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STRONG STABILITY OF 2D VISCOELASTIC POISEUILLE-TYPE FLOWS

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Abstract. We investigate L^p-stability of small viscoelastic Poiseuille-type flows in two dimensions stemming from a model considered in Lin, Liu, and Zhang (2005). We show global existence and exponential decay of the perturbed flows whenever the initial perturbation and the height of the layer are sufficiently small.

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1 Introduction

Viscoelasticity describes a property of materials exhibiting both viscous and elastic characteristics under deformation. Such a material may show elastic behavior as well as fluid properties. We are interested in stability of viscoelastic Poiseuille-type flows in two space dimensions in a layer. A Poiseuille-type flow has a horizontal flow-profile that is completely determined by the vertical component.

There is an earlier work by Endo, Götz, Liu and the first author [6], where they used an energy argument to prove L^2-type stability results for a small Poiseuille-type flow subject to

$$\begin{cases} \partial_t F + u \cdot \nabla F &= F \nabla u, \\ \operatorname{div} u &= 0, \\ \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla \pi &= \operatorname{div} F^T F, \end{cases} \tag{1}$$

where F denotes the deformation tensor, u the velocity, π the pressure and ν the viscosity. The present paper considers a similar problem in an L^p-setting.

We shall show stability of small viscoelastic Poiseuille-type flows in layer domains under a periodicity condition in one direction and under the assumption that the height of the layer is sufficiently small.

The above viscoelastic model is due to considerations by Lin, Liu and Zhang [10]. There, the authors use weak theory to obtain local-in-time smooth solutions in bounded domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ with smooth boundary, the whole space $\mathbb{R}^2$ and $\mathbb{R}^3$ or a periodic box. They show global-in-time existence of solutions with small initial data in the case of $\mathbb{R}^2$ and the periodic box. Existence of local-in-time strong solutions has been shown by Kreml and Pokorný [9]. Using maximal regularity methods, Geissert, Götz and Nesensohn [7] proved for a related model large-data local L^p well-posedness as well as existence and uniqueness of strong solutions for arbitrarily large times in a variety of domains.

Stability of a flow parallel to the boundary like the Poiseuille flow or the Couette flow is a very important topic in fluid mechanics. In fact it is known that the Couette flow for the incompressible Navier-Stokes equations in a layer domain is stable under a small perturbation, irrespective of how large its velocity is [8]; see [12] for a pioneering work.

Let us briefly explain our approach. By a change of variables introduced in [10], the problem is transformed into a parabolic quasilinear evolution equation for the velocity coupled with a damped quasilinear transport equation for the deformation tensor. We then show unique global-in-time existence of the perturbed flow for small initial perturbations. Our main ingredient is the maximal regularity estimate of the generalized Stokes problem with inhomogeneous divergence, which is due to Abels [1] in the non-periodic case. The adapted version for the periodic layer has recently been given in [13]. Let us emphasize at this point, that it is not enough to have the (strong) maximal regularity estimates, but that we also have to use an *a priori* estimate in weaker norms. This is due to the fact that we cannot use a simple Banach fixed point argument, as it is not possible to show contraction in the high norms of the expected solution spaces. To overcome this difficulty, we use as another vital ingredient a fixed point argument due to Kreml and Pokorný [9], see Lemma 1 below.

In the remaining part of the introduction we first rewrite the model for viscoelastic fluids in terms of a stream function for the deformation tensor. Then we establish a model for a perturbation of a Poiseuille-type flow. In the third part we apply a transformation to the equation for the stream function which reveals the hidden dampening term of the equation. The main result is stated at the end of the introduction.

1.1 Viscoelastic Fluids

We consider the general system (1) describing the flow of viscoelastic fluids set in a two-dimensional layer $\Omega_d = \mathbb{R} \times (0, d)$ and demand that for $F_0 := F|_{t=0}$ we have

$$\det F_0 = 1, \quad \operatorname{div} F_0 = 0 \quad \text{ in } \Omega. \tag{2}$$

By formally taking the matrix-valued divergence on the first equation in (1), we obtain by employing Einstein's sum convention

$$\partial_t \partial_j F_{ij} + (\partial_j u_k)\partial_k F_{ij} + u_k \partial_k \partial_j F_{ij} = (\partial_j F_{ik})\partial_k u_j + F_{ik}\partial_j \partial_k u_j \,.$$

Using $\operatorname{div} u = 0$ in the second term of the right-hand side and noting that the second term on the left-hand side equals the first term on the right-hand side, we obtain the following equation for $\operatorname{div} F$:

$$\partial_t \operatorname{div} F + (u \cdot \nabla)\operatorname{div} F = 0, \qquad \text{in } (0, T) \times \Omega_d. \tag{3}$$

Therefore, the initial datum $\operatorname{div} F_0 = 0$ is merely transported and hence

$$\operatorname{div} F = 0 \qquad \text{in } (0, T) \times \Omega_d. \tag{4}$$

By an analogous argumentation it follows

$$\det F = 1 \qquad \text{in } (0, T) \times \Omega_d. \tag{5}$$

In two space dimensions, we obtain for a solenoidal matrix field an $\mathbb{R}^2$-valued stream function ζ_0 such that

$$F_0 = \nabla^{\perp}\zeta_0 = \begin{pmatrix} -\partial_2 \zeta_{0_1} & \partial_1 \zeta_{0_1} \\ -\partial_2 \zeta_{0_2} & \partial_1 \zeta_{0_2} \end{pmatrix}.$$

Moreover, if this quantity is propagated in time subject to the transport equation

$$\begin{aligned} \partial_t \zeta + u \cdot \nabla \zeta &= 0, \\ \zeta(0) &= \zeta_0, \end{aligned} \tag{6}$$

then for $F = \nabla^{\perp}\zeta$, the first equation of (1), is fulfilled, see [10]. This system is much more friendly to analyze and hence we will in the following consider the function ζ instead of F. With this new variable, it is computed in [10] that

$$\operatorname{div} F^T F = \frac{1}{2}\nabla|\nabla\zeta|^2 - \Delta\zeta_1 \nabla\zeta_1 - \Delta\zeta_2 \nabla\zeta_2. \tag{7}$$

Notice that the first term is a gradient that can be absorbed into the pressure function in the momentum balance equation in (1). Therefore, let us introduce a new pressure function $\tilde{\pi} = \pi - \frac{1}{2}|\nabla\zeta|^2$, which is again denoted by π in the following. With this we end up with an equivalent system that is valid in two space-dimensions for $(u, F, \pi) = (u, \nabla^\perp\zeta, \pi)$, when we apply Einstein's sum convention:

$$\begin{cases} \partial_t\zeta + u\cdot\nabla\zeta = 0, \\ \operatorname{div} u = 0, \\ \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla\pi = -\Delta\zeta_k\nabla\zeta_k. \end{cases} \tag{8}$$

We now want to construct a suitable Poiseuille-type flow solution $\bar{u}$ to (1) or equivalently (8), *i.e.*, a solution with horizontal flow-profile that is completely determined by the vertical component. Hence, we assume that $\bar{u}$ takes the form

$$\bar{u}(t,x) = \begin{pmatrix} \psi(t,x_2) \\ 0 \end{pmatrix},$$

with homogeneous Dirichlet boundary conditions. Then the divergence condition in (1) is trivially fulfilled.

