\ u_S = u_b & \text{on } H^{1/2}(\Gamma)^n. \end{cases} \tag{S}$$

We refer to [20] for details on the Stokes problem, (i.e. more physical background and corresponding mathematical analysis). Taking the divergence of the first equation, we are led to

$$\operatorname{div} F = \operatorname{div}(-\Delta u_S + \nabla p_S) = -\Delta(\operatorname{div} u_S) + \Delta p_S = \Delta p_S \tag{1.1}$$

in distributions sense. This is often called pressure-Poisson equation and is used in MAC, SMAC or projection method (cf. [1, 4, 7, 12, 13, 15, 17, 19], e.g.). Bearing this in mind, we consider a similar problem: Find $u_{PP} \in H^1(\Omega)^n$ and $p_{PP} \in H^1(\Omega)$ satisfying

$$\begin{cases} -\Delta u_{PP} + \nabla p_{PP} = F & \text{in } H^{-1}(\Omega)^n, \\ -\Delta p_{PP} = -\operatorname{div} F & \text{in } H^{-1}(\Omega), \\ u_{PP} = u_b & \text{on } H^{1/2}(\Gamma)^n, \\ p_{PP} = p_b & \text{on } H^{1/2}(\Gamma). \end{cases} \tag{PP}$$

with $p_b \in H^{1/2}(\Gamma)$. Let this problem be called pressure-Poisson problem. This idea using (1.1) instead of $\operatorname{div} u_S = 0$ is useful to calculate the pressure numerically in the Navier-Stokes equation. For example, the idea is used in both the MAC, SMAC and projection methods [1, 4, 7, 12, 13, 15, 17, 19]. Dirichlet boundary condition for pressure can be found in many circumstances such as outflow boundary [3, 21]. (See also [5, 6, 16].)

In this Note, we prepare on an "interpolation" between these problems (S) and (PP), i.e. we introduce an intermediate problem: For $\varepsilon > 0$, find $u_\varepsilon \in H^1(\Omega)^n$ and $p_\varepsilon \in H^1(\Omega)$ which satisfy

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = F & \text{in } H^{-1}(\Omega)^n, \\ -\varepsilon\Delta p_\varepsilon + \operatorname{div} u_\varepsilon = -\varepsilon \operatorname{div} F & \text{in } H^{-1}(\Omega), \\ u_\varepsilon = u_b & \text{on } H^{1/2}(\Gamma)^n, \\ p_\varepsilon = p_b & \text{on } H^{1/2}(\Gamma). \end{cases} \tag{ES}$$

Let this problem be called ε-Stokes problem. In [8, 11, 14], they treat this problem as approximation of the Stokes problem to avoid numerical instabilities. The ε-Stokes problem (ES) formally approximates the Stokes problem (S) as $\varepsilon \to 0$ and the pressure-Poisson problem (PP) as $\varepsilon \to \infty$ (Figure 1). We show here that (ES) is a natural link between (S) and (PP) in Proposition 2.7. The aim of this Note is to give a precise asymptotic estimates for (ES) when ε tends to zero or ∞.

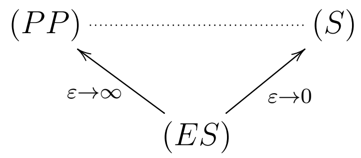

Figure 1: Sketch of the connections between the problems (S), (PP) and (ES).

2 Well-posedness

2.1 Notation

We set

$$C_0^\infty(\Omega)^n := \{f \in C^\infty(\Omega)^n \mid \text{supp}(f) \text{ is compact subset in } \Omega\},$$
$$L^2(\Omega)/\mathbb{R} := \left\{u \in L^2(\Omega) \;\middle|\; \int_\Omega u = 0\right\}.$$

For $m = 1$ or n, $H^{-1}(\Omega)^m = (H_0^1(\Omega)^m)^*$ is equipped with the norm $||f||_{H^{-1}(\Omega)^m} := \sup_{\varphi\in S_m}\langle f, \varphi\rangle$ for $f \in H^{-1}(\Omega)^m$, where $S_m = \{\varphi \in H_0^1(\Omega)^m \mid ||\nabla\varphi||_{L^2(\Omega)^{n\times m}} = 1\}$. We define $[p] := p - (1/|\Omega|)\int_\Omega p$ and $||p||_{L^2(\Omega)/\mathbb{R}} := \inf_{a\in\mathbb{R}} ||p - a||_{L^2(\Omega)} = ||[p]||_{L^2(\Omega)}$ for all $p \in L^2(\Omega)$, where $|\Omega|$ is the volume of Ω.