In order to adequately determine the corresponding deformation tensor $\bar{F}$ or, equivalently, the corresponding stream function $\bar{\zeta}$, we introduce the flow map $x_i(t,\xi)$, $0 \le t < T$, corresponding to Lagrangian coordinates ξ. These flow maps are given by the system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt}x_1(t,\xi) &= \bar{u}_1(t, x_1(t,\xi), x_2(t,\xi)) = \psi(t, x_2(t,\xi)), \qquad && x_1(0) = \xi_1, \\ \frac{d}{dt}x_2(t,\xi) &= \bar{u}_2(t, x_1(t,\xi), x_2(t,\xi)) = 0, && x_2(0) = \xi_2, \end{aligned}$$

which can easily be solved by

$$\begin{aligned} x_1(t,\xi) &= \xi_1 + \int_0^t \psi(s, x_2(s,\xi))\,\mathrm{d}s = \xi_1 + \int_0^t \psi(s,\xi_2)\,\mathrm{d}s, \\ x_2(t,\xi) &= \xi_2, \end{aligned}$$

as long as ψ admits sufficient regularity. Let us abbreviate

$$\phi(t,x_2) = \int_0^t \psi(s,x_2)\,\mathrm{d}s. \tag{9}$$

Then, we can calculate the deformation tensor and the resulting elastic force

$$\bar{F} = \begin{pmatrix} 1 & 0 \\ \partial_2\phi & 1 \end{pmatrix}, \quad \bar{F}^T\bar{F} = \begin{pmatrix} 1 + (\partial_2\phi)^2 & \partial_2\phi \\ \partial_2\phi & 1 \end{pmatrix} \quad \text{and} \quad \operatorname{div}\bar{F}^T\bar{F} = \begin{pmatrix} \partial_2^2\phi \\ 0 \end{pmatrix}.$$

Note here, that with $x_2(t,\xi) = \xi_2$ it is also $\frac{\partial}{\partial\xi_2} = \frac{\partial}{\partial x_2} = \partial_2$. Let us also remark at this point that $\operatorname{div}\bar{F} = 0$.

The stream function $\bar{\zeta}$ corresponding to $\bar{F}$ may be chosen as

$$\bar{\zeta}(t,x) = \begin{pmatrix} -x_2 \\ x_1 - \phi(t,x_2) \end{pmatrix} \tag{10}$$

solving the system

$$\begin{cases} \partial_t\bar{\zeta} + \bar{u}\cdot\nabla\bar{\zeta} &= 0, \quad \text{in} \quad (0,T)\times\Omega_d, \\ \bar{\zeta}(0,x) &= x^\perp, \quad \text{for} \quad x\in\Omega_d. \end{cases}$$

Here we have used the notation $x^\perp := \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$. We note that while $\bar{\zeta}$ is not constant in the horizontal variable, derivatives of $\bar{\zeta}$ are.

We insert the elastic force into the balance of momentum for $\bar{u}$, *i.e.*,

$$\partial_t\bar{u} - \mu\Delta\bar{u} + \bar{u}\cdot\nabla\bar{u} + \nabla\bar{\pi} = \operatorname{div}\bar{F}^T\bar{F}, \quad \text{in} \quad (0,T)\times\Omega_d,$$

which yields the equivalent formulation

$$\left.\begin{aligned} \partial_t\psi + \partial_1\bar{\pi} = \mu\partial_2^2\psi + \partial_2^2\phi, \\ \partial_2\bar{\pi} = 0. \end{aligned}\right\} \quad \text{in} \quad (0,T)\times\Omega_d.$$

We conclude from the second equation that the pressure is a function depending only on the horizontal variable $\bar{\pi} = \bar{\pi}(t,x_1)$. Since ψ and ϕ depend only on t and x_2, the first equation implies that $\partial_1\bar{\pi}$ is a function of time only, *i.e.*, $\partial_1\bar{\pi}(t,x_1) = -h(t)$ for some function h. Inserting this into the system yields

$$\partial_t\psi - \partial_2^2\phi = \mu\partial_2^2\psi + h, \quad \text{in} \quad (0,d).$$

Finally, by the definition of ϕ it is $\psi(t,x_2) = \partial_t\phi(t,x_2)$ and moreover, the homogeneous Dirichlet boundary conditions for $\bar{u}$ carry over to ϕ, i.e. $\phi(t,0) = \phi(t,1) = 0$. At initial time we have $\phi(0,x_2) = 0$ and $\partial_t\phi(0,x_2) = \psi(0,x_2) = \psi_0(x_2)$ for some function ψ_0 that will be given satisfying homogeneous Dirichlet conditions.

With this, we end up with a viscous wave equation in one dimension

$$\begin{cases} \partial_t^2\phi - \partial_2^2\phi = \mu\partial_t\partial_2^2\phi + h, & \text{in} \quad [0,T)\times(0,d), \\ \phi(t,0) = \phi(t,d) = 0, & \text{for} \quad t\in(0,T), \\ \phi(0) = 0, \quad \partial_t\phi(0) = \psi_0, & \text{in} \quad (0,d). \end{cases} \tag{11}$$

For sufficiently regular data (ψ_0, h), this equation is uniquely solvable for all times, see e.g. [6, Proposition 3.1] and cf. [2], [3], [5, Exercise 2 and 3, pp. 582]. Inserting the function $\psi = \partial_t\phi$ into the ansatz for $\bar{u}$, we receive a solution $(\bar{u}, \bar{\zeta}, \bar{\pi})$ of the system

$$\begin{cases} \partial_t\bar{\zeta} + \bar{u}\cdot\nabla\bar{\zeta} = 0, & \text{in} \quad (0,T)\times\Omega_d, \\ \operatorname{div}\bar{u} = 0, & \text{in} \quad (0,T)\times\Omega_d, \\ \partial_t\bar{u} - \mu\Delta\bar{u} + \bar{u}\cdot\nabla\bar{u} + \nabla\bar{\pi} = -\Delta\bar{\zeta}_k\nabla\bar{\zeta}_k, & \text{in} \quad (0,T)\times\Omega_d, \\ \bar{u} = 0, & \text{on} \quad (0,T)\times\partial\Omega_d, \\ \bar{\zeta}(0) = x^\perp, & \text{for} \quad x\in\Omega_d, \\ \bar{u}(0) = (\psi_0, 0)^T, & \text{in} \quad \Omega_d. \end{cases}$$

Due to the homogeneous Dirichlet boundary conditions for $\bar{u}$ and the advective nature of the equation for $\bar{\zeta}$, it is $\bar{\zeta}|_{\partial\Omega_d} = x^\perp$ for all times.

1.2 Perturbation of the flow through the layer

It is our aim to examine the stability of system (8) (or equivalently (1)) with respect to the Poiseuille-type flow $(\bar{u}, \bar{\zeta}, \bar{\pi})$ constructed in the previous section. For this, we introduce the perturbation

$$(v, \alpha, p) = (u, \zeta, \pi) - (\bar{u}, \bar{\zeta}, \bar{\pi})$$

of the solution (u, ζ, π) to (8) around the Poiseuille-type flow $(\bar{u}, \bar{\zeta}, \bar{\pi})$. We denote by G the deformation tensor associated to ζ and by F the deformation tensor associated to α.

We are interested in solutions (u, ζ, π) that satisfy homogeneous Dirichlet boundary conditions $u|_{\partial\Omega_d} = 0$, and have initial values ζ_0 and u_0. Let u_0 satisfy the compatibility condition

$$\operatorname{div} u_0 = 0.$$

Let us moreover assume that the initial stream function satisfies

$$\zeta_0|_{\partial\Omega_d} = x^\perp \quad \text{and} \quad (\partial_1 \zeta_{0_1})(\partial_2 \zeta_{0_2}) - (\partial_1 \zeta_{0_2})(\partial_2 \zeta_{0_1}) = 1.$$

The first assumption together with the homogeneous Dirichlet boundary conditions for u guarantees $\zeta|_{\partial\Omega_d} = x^\perp$ for all times. The second assumption is a reformulation of the incompressibility condition $\det G_0 = 1$, which by (5) ensures $\det G = 1$ for all times. Therefore it holds

$$\begin{aligned} 1 = \det G = \det(F + \bar{F}) &= (F_{11} + 1)(F_{22} + 1) - (F_{21} + \partial_2\phi)F_{12} \\ &= \det F + \operatorname{tr} F + 1 - \partial_2\phi F_{12}, \end{aligned}$$

due to the structure of $\bar{F}$. Consequently

$$\det F = (\partial_2\phi)F_{12} - \operatorname{tr} F = (\partial_2\phi)\partial_1\alpha_1 + \partial_2\alpha_1 - \partial_1\alpha_2. \tag{12}$$

On the other hand

$$\det F = F_{11}F_{22} - F_{12}F_{21} = (\partial_1\alpha_2)\partial_2\alpha_2 - (\partial_2\alpha_1)\partial_1\alpha_2 \tag{13}$$

Putting (12) and (13) together implies

$$\operatorname{div} \alpha^\perp = \partial_2\alpha_1 - \partial_1\alpha_2 = \partial_1\alpha_1\partial_2\alpha_2 - \partial_2\alpha_1\partial_1\alpha_2 - \partial_2\phi\partial_1\alpha_1. \tag{14}$$

This quadratic structure of the divergence of $\alpha^\perp$ will be crucial later on in the application of a fixed point argument. Lastly, we assume that the perturbations v, α and p are periodic in the x_1-variable with some fixed period $L > 0$.

Then (v, α, p) solves

$$\begin{cases} \partial_t\alpha + v\cdot\nabla\alpha + \bar{u}\cdot\nabla\alpha = -v\cdot\nabla\bar{\zeta} & \text{in } (0,T)\times\Omega_d, \\ \operatorname{div} v = 0 & \text{in } (0,T)\times\Omega_d, \\ \partial_t v - \nu\Delta v + v\cdot\nabla v + v\cdot\nabla\bar{u} + \bar{u}\cdot\nabla v + \nabla p & \\ \quad = -\Delta\alpha_k\nabla\alpha_k - \Delta\bar{\zeta}_k\nabla\alpha_k - \Delta\alpha_k\nabla\bar{\zeta}_k & \text{in } (0,T)\times\Omega_d, \\ \alpha|_{\partial\Omega_d} = 0, & \text{in } (0,T), \\ v|_{\partial\Omega_d} = 0, & \text{in } (0,T), \\ \alpha(0) = \zeta_0(x) - x^\perp & \text{for } x \in \Omega_d, \\ v(0) = u_0 - (\psi_0, 0)^T & \text{in } \Omega_d. \end{cases} \tag{15}$$

1.3 Change of variables

Using the definition of $\bar\zeta$ in (10), we obtain

$$\nabla\bar\zeta = \begin{pmatrix} 0 & 1 \\ -1 & -\partial_2\phi(t,x_2)\end{pmatrix}, \quad \Delta\bar\zeta = \begin{pmatrix} 0 \\ -\partial_2^2\phi(t,x_2)\end{pmatrix}.$$

Writing $\alpha^\perp = \begin{pmatrix} -\alpha_2 \\ \alpha_1\end{pmatrix}$, we deduce

$$-\Delta\alpha_k\nabla\bar\zeta_k - \Delta\bar\zeta_k\nabla\alpha_k = \Delta\alpha^\perp + \nabla\phi\Delta\alpha_2 + \partial_2^2\phi\nabla\alpha_2.$$

If inserted into the momentum equation, this yields

$$\begin{aligned}\partial_t v - \nu\Delta(v + \frac{1}{\nu}\alpha^\perp) + v\cdot\nabla v + v\cdot\nabla\bar u + \bar u\cdot\nabla v + \nabla p \\ = -\Delta\alpha_k\nabla\alpha_k + \nabla\phi\Delta\alpha_2 + \partial_2^2\phi\nabla\alpha_2,\end{aligned}$$

which suggests to introduce a new variable w to replace v:

$$w := v + \frac{1}{\nu}\alpha^\perp. \tag{16}$$

The next step is to determine the right system that defines w. It is easy to see with

$$-v\cdot\nabla\bar\zeta = \begin{pmatrix} v_2 \\ -v_1 + \partial_2\phi v_2\end{pmatrix},$$

that $\frac{1}{\nu}\alpha^\perp$ satisfies

$$\partial_t\big(\frac{1}{\nu}\alpha^\perp\big) + v\cdot\nabla\big(\frac{1}{\nu}\alpha^\perp\big) + \bar u\cdot\nabla\big(\frac{1}{\nu}\alpha^\perp\big) = \frac{1}{\nu}v + \frac{1}{\nu}v_2\nabla\phi^\perp.$$