Let $\gamma_0 \in B(H^1(\Omega), H^{1/2}(\Gamma))$ be the standard trace operator. It is known that (see e.g. [20, pp.10–11, Lemma 1.3]) there exists a linear continuous operator $\gamma_\nu : H^1(\Omega)^n \to H^{-1/2}(\Gamma)$ such that $\gamma_\nu u = u\cdot\nu|_\Gamma$ for all $u \in C^\infty(\overline{\Omega})^n$, where ν is the unit outward normal for Γ and $H^{-1/2}(\Gamma) := H^{1/2}(\Gamma)^*$. Then, the following generalized Gauss divergence formula holds:

$$\int_\Omega u \cdot \nabla\omega + \int_\Omega (\operatorname{div} u)\omega = \langle\gamma_\nu u, \gamma_0\omega\rangle \quad \text{for all } u \in H^1(\Omega)^n, \omega \in H^1(\Omega).$$

We recall the following Theorem 2.1 that plays an important role in the proof of the existence of pressure solution of Stokes problem; see [18, pp.187–190, Lemme 7.1, $l = 0$] and [9, pp.111–115, Theorem 3.2 and Remark 3.1 (Ω is C^1 class)] for the proof.

Theorem 2.1. *There exists a constant $c > 0$ such that*

$$||f||_{L^2(\Omega)} \leq c(||f||_{H^{-1}(\Omega)} + ||\nabla f||_{H^{-1}(\Omega)})$$

for all $f \in L^2(\Omega)$.

The following result follows from Theorem 2.1.

Theorem 2.2. [10, pp.20–21] *There exists a constant $c > 0$ such that*

$$||f||_{L^2(\Omega)/\mathbb{R}} \leq c||\nabla f||_{H^{-1}(\Omega)^n}$$

for all $f \in L^2(\Omega)$.

2.2 Well-posedness

Theorem 2.3. *For* $F \in L^2(\Omega)^n$ *and* $u_b \in H^{1/2}(\Gamma)^n$, *there exists a unique pair of functions* $(u_S, p_S) \in H^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ *satisfying (S).*

See [20, pp.31–32, Theorem 2.4 and Remark 2.5] for the proof.

Theorem 2.4. *For* $F \in L^2(\Omega)^n, u_b \in H^{1/2}(\Gamma)^n$ *and* $p_b \in H^{1/2}(\Gamma)$, *there exists a unique pair of functions* $(u_{PP}, p_{PP}) \in H^1(\Omega)^n \times H^1(\Omega)$ *satisfying (PP).*

Proof. From the second and fourth equations of (PP), $p_{PP} \in H^1(\Omega)$ is uniquely determined. Then $u_{PP} \in H^1(\Omega)^n$ is also uniquely determined from the first and third equations. □

Corollary 2.5. *If the solution* $(u_{PP}, p_{PP}) \in H^1(\Omega)^n \times H^1(\Omega)$ *of (PP) satisfies* $\operatorname{div} u_{PP} = 0$, *by Theorem 2.3,* $u_S = u_{PP}$ *and* $p_S = [p_{PP}]$ *hold.*

Theorem 2.6. *For* $\varepsilon > 0, F \in L^2(\Omega)^n, u_b \in H^{1/2}(\Gamma)^n$ *and* $p_b \in H^{1/2}(\Gamma)$, *there exists a unique pair of functions* $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega)^n \times H^1(\Omega)$ *satisfying the problem (ES).*

Proof. We pick $u_1 \in H^1(\Omega)^n$ and $p_0 \in H^1(\Omega)$ with $\gamma_0 u_1 = u_b, \gamma_0 p_0 = p_b$. Since div : $H_0^1(\Omega)^n \to L^2(\Omega)/\mathbb{R}$ is surjective [10, p.24, Corollary 2.4, 2°)] and [20, p.32, Lemma 2.4, Chapter 1], there exists $u_2 \in H_0^1(\Omega)^n$ such that $\operatorname{div} u_2 = \operatorname{div} u_1$. We put $u_0 := u_1 - u_2$, and then $\gamma_0 u_0 = u_b$ and $\operatorname{div} u_0 = 0$ in Ω. To simplify the notation, we set $u := u_\varepsilon - u_0 (\in H_0^1(\Omega)^n), p := p_\varepsilon - p_0 (\in H_0^1(\Omega)), f \in H^{-1}(\Omega)^n$ and $g \in H^{-1}(\Omega)$ such that $\langle f, v\rangle = \int_\Omega Fv - \int_\Omega \nabla u_0 : \nabla v - \int_\Omega (\nabla p_0) \cdot v \ (v \in H_0^1(\Omega)^n), \langle g, q\rangle = \int_\Omega F \cdot \nabla q - \int_\Omega \nabla p_0 \cdot \nabla q \ (q \in H_0^1(\Omega))$. Then we have

$$\begin{cases} \displaystyle\int_\Omega \nabla u : \nabla \varphi + \int_\Omega (\nabla p) \cdot \varphi = \langle f, \varphi\rangle & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon \int_\Omega \nabla p \cdot \nabla \psi + \int_\Omega (\operatorname{div} u)\psi = \varepsilon\langle g, \psi\rangle & \text{for all } \psi \in H_0^1(\Omega). \end{cases} \tag{2.2}$$

Adding the equations in (2.2), we get

$$(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix})_\varepsilon = \langle f, \varphi\rangle + \varepsilon\langle g, \psi\rangle.$$

Here, we denote

$$(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix})_\varepsilon := \int_\Omega \nabla u : \nabla \varphi + \varepsilon \int_\Omega \nabla p \cdot \nabla \psi + \int_\Omega (\nabla p) \cdot \varphi + \int_\Omega (\operatorname{div} u)\psi.$$

We check that $(*, *)_\varepsilon$ is a continuous coercive bilinear form on $H_0^1(\Omega)^n \times H_0^1(\Omega)$. The bilinearity and continuity of $(*, *)_\varepsilon$ are obvious. The coercivity of $(*, *)_\varepsilon$ is obtained in the following way: Let ${}^t(u, p) \in H_0^1(\Omega)^n \times H_0^1(\Omega)$. We have the following sequence of inequalities;

$$\begin{aligned} (\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix})_\varepsilon &= \int_\Omega \nabla u : \nabla u + \varepsilon \int_\Omega \nabla p \cdot \nabla p + \int_\Omega \operatorname{div}(up) \\ &= ||\nabla u||^2_{L^2(\Omega)} + \varepsilon ||\nabla p||^2_{L^2(\Omega)} \\ &\geq \min\{1, \varepsilon\}(||\nabla u||^2_{L^2(\Omega)} + ||\nabla p||^2_{L^2(\Omega)}) \\ &\geq c \min\{1, \varepsilon\}(||u||^2_{H^1(\Omega)^n} + ||p||^2_{H^1(\Omega)}) \end{aligned}$$

by the Poincaré inequality. Summarizing, $(*,*)_\varepsilon$ is a continuous coercive bilinear form and $H_0^1(\Omega)^{n+1}$ is a Hilbert space. Therefore, the conclusion of Theorem 2.6 follows based on the Lax-Milgram Theorem. $\square$

From now on, let the solutions of (S), (PP) and (ES) be denoted by (u_S, p_S), (u_{PP}, p_{PP}) and $(u_\varepsilon, p_\varepsilon)$, respectively.

Proposition 2.7. *Suppose that $p_S \in H^1(\Omega)$. Then there exists a constant $c > 0$ independent of ε such that*

$$||u_S - u_{PP}||_{H^1(\Omega)^n} \le c||\gamma_0 p_S - p_b||_{H^{1/2}(\Gamma)}, \quad ||u_S - u_\varepsilon||_{H^1(\Omega)^n} \le c||\gamma_0 p_S - p_b||_{H^{1/2}(\Gamma)}.$$

In particular, if $\gamma_0 p_S = p_b$, then $p_{PP} = p_\varepsilon = p_S$ hold for all $\varepsilon > 0$.

Proof. From (S) and (PP), we have

$$\begin{cases} \displaystyle\int_\Omega \nabla(u_S - u_{PP}) : \nabla\varphi = -\int_\Omega (\nabla(p_S - p_{PP})) \cdot \varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\int_\Omega \nabla(p_S - p_{PP}) \cdot \nabla\psi = 0 & \text{for all } \psi \in H_0^1(\Omega). \end{cases} \tag{2.3}$$

Putting $\varphi := u_S - u_{PP} \in H_0^1(\Omega)^n$ in (2.3), we get

$$\begin{aligned} ||\nabla(u_S - u_{PP})||^2_{L^2(\Omega)^{n\times n}} &= -\int_\Omega (\nabla(p_S - p_{PP})) \cdot (u_S - u_{PP}) \\ &\le ||\nabla(p_S - p_{PP})||_{L^2(\Omega)^n} ||u_S - u_{PP}||_{L^2(\Omega)^n}, \end{aligned}$$

and then

$$||u_S - u_{PP}||_{H^1(\Omega)^n} \le c_1 ||\nabla(p_S - p_{PP})||_{L^2(\Omega)^n} \tag{2.4}$$