We add this equation to the system for v and insert the divergence relation (14). Then, after rescaling to $(0,T/d)\times\Omega$ with $\Omega := \mathbb{T}\times(0,1)$ (where we encode the periodicity of α and v in the horizontal variable by replacing the real line with a torus), we receive with the notation

$$K(t,x) := \begin{pmatrix} d & 0 \\ \partial_2\phi(t,x) & d\end{pmatrix}.$$

a new system for the variables α and w

$$\begin{cases} \partial_t\alpha + (\bar u + w + \frac{1}{\nu}\alpha^\perp)\cdot\nabla\alpha + \frac{1}{\nu}K\alpha = -dw^\perp + w_2\nabla\phi \\ \operatorname{div} w = \frac{1}{d\nu}(-\partial_1\alpha_1\partial_2\alpha_2 + \partial_2\alpha_1\partial_1\alpha_2 + \partial_2\phi\partial_1\alpha_1), \\ \partial_t w - \frac{\nu}{d}\Delta w + \nabla p = -\frac{1}{\nu}\alpha\cdot\nabla\bar u - \frac{1}{d^2}\Delta\alpha_k\nabla\alpha_k - \frac{1}{\nu^2}\alpha_1\nabla\phi^\perp \\ \qquad + \frac{1}{d^2}\partial_2^2\phi\nabla\alpha_2 - \frac{d}{\nu^2}\alpha^\perp + \frac{1}{d^2}\nabla\phi\Delta\alpha_2 \\ \qquad - \bar u\cdot\nabla w + \frac{1}{\nu}w_2\nabla\phi^\perp + \frac{d}{\nu}w - w\cdot\nabla w \\ \qquad + \frac{1}{\nu}\alpha^\perp\cdot\nabla w + w\cdot\nabla\bar u \end{cases} \tag{P}$$

in $(0, T/d) \times \Omega$, with boundary and initial conditions

$$\begin{cases} (w, \alpha) = (0, 0), & \text{on } (0, T/d) \times \partial\Omega, \\ \alpha(0) = \alpha_0 := \zeta_0(x) - x^\perp, & \text{in } \Omega. \\ w(0) = w_0 := u_0 - (\psi_0, 0)^T + \dfrac{1}{\nu} a_0^\perp, & \text{in } \Omega. \end{cases}$$

1.4 Formulation of the Main Result

We give our main result on strong stability in terms of function spaces that are introduced in Section 2.

Main Theorem. *Let $T \in (0, \infty]$, $p \in (4, \infty)$ and $\nu > 0$. There is $\tau > 0$ such that for all $d \in (0, \tau\nu)$ there are $\varepsilon, \delta, \kappa > 0$, such that if*

$$\|\bar{u}\|_{\mathbb{E}_\delta} + |||\nabla\phi, \nabla^2\phi, \nabla^3\phi||| \leq \kappa,$$

the perturbed Poiseuille problem (P) *with data $(\alpha_0, w_0) \in B_\varepsilon(0) \subseteq D(\Delta_p) \times \mathsf{Tr}_{\mathbb{E}}$ satisfying the compatibility condition*

$$\operatorname{div} w_0 = \frac{1}{\nu} \operatorname{div} \alpha_0^\perp \qquad \text{in } \mathsf{Tr}_{\mathbb{G}},$$

admits a unique small solution

$$(\alpha, w, \nabla p) \in \mathbb{A}_\delta \times \mathbb{E}_\delta \times \mathbb{F}_\delta.$$

Furthermore, there is a constant $c > 0$ such that for all such compatible

$$(\alpha_0, w_0), (\bar{\alpha}_0, \bar{w}_0) \in B_\varepsilon(0)$$

the weak estimate

$$\begin{aligned} \|\nabla\alpha - \nabla\bar{\alpha}\|_{\mathbb{F}^{p/2,p}} + &\|\nabla w - \nabla\bar{w}\|_{\mathbb{F}^{p/2,p}} \\ &\leq c(\|\nabla\alpha_0 - \nabla\bar{\alpha}_0\|_{L^p(\Omega)^2} + \|\nabla w_0 - \nabla\bar{w}_0\|_{\left(\mathsf{Tr}_{\mathbb{E}}^{(p/2)',p'}\right)'}) \end{aligned} \tag{17}$$

is valid.

Let us give some comments on the result. Formally, $\mathbb{E}_\delta$ controls the L^p-norm up to first time and second spatial derivatives, while $\mathbb{A}_\delta$ controls additionally the L^p norm of the mixed derivative $\partial_t\nabla$. One might be surprised that $\mathbb{A}_\delta$ does not control derivatives of the form $\partial_t\nabla^2$ (since only then the trace space of $\mathbb{A}_\delta$ would be given by $D(\Delta_q)$). This is because $\mathbb{A}_\delta$ has to be stable under transition between Eulerian and Lagrangian coordinates, see Proposition 5. On the other hand, control of the $\partial_t\nabla$ terms is necessary, since $\mathbb{A}_\delta$ has to be chosen such that one can make use of the quadratic structure of the divergence of w in problem (P), see Lemma 7. Moreover, one cannot expect a smoothing effect for α, which forces us to assume control of the full second derivatives for the initial value α_0.

Also, a comment on estimate (17) is in order. It is here that one of the main obstacles and at the same time one of the vital ingredients of our proof are visible in the result. Namely, a simple application of the contraction mapping principle is not possible due to the hyperbolic nature of the first equation in (P). Instead, we apply a fixed point argument where contraction has to be shown only in weak norms, see Lemma 1. These low norms we choose to be dual norms, since then we can apply the theory of very weak solutions, which can handle well inhomogeneous divergence data by design, see Proposition 3.

Our plan of the remaining part of this paper is as follows. After introducing the appropriate function spaces in Section 2, we develop the necessary linear theory in Section 3. We then treat the quasilinear transport equation in Lagrangian coordinates with a delicate fixed point argument in Section 4. Due to the dampening term in the transport equation we obtain exponential decay of α. Section 5 is devoted to estimating the corresponding nonlinearities. Finally, in Section 6, we use an argumentation similar to CLÉMENT-LI [4] in order to derive our result via a fixed point argument, using the quadratic structure of both the divergence and the right-hand side of the momentum equation.