follows. We pick up $p_0 \in H^1(\Omega)$ such that $\gamma_0 p_0 = p_b$. From the fourth equation of (PP) and the second equation of (2.3), we obtain $p_{PP} - p_0 \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla(p_{PP} - p_0) \cdot \nabla\psi = \int_\Omega \nabla(p_S - p_0) \cdot \nabla\psi,$$

and, by Stampacchia Theorem [2, Theorem 5.6], it follows that

$$\begin{aligned} &\min_{\psi \in H_0^1(\Omega)^n} \left(\frac{1}{2} \|\nabla\psi\|^2_{L^2(\Omega)^n} - \int_\Omega \nabla(p_S - p_0) \cdot \nabla\psi \right) \\ = \;& \frac{1}{2} \|\nabla(p_{PP} - p_0)\|^2_{L^2(\Omega)^n} - \int_\Omega \nabla(p_S - p_0) \cdot \nabla(p_{PP} - p_0) \\ = \;& \frac{1}{2} \|\nabla p_{PP}\|^2_{L^2(\Omega)^n} - \frac{1}{2} \|\nabla p_0\|^2_{L^2(\Omega)^n} - \int_\Omega \nabla p_S \cdot \nabla p_{PP} + \int_\Omega \nabla p_S \cdot \nabla p_0. \end{aligned}$$

Hence,

$$\frac{1}{2} \|\nabla(p_S - p_{PP})\|^2_{L^2(\Omega)^n} = \min_{\psi \in H_0^1(\Omega)^n} \left(\frac{1}{2} \|\nabla(p_S - p_0 - \psi)\|^2_{L^2(\Omega)^n} \right).$$

Since γ_0 is surjective and the space $\mathrm{Ker}(\gamma_0) = H_0^1(\Omega)$, $H^1(\Omega)/H_0^1(\Omega)$ and $H^{1/2}(\Gamma)$ are isomorphic, there exists a constant $c_2 > 0$ such that $\|q\|_{H^1(\Omega)/H_0^1(\Omega)} \le c_2\|\gamma_0 q\|_{H^{1/2}(\Gamma)}$ for all $q \in H^1(\Omega)$. Hence, we obtain

$$\begin{aligned}\|\nabla(p_S - p_{PP})\|_{L^2(\Omega)^n} &\le \min_{\psi\in H_0^1(\Omega)^n} \|\nabla(p_S - p_0 - \psi)\|_{L^2(\Omega)^n} \\ &\le \min_{\psi\in H_0^1(\Omega)^n} \|p_S - p_0 - \psi\|_{H^1(\Omega)} \\ &= \|p_S - p_{PP}\|_{H^1(\Omega)/H_0^1(\Omega)} \\ &\le c_2\|\gamma_0 p_S - \gamma_0 p_0\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

Together with (2.4) and the assumption $\gamma_0 p_0 = p_b$, we obtain $\|u_S - u_{PP}\|_{H^1(\Omega)^n} \le c_1 c_2\|\gamma_0 p_S - p_b||_{H^{1/2}(\Gamma)}$.

Let $w_\varepsilon := u_S - u_\varepsilon \in H_0^1(\Omega)^n, r_\varepsilon := p_{PP} - p_\varepsilon \in H_0^1(\Omega)$. By (S), (PP) and (ES), we have

$$\begin{cases} \displaystyle\int_\Omega \nabla w_\varepsilon : \nabla\varphi + \int_\Omega (\nabla r_\varepsilon)\cdot\varphi = -\int_\Omega (\nabla(p_S - p_{PP}))\cdot\varphi & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \displaystyle\varepsilon\int_\Omega \nabla r_\varepsilon\cdot\nabla\psi + \int_\Omega (\operatorname{div} w_\varepsilon)\psi = 0 & \text{for all } \psi \in H_0^1(\Omega). \end{cases} \tag{2.5}$$

Putting $\varphi := w_\varepsilon$ and $\psi := r_\varepsilon$ and adding two equations of (2.5), we get

$$||\nabla w_\varepsilon||^2_{L^2(\Omega)^{n\times n}} + \varepsilon||\nabla r_\varepsilon||^2_{L^2(\Omega)^n} = -\int_\Omega (\nabla(p_S - p_{PP}))\cdot w_\varepsilon \le ||\nabla(p_S - p_{PP})||_{L^2(\Omega)^n}||w_\varepsilon||_{L^2(\Omega)^n}$$

from $\int_\Omega(\nabla r_\varepsilon)\cdot w_\varepsilon = -\int_\Omega(\operatorname{div} w_\varepsilon)r_\varepsilon$. Thus it leads $||w_\varepsilon||_{H^1(\Omega)^n} \le c_3||\nabla(p_S - p_{PP})||_{L^2(\Omega)^n}$. Hence we obtain $||u_S - u_\varepsilon||_{H^1(\Omega)^n} = ||w_\varepsilon||_{H^1(\Omega)^n} \le c_2 c_3||\gamma_0 p_S - p_b||_{H^{1/2}(\Gamma)}$. □

Proposition 2.8. *Under the hypotheses of Proposition 2.7, if $\tilde p \in H^1(\Omega)$ satisfies $\gamma_0\tilde p = p_b$, then we have*

$$||\nabla(\tilde p - p_{PP})||_{L^2(\Omega)^n} \le ||\nabla(\tilde p - p_S)||_{L^2(\Omega)^n}.$$

Proof. By the second equation of (2.3) and $\tilde p - p_{PP} \in H_0^1(\Omega)$, it yields

$$\int_\Omega \nabla(p_S - p_{PP})\cdot\nabla(\tilde p - p_{PP}) = 0.$$

Hence we obtain

$$\begin{aligned}||\nabla(\tilde p - p_{PP})||^2_{L^2(\Omega)^n} &= \int_\Omega \nabla(\tilde p - p_S + p_S - p_{PP})\cdot\nabla(\tilde p - p_{PP}) \\ &\le ||\nabla(\tilde p - p_S)||_{L^2(\Omega)^n}||\nabla(\tilde p - p_{PP})||_{L^2(\Omega)^n}.\end{aligned}$$

Therefore, $||\nabla(\tilde p - p_{PP})||_{L^2(\Omega)^n} \le ||\nabla(\tilde p - p_S)||_{L^2(\Omega)^n}$ holds. □

Remark 2.9. If $p_S \in H^1(\Omega)$, then we have

$$||\nabla(p_\varepsilon - p_{PP})||_{L^2(\Omega)^n} \le ||\nabla(p_\varepsilon - p_S)||_{L^2(\Omega)^n}$$

for all $\varepsilon > 0$, (from Proposition 2.8). Hence, if $(\nabla p_\varepsilon)_{\varepsilon>0}$ converges strongly to ∇p_S in $L^2(\Omega)^n$, then there exists a constant $c \in \mathbb{R}$ such that $u_{PP} = u_S$ and $p_{PP} = p_S + c$, which imply $\gamma_0 p_S = p_b + c$ for some $c \in \mathbb{R}$. In other words, if $p_S \in H^1(\Omega)$ satisfies $\gamma_0 p_S \ne p_b + c$ for all $c \in \mathbb{R}$, then ∇p_ε does not converge to ∇p_S in $L^2(\Omega)^n$ as $\varepsilon \to 0$.

3 Links between (ES) and (PP)

Theorem 3.1. *There exists a constant $c > 0$ independent of ε satisfying*

$$||u_\varepsilon - u_{PP}||_{H^1(\Omega)^n} \leq \frac{c}{\varepsilon}||\operatorname{div} u_{PP}||_{H^{-1}(\Omega)}, \quad ||p_\varepsilon - p_{PP}||_{H^1(\Omega)} \leq \frac{c}{\varepsilon}||\operatorname{div} u_{PP}||_{H^{-1}(\Omega)}.$$

for all $\varepsilon > 0$. In particular, we have

$$||u_\varepsilon - u_{PP}||_{H^1(\Omega)^n} \to 0,\ ||p_\varepsilon - p_{PP}||_{H^1(\Omega)} \to 0 \quad \text{as } \varepsilon \to \infty.$$

Proof. From (PP) and (ES), we have

$$\begin{cases} \int_\Omega \nabla(u_\varepsilon - u_{PP}) : \nabla\varphi + \int_\Omega (\nabla(p_\varepsilon - p_{PP}))\cdot\varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \varepsilon\int_\Omega \nabla(p_\varepsilon - p_{PP})\cdot\nabla\psi + \int_\Omega(\operatorname{div}(u_\varepsilon - u_{PP}))\psi = -\int_\Omega(\operatorname{div} u_{PP})\psi & \text{for all } \psi \in H_0^1(\Omega). \end{cases} \tag{3.6}$$