2 Function Spaces

From now on, we will always assume $T \in (0,\infty]$, $r, q \in [1,\infty]$ and $p \in (4,\infty)$. The restriction to large p is due to embedding properties, see (18) below. Define the spaces

$$\begin{aligned}
\mathbb{A}^{r,q} &:= L^r(0,T; D(\Delta_q)) \cap W^{1,r}(0,T; W^{1,q}(\Omega)^2),\\
\mathbb{E}^{r,q} &:= L^r(0,T; D(\Delta_q)) \cap W^{1,r}(0,T; L^q(\Omega)^2),\\
\mathbb{F}^{r,q} &:= L^r(0,T; L^q(\Omega)^2),\\
\mathbb{G}^{r,q} &:= L^r(0,T; W^{1,q}(\Omega)) \cap W^{1,r}(0,T; W_0^{-1,q}(\Omega)),
\end{aligned}$$

where

$$D(\Delta_q) := W^{2,q}(\Omega)^2 \cap W_0^{1,q}(\Omega)^2,$$

and where $W_0^{-1,q}(\Omega)$ is the dual space of $W^{1,q'}(\Omega)$ with q' being the Hölder conjugate exponent of q. In analogy, the dual space of $W_0^{1,q'}(\Omega)$ will be denoted by $W^{-1,q}(\Omega)$. Moreover, if $r = q = p$, we simply write $\mathbb{A} := \mathbb{A}^{p,p}$ and similarly for the spaces $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$.

In fact, as we are aiming for exponential decay, we will work in time weighted spaces. More precisely, for $\delta > 0$ consider the spaces

$$\begin{aligned}
\mathbb{A}_\delta^{r,q} &:= \{\alpha \in \mathbb{A}^{r,q} : e^{t\delta}\alpha \in \mathbb{A}^{r,q}\}, & \|\alpha\|_{\mathbb{A}_\delta^{r,q}} &:= \|e^{t\delta}\alpha\|_{\mathbb{A}^{r,q}}.\\
\mathbb{E}_\delta^{r,q} &:= \{w \in \mathbb{E}^{r,q} : e^{t\delta}w \in \mathbb{E}^{r,q}\}, & \|w\|_{\mathbb{E}_\delta^{r,q}} &:= \|e^{t\delta}w\|_{\mathbb{E}^{r,q}},\\
\mathbb{F}_\delta^{r,q} &:= \{f \in \mathbb{F}^{r,q} : e^{t\delta}f \in \mathbb{F}^{r,q}\}, & \|f\|_{\mathbb{F}_\delta^{r,q}} &:= \|e^{t\delta}f\|_{\mathbb{F}^{r,q}},\\
\mathbb{G}_\delta^{r,q} &:= \{g \in \mathbb{G}^{r,q} : e^{t\delta}g \in \mathbb{G}^{r,q}\}, & \|g\|_{\mathbb{G}_\delta^{r,q}} &:= \|e^{t\delta}g\|_{\mathbb{G}^{r,q}}.
\end{aligned}$$

The corresponding trace spaces to $\mathbb{E}_\delta^{r,q}$ and $\mathbb{G}_\delta^{r,q}$ are given, respectively, by

$$\mathsf{Tr}_{\mathbb{E}}^{r,q} = (L^q(\Omega)^2, D(\Delta_q))_{1-1/r,r}, \quad \mathsf{Tr}_{\mathbb{G}}^{r,q} := (W_0^{-1,q}(\Omega), W^{1,q}(\Omega))_{1-1/r,r}.$$

Note that these trace spaces are independent of the exponential time weights; see, for example, [11, Remark 1.16]. Moreover, due to $p > 4$ we have

$$\mathbb{A}_\delta \hookrightarrow \mathbb{E}_\delta \hookrightarrow BUC([0,T); BUC^1(\bar{\Omega})^2), \qquad W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega). \tag{18}$$

For notational convenience, we will assume without loss of generality that the corresponding embedding constants are bounded by 1, which is possible since we will always think of the time $T > 0$ being large. Furthermore, for finite T, $\mathbb{A}^{r,q} = \mathbb{A}^{r,q}_\delta$, $\mathbb{E}^{r,q} = \mathbb{E}^{r,q}_\delta$, $\mathbb{F}^{r,q} = \mathbb{F}^{r,q}_\delta$ and $\mathbb{G}^{r,q} = \mathbb{G}^{r,q}_\delta$ with equivalent norms. For brevity, set for $f \in \mathbb{F}^{\infty,\infty}$

$$|||f||| := \|f\|_{\mathbb{F}^{\infty,\infty}}.$$

In order to treat our problem at hand, we will want to use a fixed point argument. As mentioned in the introduction, it turns out that we cannot show contraction in the high norms that one expects for the solutions and therefore need a suitable variant of the Banach fixed point theorem, see [9, Lemma 2.5].

Lemma 1. *Let X be a reflexive Banach space or let X have a separable pre-dual. Let H be a nonempty, convex, closed and bounded subset of X and let $X \hookrightarrow Y$, where Y is a Banach space. Let $T : X \to X$ map H into H and let there be $\rho < 1$ such that for all $u, v \in H$ we have a contraction in the lower norms, that is*

$$\|Tu - Tv\|_Y \leq \rho \|u - v\|_Y.$$

Then there exists a unique fixed point of T in H.

3 Solvability of a Generalized Stokes Problem

In this section we investigate the following generalized linear Stokes problem

$$\begin{cases} \partial_t w - \Delta w + \nabla p &= f, \quad \text{in } (0,T) \times \Omega, \\ \operatorname{div} w &= g, \quad \text{in } (0,T) \times \Omega, \\ w|_{\partial\Omega} &= 0, \quad \text{on } (0,T) \times \partial\Omega, \\ w|_{t=0} &= w_0 \quad \text{on } \Omega. \end{cases} \tag{S}$$

We use the analysis of the partially periodic Stokes operator and its reduced counterpart that has been carried out in [13] to treat this system.

Proposition 2. *Let $r, q \in (1,\infty)$. There is $\delta_0 > 0$ such that for every $0 \leq \delta \leq \delta_0$ Problem* (S) *admits a unique solution $(w, \nabla p) \in \mathbb{E}^{r,q}_\delta \times \mathbb{F}^{r,q}_\delta$ if*

$$f \in \mathbb{F}^{r,q}_\delta, \quad g \in \mathbb{G}^{r,q}_\delta, \quad w_0 \in \mathsf{Tr}^{r,q}_{\mathbb{E}},$$

satisfy the compatibility condition

$$\operatorname{div} w_0 = g(0) \qquad \text{in } \mathsf{Tr}^{r,q}_{\mathbb{E}}.$$

Moreover, there is an $M > 0$ such that

$$\|w\|_{\mathbb{E}^{r,q}_\delta} + \|\nabla p\|_{\mathbb{F}^{r,q}_\delta} \leq M(\|w_0\|_{\mathsf{Tr}^{r,q}_{\mathbb{E}}} + \|f\|_{\mathbb{F}^{r,q}_\delta} + \|g\|_{\mathbb{G}^{r,q}_\delta}). \tag{19}$$

Proof. For $\delta = 0$, the assertion is contained in [13, Theorem 1.3]. We note that as a particular instance, this shows that the (partially periodic) Stokes operator A with $D(A) = X_1 \cap L^p_\sigma(\Omega)$ – the closure of divergence-free test functions in $L^p(\Omega)^2$ – possesses maximal L^p regularity on $(0,T)$ for all $0 < T \leq \infty$. By [13, Theorem 1.4], A is invertible, and so maximal L^p regularity remains valid even for the slightly shifted operator $A - \delta_0$ for some small $\delta_0 > 0$.