Putting $\varphi := u_\varepsilon - u_{PP} \in H_0^1(\Omega)^n$ and $\psi := p_\varepsilon - p_{PP} \in H_0^1(\Omega)$ and adding two equations of (3.6), we obtain

$$||\nabla(u_\varepsilon - u_{PP})||^2_{L^2(\Omega)^{n\times n}} + \varepsilon||\nabla(p_\varepsilon - p_{PP})||^2_{L^2(\Omega)^n} \leq ||\operatorname{div} u_{PP}||_{H^{-1}(\Omega)}||\nabla(p_\varepsilon - p_{PP})||_{L^2(\Omega)^n},$$

where we have used $\int_\Omega(\nabla(p_\varepsilon - p_{PP}))\cdot(u_\varepsilon - u_{PP}) = -\int_\Omega(\operatorname{div}(u_\varepsilon - u_{PP}))(p_\varepsilon - p_{PP})$. Thus

$$||\nabla(p_\varepsilon - p_{PP})||_{L^2(\Omega)^n} \leq \frac{1}{\varepsilon}||\operatorname{div} u_{PP}||_{H^{-1}(\Omega)}$$

follows. In addition, by (3.6) and the Poincaré inequality, we have

$$\begin{aligned} ||\nabla(u_\varepsilon - u_{PP})||^2_{L^2(\Omega)^n} &= -\int_\Omega(\nabla(p_\varepsilon - p_{PP}))\cdot(u_\varepsilon - u_{PP}) \\ &\leq ||\nabla(p_\varepsilon - p_{PP})||_{L^2(\Omega)^n}||u_\varepsilon - u_{PP}||_{L^2(\Omega)^n} \\ &\leq c||\nabla(p_\varepsilon - p_{PP})||_{L^2(\Omega)^n}||\nabla(u_\varepsilon - u_{PP})||_{L^2(\Omega)^{n\times n}}, \end{aligned}$$

and then $||\nabla(u_\varepsilon - u_{PP})||_{L^2(\Omega)^n} \leq (c/\varepsilon)||\operatorname{div} u_{PP}||_{H^{-1}(\Omega)}$ follows. □

Corollary 3.2. *If u_{PP} satisfies $\operatorname{div} u_{PP} = 0$, then $u_\varepsilon = u_{PP}$ and $p_\varepsilon = p_{PP}$ hold for all $\varepsilon > 0$. Furthermore, $u_S = u_\varepsilon = u_{PP}$ and $p_S = [p_\varepsilon] = [p_{PP}]$ hold for all $\varepsilon > 0$.*

4 Links between (ES) and (S)

Lemma 4.1. *If $v \in H^1(\Omega)^n, q \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)^n$ satisfy*

$$\int_\Omega \nabla v : \nabla\varphi + \langle\nabla q, \varphi\rangle = \langle f, \varphi\rangle \quad \text{for all } \varphi \in H_0^1(\Omega)^n,$$

then there exists a constant $c > 0$ such that

$$||q||_{L^2(\Omega)/\mathbb{R}} \leq c(||\nabla v||_{L^2(\Omega)^{n\times n}} + ||f||_{H^{-1}(\Omega)^n}).$$

Proof. Let c be the constant arising in Theorem 2.2. Then we have

$$\begin{aligned}||q||_{L^2(\Omega)/\mathbb{R}} \leq\ & c||\nabla q||_{H^{-1}(\Omega)^n} = c \sup_{\varphi \in S_n} |\langle \nabla q, \varphi\rangle| \\ \leq\ & c \sup_{\varphi \in S_n} (|\int_\Omega \nabla v : \nabla \varphi| + |\langle f, \varphi\rangle|) \\ \leq\ & c(||\nabla v||_{L^2(\Omega)^{n\times n}} + ||f||_{H^{-1}(\Omega)^n}).\end{aligned}$$

□

Theorem 4.2. *There exists a constant $c > 0$ independent of ε such that*

$$||u_\varepsilon||_{H^1(\Omega)^n} \leq c, \quad ||p_\varepsilon||_{L^2(\Omega)/\mathbb{R}} \leq c \quad \textit{for all } \varepsilon > 0.$$

Furthermore, we have

$$u_\varepsilon - u_S \rightharpoonup 0 \textit{ weakly in } H_0^1(\Omega)^n,\ [p_\varepsilon] - p_S \rightharpoonup 0 \textit{ weakly in } L^2(\Omega)/\mathbb{R} \quad \text{as } \varepsilon \to 0.$$

Proof. We use the notations $u_0 \in H^1(\Omega)^n, p_0 \in H^1(\Omega), f \in H^{-1}(\Omega)^n$ and $g \in H^{-1}(\Omega)$ in Theorem 2.6. We put $\tilde{u}_\varepsilon := u_\varepsilon - u_0 \in H_0^1(\Omega)^n, \tilde{p}_\varepsilon := p_\varepsilon - p_0 \in H_0^1(\Omega)$. Then we have

$$\begin{cases} \int_\Omega \nabla \tilde{u}_\varepsilon : \nabla \varphi + \int_\Omega (\nabla \tilde{p}_\varepsilon) \cdot \varphi = \langle f, \varphi\rangle & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \varepsilon \int_\Omega \nabla \tilde{p}_\varepsilon \cdot \nabla \psi + \int_\Omega (\operatorname{div} \tilde{u}_\varepsilon)\psi = \varepsilon\langle g, \psi\rangle & \text{for all } \psi \in H_0^1(\Omega). \end{cases} \tag{4.7}$$

Putting $\varphi := \tilde{u}_\varepsilon, \psi := \tilde{p}_\varepsilon$ and adding the two equations of (4.7), we get

$$||\nabla \tilde{u}_\varepsilon||^2_{L^2(\Omega)^{n\times n}} + \varepsilon||\nabla \tilde{p}_\varepsilon||^2_{L^2(\Omega)^n} \leq ||f||_{H^{-1}(\Omega)^n}||\nabla \tilde{u}_\varepsilon||_{L^2(\Omega)^{n\times n}} + \varepsilon||g||_{H^{-1}(\Omega)}||\nabla \tilde{p}_\varepsilon||_{L^2(\Omega)^n}$$

since $\int_\Omega (\nabla \tilde{p}_\varepsilon)\cdot\tilde{u}_\varepsilon = -\int_\Omega (\operatorname{div} \tilde{u}_\varepsilon)\tilde{p}_\varepsilon$. It leads that $(||\tilde{u}_\varepsilon||_{H^1(\Omega)^n})_{0<\varepsilon<1}$ and $(||\sqrt{\varepsilon}\tilde{p}_\varepsilon||_{H^1(\Omega)})_{0<\varepsilon<1}$ are bounded. In addition,

$$||\tilde{p}_\varepsilon||_{L^2(\Omega)/\mathbb{R}} \leq c(||\nabla \tilde{u}_\varepsilon||_{L^2(\Omega)^{n\times n}} + ||f||_{H^{-1}(\Omega)^n})$$

by Lemma 4.1, which implies that $(||\tilde{p}_\varepsilon||_{L^2(\Omega)/\mathbb{R}})_{0<\varepsilon<1}$ is bounded. By Theorem 3.1, $(||u_\varepsilon||_{H^1(\Omega)^n})_{\varepsilon\geq 1}$ and $(||\tilde{p}_\varepsilon||_{L^2(\Omega)/\mathbb{R}})_{\varepsilon\geq 1}$ are bounded, and then $(||u_\varepsilon||_{H^1(\Omega)^n})_{\varepsilon>0}$ and $(||\tilde{p}_\varepsilon||_{L^2(\Omega)/\mathbb{R}})_{\varepsilon>0}$ are bounded.

Since $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ is reflexive and $(\tilde{u}_\varepsilon, [\tilde{p}_\varepsilon])_{0<\varepsilon<1}$ is bounded in $H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$, there exist $(u, p) \in H_0^1(\Omega)^n \times (L^2(\Omega)/\mathbb{R})$ and a subsequence of pair $(\tilde{u}_{\varepsilon_k}, \tilde{p}_{\varepsilon_k})_{k\in\mathbb{N}} \subset H_0^1(\Omega)^n \times H_0^1(\Omega)$ such that

$$\tilde{u}_{\varepsilon_k} \rightharpoonup u \text{ weakly in } H_0^1(\Omega)^n,\ [\tilde{p}_{\varepsilon_k}] \rightharpoonup p \text{ weakly in } L^2(\Omega)/\mathbb{R} \quad \text{as } k \to \infty.$$

Hence, from (4.7) with $\varepsilon := \varepsilon_k$, taking $k \to \infty$, we obtain

$$\begin{cases} \int_\Omega \nabla u : \nabla \varphi + \langle \nabla p, \varphi\rangle = \langle f, \varphi\rangle & \text{for all } \varphi \in H_0^1(\Omega)^n \\ \int_\Omega (\operatorname{div} u)\psi = 0 & \text{for all } \psi \in H_0^1(\Omega), \end{cases} \tag{4.8}$$

where

$$|\varepsilon_k \int_\Omega \nabla\tilde{p}_{\varepsilon_k} \cdot \nabla\psi| \leq \sqrt{\varepsilon_k}||\sqrt{\varepsilon}\tilde{p}_\varepsilon||_{H^1(\Omega)}||\psi||_{H^1(\Omega)} \to 0,$$

$$\int_\Omega \nabla\tilde{p}_{\varepsilon_k} \cdot \varphi = -\int_\Omega [\tilde{p}_{\varepsilon_k}] \operatorname{div}\varphi \to -\int_\Omega p \operatorname{div}\varphi = \langle \nabla p, \varphi \rangle$$