Let now $0 \leq \delta \leq \delta_0$. Observe that since $(e^{t\delta} f, e^{t\delta} g, w_0) \in \mathbb{F} \times \mathbb{G} \times \mathsf{Tr}^{r,q}_{\mathbb{E}}$, the result in [13] gives a unique solution $(v, p_v) \in \mathbb{E} \times \mathbb{F}$ to

$$\begin{cases} \partial_t v - \Delta v + \nabla p_v &= e^{t\delta} f, \quad \text{in } (0,T) \times \Omega, \\ \operatorname{div} v &= e^{t\delta} g, \quad \text{in } (0,T) \times \Omega, \\ v|_{\partial\Omega} &= 0, \quad \text{on } (0,T) \times \partial\Omega, \\ v|_{t=0} &= w_0 \quad \text{on } \Omega, \end{cases}$$

with a corresponding estimate

$$\|v\|_{\mathbb{E}^{r,q}} + \|\nabla p_v\|_{\mathbb{F}^{r,q}} \leq C(\|w_0\|_{\mathsf{Tr}^{r,q}_{\mathbb{E}}} + \|f\|_{\mathbb{F}^{r,q}_\delta} + \|g\|_{\mathbb{G}^{r,q}_\delta}). \tag{20}$$

On the other hand, since $A - \delta_0$ possesses maximal L^p regularity,

$$\begin{cases} \partial_t u - \Delta u - \delta u + \nabla p_u &= \delta v, \quad \text{in } (0,T) \times \Omega, \\ \operatorname{div} u &= 0, \quad \text{in } (0,T) \times \Omega, \\ u|_{\partial\Omega} &= 0, \quad \text{on } (0,T) \times \partial\Omega, \\ u|_{t=0} &= 0, \quad \text{on } \Omega \end{cases}$$

is uniquely solvable for every $0 \leq \delta < \delta_0$ and there is a $C > 0$ such that (without loss of general $\delta_0 \leq 1$)

$$\|u\|_{\mathbb{E}^{r,q}} + \|\nabla p_u\|_{\mathbb{F}^{r,q}} \leq C\|v\|_{\mathbb{F}^{r,q}} \leq C\|v\|_{\mathbb{E}^{r,q}}. \tag{21}$$

But then

$$w := e^{-t\delta}(v + u), \quad \nabla p := e^{-t\delta}(\nabla p_v + \nabla p_u)$$

is the unique solution to (S) and by (20) and (21) there is a $C > 0$ such that

$$\begin{aligned} \|w\|_{\mathbb{E}^{r,q}_\delta} + \|\nabla p\|_{\mathbb{F}^{r,q}_\delta} &\leq \|u\|_{\mathbb{E}^{r,q}} + \|v\|_{\mathbb{E}^{r,q}} + \|\nabla p_u\|_{\mathbb{F}^{r,q}} + \|\nabla p_v\|_{\mathbb{F}^{r,q}} \\ &\leq C\|v\|_{\mathbb{E}^{r,q}} + \|\nabla p_v\|_{\mathbb{F}^{r,q}} \\ &\leq C(\|w_0\|_{\mathsf{Tr}^{r,q}_{\mathbb{E}}} + \|f\|_{\mathbb{F}^{r,q}_\delta} + \|g\|_{\mathbb{G}^{r,q}_\delta}). \end{aligned}$$

This is the assertion. □

Besides strong solutions to (S) with corresponding estimates, Lemma 1 suggests that weaker notions of estimates have to be investigated as well. Here, it turns out that it is convenient to work with a dual version of estimate (19). These dual estimates are closely related to the concept of very weak solutions, as pointed out by Schumacher [14, 15]. Therefore, we usually refer to them as very weak estimates.

Proposition 3. *In the situation of Proposition 2 with $\delta > 0$, it holds for all $s \in (1, r]$*

$$f \in L^s(0,T;D(\Delta_{q'})'), \quad g \in L^s(0,T;W_0^{-1,q}(\Omega)^2), \quad w_0 \in (\mathsf{Tr}_{\mathbb{E}}^{s',q'})',$$

and we have the estimate

$$\|w\|_{\mathbb{F}^{s,q}} \leq M(\|w_0\|_{\left(\mathsf{Tr}_{\mathbb{E}}^{s',q'}\right)'} + \|f\|_{L^s(0,T;D(\Delta_{q'})')} + \|g\|_{L^s(0,T;W_0^{-1,q}(\Omega)^2)}). \tag{22}$$

Proof. Let us first show the estimate for $T < \infty$. Let $v \in (\mathbb{F}^{s,q})^* = \mathbb{F}^{s',q'}$ and use Proposition 2 to obtain a unique solution

$$(\varphi, \nabla\psi) \in \mathbb{E}^{s',q'} \times \mathbb{F}^{s',q'}$$

to the backwards-in-time problem

$$\begin{cases} -\partial_t\varphi - \Delta\varphi - \nabla\psi &= v, \quad \text{in } (0,T) \times \Omega, \\ \operatorname{div}\varphi &= 0, \quad \text{in } (0,T) \times \Omega, \\ \varphi|_{\partial\Omega} &= 0, \quad \text{on } (0,T) \times \partial\Omega, \\ \varphi|_{t=T} &= 0 \quad \text{on } \Omega. \end{cases}$$

This solution enjoys the estimate

$$\|\varphi\|_{\mathbb{E}^{s',q'}} + \|\nabla\psi\|_{\mathbb{F}^{s',q'}} \leq M\|v\|_{\mathbb{F}^{s',q'}},$$

which we will use in the weaker form

$$\|\varphi(0)\|_{\mathsf{Tr}_{\mathbb{E}}^{s',q'}} + \|\varphi\|_{L^{s'}(D(\Delta_{q'}))} + \|\nabla\psi\|_{\mathbb{F}^{s',q'}} \leq M\|v\|_{\mathbb{F}^{s',q'}}.$$

This latter estimate is indeed weaker, since $\mathsf{Tr}_{\mathbb{E}}^{s',q'}$ is the trace space of $\mathbb{E}^{s',q'}$. With this decomposition, we note that w satisfies

$$\begin{aligned}(w,v)_{T,\Omega} &= -(w,\partial_t\varphi)_{T,\Omega} - (w,\Delta\varphi)_{T,\Omega} - (w,\nabla\psi)_{T,\Omega} \\ &= (f,\varphi)_{T,\Omega} + (w_0,\varphi(0))_\Omega + (g,\psi)_{T,\Omega},\end{aligned}$$

whence the estimate (22) readily follows.