as $k \to \infty$. The first equation of (4.8) implies that

$$-\Delta(u+u_0) + \nabla(p+p_0) = F \quad \text{in } H^{-1}(\Omega)^n.$$

From the second equation of (4.8), $\operatorname{div}(u+u_0) = 0$ follows. Hence, we obtain $u_S = u + u_0$ and $p_S = p + [p_0]$. Then we have

$$u_{\varepsilon_k} - u_S = u_{\varepsilon_k} - u - u_0 = \tilde{u}_{\varepsilon_k} - u_S \rightharpoonup 0 \text{ weakly in } H_0^1(\Omega)^n,$$

$$[p_{\varepsilon_k}] - p_S = [p_{\varepsilon_k} - p - p_0] = [\tilde{p}_{\varepsilon_k}] - p \rightharpoonup 0 \text{ weakly in } L^2(\Omega)/\mathbb{R}.$$

An arbitrarily chosen subsequence of $((u_\varepsilon, [p_\varepsilon]))_{0<\varepsilon<1}$ has a subsequence which converges to (u_S, p_S), so we can conclude the proof. □

Theorem 4.3. *Suppose that $p_S \in H^1(\Omega)$. Then we have*

$$u_\varepsilon - u_S \to 0 \textit{ strongly in } H_0^1(\Omega)^n, \ [p_\varepsilon] - p_S \to 0 \textit{ strongly in } L^2(\Omega)/\mathbb{R} \quad \text{as } \varepsilon \to 0.$$

Proof. We have

$$\begin{cases} \int_\Omega \nabla(u_\varepsilon - u_S) : \nabla\varphi + \int_\Omega (\nabla(p_\varepsilon - p_S)) \cdot \varphi = 0 & \text{for all } \varphi \in H_0^1(\Omega)^n, \\ \varepsilon \int_\Omega \nabla(p_\varepsilon - p_S) \cdot \nabla\psi + \int_\Omega (\operatorname{div} u_\varepsilon)\psi = 0 & \text{for all } \psi \in H_0^1(\Omega). \end{cases} \tag{4.9}$$

We use the notations $p_0 \in H^1(\Omega)$ in Theorem 2.6. Putting $\varphi := u_\varepsilon - u_S \in H_0^1(\Omega)^n, \psi := p_\varepsilon - p_0 \in H_0^1(\Omega)$ and $\tilde{p}_S := p_S - p_0 \in H^1(\Omega)$, we get

$$\begin{aligned} & ||\nabla(u_\varepsilon - u_S)||^2_{L^2(\Omega)^{n\times n}} + \varepsilon||\nabla(p_\varepsilon - p_S)||^2_{L^2(\Omega)^n} \\ = \ & \int_\Omega (\nabla\tilde{p}_S) \cdot (u_\varepsilon - u_S) - \varepsilon \int_\Omega \nabla(p_\varepsilon - p_S) \cdot \nabla\tilde{p}_S \\ \leq \ & ||\nabla\tilde{p}_S||_{L^2(\Omega)^n}||u_\varepsilon - u_S||_{L^2(\Omega)^n} + \varepsilon||\nabla(p_\varepsilon - p_S)||_{L^2(\Omega)^n}||\nabla\tilde{p}_S||_{L^2(\Omega)^n}. \end{aligned}$$

By Corollary 4.2 and the Rellich-Kondrachov Theorem, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that

$$u_{\varepsilon_k} \to u_S \text{ strongly in } L^2(\Omega)^n.$$

So, we can write that

$$\begin{aligned} & ||\nabla(u_{\varepsilon_k} - u_S)||^2_{L^2(\Omega)^{n\times n}} \\ \leq \ & ||\nabla\tilde{p}_S||_{L^2(\Omega)^n}||u_{\varepsilon_k} - u_S||_{L^2(\Omega)^n} + \sqrt{\varepsilon_k}||\sqrt{\varepsilon_k}\nabla(p_{\varepsilon_k} - p_S)||_{L^2(\Omega)^n}||\nabla\tilde{p}_S||_{L^2(\Omega)^n}. \end{aligned}$$

It implies that

$$||[p_{\varepsilon_k}] - p_S||_{L^2(\Omega)/\mathbb{R}} = ||p_{\varepsilon_k} - p_S||_{L^2(\Omega)/\mathbb{R}} \leq c||\nabla(u_{\varepsilon_k} - u_S)||_{L^2(\Omega)^{n\times n}} \to 0$$

by Lemma 4.1. An arbitrarily chosen subsequence of $((u_\varepsilon, [p_\varepsilon]))_{0<\varepsilon<1}$ has a subsequence which converges to (u_S, p_S), so we can conclude the proof. □

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