If $T = \infty$, we choose a function $v \in L^{r'}(0,T;L^{p'}(\Omega)^2)$ with $\operatorname{supp} v \subset (0,N) \times \overline{\Omega}$ for some $N \in \mathbb{N}$ and let $N \to \infty$, see [14, Theorem 9.2.1] for details. □

4 Transport equation

To treat the first equation in the Perturbation Equation (P), we work in Lagrangian coordinates. For a fixed velocity field $\widetilde{u}$, we consider $x = x(t,\xi)$ to be the solution of

$$\frac{d}{dt}x = \widetilde{u}(t,x), \quad t \geq 0, \quad x_{|t=0} = \xi.$$

The transformation $x = x(t,\xi)$ connects the Eulerian coordinate $x = (x_1, x_2)$ and the Lagrangian coordinate $\xi = (\xi_1, \xi_2)$ of the same fluid particle. Moreover,

$$x = \xi + \int_0^t u(s,\xi)\,\mathrm{d}s =: X_u(t,\xi),$$

where $u(t,\xi) := \widetilde{u}(t, X_u(t,\xi))$. If $\widetilde{u}$ is Lipschitz in x and $\int_0^t \|\nabla\widetilde{u}\|_\infty \,\mathrm{d}s < \infty$ for $t \geq 0$, then $x = X_u(t,\xi)$ is well defined for any $t \geq 0$, $\xi \in \Omega$.

We would like to show that for small drifts u, the spaces $\mathbb{F}$, $\mathbb{E}$ and $\mathbb{A}$ are stable under transformation between Eulerian and Lagrangian coordinates. To this end, we first prove the following lemma on multiplicative estimates.

Lemma 4. *Let $\delta > 0$ and assume we are given two functions f and g such that $f \in \mathbb{F}_\delta$ and $g \in \mathbb{F}^{p,\infty}$. Then $h \in \mathbb{F}$, where*

$$h(t,x) := g(t,x) \int_0^t f(s,x)\ \mathrm{d}s,$$

and it holds the estimate

$$\|h\|_{\mathbb{F}} \leq \frac{1}{(\delta p')^{1/p'}} \|f\|_{\mathbb{F}_\delta} \|g\|_{\mathbb{F}^{p,\infty}}. \tag{23}$$

Proof. We observe

$$\|h\|_{\mathbb{F}} \leq \| \int_0^t f\ \mathrm{d}s \|_{\mathbb{F}^{\infty,p}} \|g\|_{\mathbb{F}^{p,\infty}}.$$

With Minkowski's inequality we obtain for the first factor

$$\begin{aligned} \| \int_0^t f\ \mathrm{d}s\|_{\mathbb{F}^{\infty,p}} &= \operatorname*{ess\,sup}_{t\in(0,T)} |(\int_\Omega | \int_0^t f\ \mathrm{d}s|^p\,\mathrm{d}x)^{\frac{1}{p}}| \leq \int_0^T \|f\|_{L^p(\Omega)^2}\,\mathrm{d}s \\ &= \int_0^T e^{-\delta s} \|e^{\delta s} f\|_{L^p(\Omega)^2}\,\mathrm{d}s \leq \frac{1}{(\delta p')^{1/p'}} \|f\|_{\mathbb{F}_\delta}, \end{aligned}$$

which is the assertion. □

Proposition 5. *Let $T \in (0,\infty]$, $r, q \in [1,\infty]$, $p \in (4,\infty)$ and $\delta > 0$. Assume that $u \in \mathbb{E}_\delta$ is a given velocity. There is a $\sigma > 0$ such that whenever $\|u\|_{\mathbb{E}_\delta} \leq \sigma\delta$, the coordinate transformation*

$$(t,\xi) \mapsto \Phi_u(t,\xi) := (t, X_u(t,\xi)) \tag{24}$$

for $t \in [0,T), \xi \in \Omega$, yields for $\lambda \geq 0$ the homeomorphism

$$Y_\lambda \to Y_\lambda : v \mapsto \hat{v} := v \circ \Phi_u,$$

where Y_λ may be any of the spaces $\mathbb{F}^{r,q}_\lambda$, $\mathbb{E}_\lambda$ or $\mathbb{A}_\lambda$.

Proof. Let $A = A(t,\xi)$ denote the spatial Jacobian of X_u, then, for $i,j = 1,2$, its components read $A_{ij} = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial \xi_j}(s,\xi)ds$. Hence, choosing $\sigma > 0$ small enough, $A(t,\xi)$ is a perturbation of the identity for all times $t \leq T$, since

$$\int_0^t \|\frac{\partial u_i}{\partial \xi_j}(s,\cdot)\|_\infty \,\mathrm{d}s \leq \delta^{-1}\|e^{t\delta}u\|_{\mathbb{E}} < \sigma. \tag{25}$$

Furthermore, $A^{-1} = I + V_0(\int_0^t \nabla_\xi u(s,\xi)ds)$ holds for some C^∞-function V_0 defined on matrices $K \in \mathbb{R}^{2\times 2}$, $|K| < 2\sigma$.

Consequently, for $f \in \mathbb{F}_\lambda^{r,q}$, there is a $C > 0$ such that

$$\begin{aligned} \|\hat{f}\|_{\mathbb{F}_\lambda^{r,q}} &= \left(\int_0^T \left(\int_\Omega |e^{t\lambda} f(t, X_u(t,\xi))|^q \,\mathrm{d}\xi\right)^{\frac{r}{q}} \mathrm{d}t\right)^{\frac{1}{r}} \\ &= \left(\int_0^T \left(\int_\Omega |e^{t\lambda} f(t,x))|^q |\det A^{-1}| \,\mathrm{d}x\right)^{\frac{r}{q}} \mathrm{d}t\right)^{\frac{1}{r}} \leq C\|f\|_{\mathbb{F}_\lambda^{r,q}}, \end{aligned} \tag{26}$$

since Φ_u preserves $[0,T]\times\Omega$ by the boundary condition. Similarly,

$$\|f \circ \Phi_u^{-1}\|_{\mathbb{F}_\lambda^{r,q}} \leq C\|f\|_{\mathbb{F}_\lambda^{r,q}}.$$

As we do not only want to estimate f by $\hat{f}$ and vice versa in $\mathbb{F}_\lambda$, but in the higher order spaces $\mathbb{E}_\lambda$ and $\mathbb{A}_\lambda$, we have to compute also higher order derivatives. Formally, assuming sufficient regularity of f, the following expressions hold

$$\partial_t \hat{f} = [\partial_t f + \widetilde{u}\cdot\nabla_x f]\circ\Phi_u, \tag{27}$$

$$\nabla_\xi \hat{f} = A[\nabla_x f\circ\Phi_u] = \big(I + \int_0^t \nabla_\xi u \,\mathrm{d}s\big)\nabla_x f\circ\Phi_u, \tag{28}$$

$$\nabla_\xi^2 \hat{f} = \big(\int_0^t \nabla_\xi^2 u \,\mathrm{d}s\big)\nabla_x f\circ\Phi_u + \big(I + \int_0^t \nabla_\xi u \,\mathrm{d}s\big)^2 \nabla_x^2 f\circ\Phi_u, \tag{29}$$

$$\partial_t\nabla_\xi \hat{f} = \nabla_\xi u(t,\xi)\nabla_x f\circ\Phi_u + \big(I + \int_0^t \nabla_\xi u \,\mathrm{d}s\big)(\partial_t\nabla_x f + \widetilde{u}\cdot\nabla_x^2 f)\circ\Phi_u. \tag{30}$$

By the invertibility of Φ_u we see $|||\widetilde{u}||| = |||u||| \leq \|u\|_{\mathbb{E}_\delta} < \infty$. It hence follows with (26) and the embedding (18)

$$\begin{aligned} &\|\partial_t(e^{\lambda t}\hat{f})\|_{\mathbb{F}} \leq C(\|\partial_t(e^{\lambda t} f)\|_{\mathbb{F}} + |||u|||\|\nabla(e^{\lambda t} f)\|_{\mathbb{F}}) \leq C\|f\|_{\mathbb{E}_\lambda}, \\ &\|\nabla(e^{\lambda t}\hat{f})\|_{\mathbb{F}} \leq C\|\nabla(e^{\lambda t} f)\|_{\mathbb{F}} \leq C\|f\|_{\mathbb{E}_\lambda}, \end{aligned}$$

and similarly

$$\begin{aligned} \|\partial_t\nabla(e^{\lambda t}\hat{f})\|_{\mathbb{F}} &\leq C(|||\nabla u|||\|\nabla(e^{\lambda t} f)\|_{\mathbb{F}} + \|\partial_t\nabla e^{\lambda t} f\|_{\mathbb{F}} + |||u|||\|\nabla^2(e^{\lambda t} f)\|_{\mathbb{F}}) \\ &\leq C\|f\|_{\mathbb{A}_\lambda}, \end{aligned}$$

In order to estimate also the expression in (29), we use Lemma 4 to obtain

$$\begin{aligned}\|\Big(\int_0^t \nabla_\xi^2 u \,\mathrm{d}s\Big)\nabla(e^{\lambda t} f)\|_{\mathbb{F}} &\le \frac{1}{\delta^{1/p'}}\|\nabla^2(e^{\delta t}u)\|_{\mathbb{F}}\|\nabla(e^{\lambda t} f)\|_{\mathbb{F}^{p,\infty}} \\ &\le \sigma\delta^{1/p}\|\nabla^2(e^{\lambda t} f)\|_{\mathbb{F}}.\end{aligned}$$

Therefore, we have that both transformations are linear and continuous and, in particular, the following estimates hold

$$\begin{aligned}\|\hat{v}\|_{\mathbb{E}_\lambda} \le C_\Phi \|v\|_{\mathbb{E}_\lambda}, \quad \|\check{w}\|_{\mathbb{E}_\lambda} \le C_\Phi \|w\|_{\mathbb{E}_\lambda}, \\ \|\hat{\beta}\|_{\mathbb{A}_\lambda} \le C_\Phi \|\beta\|_{\mathbb{A}_\lambda}, \quad \|\check{\gamma}\|_{\mathbb{A}_\lambda} \le C_\Phi \|\gamma\|_{\mathbb{A}_\lambda},\end{aligned} \tag{31}$$

where $C_\Phi > 0$. □

We consider the quasilinear problem

$$\begin{cases} \partial_t \alpha + (b + \frac{1}{\nu}\alpha^\perp)\cdot\nabla\alpha + \frac{1}{\nu}K\alpha &= f, \quad t \in (0,T) \\ \alpha(0) &= \alpha_0, \end{cases} \tag{T}$$

with $\alpha_0 \in D(\Delta_p)$, $b \in \mathbb{E}_\delta$ and $f \in \mathbb{E}_\delta$, and where $K : (0,T)\times\Omega \to \mathbb{R}^{2\times 2}$ denotes the matrix-valued function

$$K(t,x) := \begin{pmatrix} d & 0 \\ \partial_2\phi(t,x) & d \end{pmatrix}.$$

Proposition 6. *Let $T \in (0,\infty]$, $p \in (4,\infty)$, $d,\nu > 0$ and $\delta \in (0, \frac{d}{2\nu})$. Assume $\|\nabla\phi, \nabla^2\phi, \nabla^3\phi\| < \infty$. Furthermore, suppose $\alpha_0 \in D(\Delta_p)$, $b \in \mathbb{E}_\delta$ and $f \in \mathbb{E}_\delta$. There is an $\varepsilon > 0$ such that whenever*

$$\|\alpha_0\|_{D(\Delta_p)} + \|b\|_{\mathbb{E}_\delta} + \|f\|_{\mathbb{E}_\delta} \le \varepsilon, \tag{32}$$

then problem (T) has a unique small solution $\alpha \in \mathbb{A}_\delta$. Moreover, there is $c > 0$ such that

$$\|\alpha\|_{\mathbb{A}_\delta} \le c(\|\alpha_0\|_{D(\Delta_p)} + \|f\|_{\mathbb{E}_\delta}). \tag{33}$$

If $(\bar{\alpha}_0, \bar{b}, \bar{f})$ is another set of data satisfying (32), then for $q \in \{\frac{p}{2}, p\}$ and $r \in [q', q]$ there is a $c > 0$ such that the corresponding solutions α and $\bar{\alpha}$ fulfill the lower order Lipschitz assertion

$$\|\alpha - \bar{\alpha}\|_{\mathbb{F}^{q,p}} \le c(\|\alpha_0 - \bar{\alpha}_0\|_{L^p(\Omega)^2} + \|f - \bar{f}\|_{\mathbb{F}^{r,p}} + \|b - \bar{b}\|_{\mathbb{F}^{r,p}}). \tag{34}$$

Proof. **Step 1:** Linear transport in Lagrangian coordinates
Formally, in Lagrangian coordinates with fixed velocity $u \in \mathbb{E}^\delta$, the transport problem (T) reads

$$\begin{cases} \partial_t \hat{\alpha} + \frac{1}{\nu}\hat{K}\hat{\alpha} &= \hat{f}, \quad t \in (0,T) \\ \hat{\alpha}(0) &= \alpha_0, \end{cases} \tag{$\hat{T}$}$$

where $\hat{f} \in \mathbb{E}_\delta$. By standard methods, there is a unique global-in-time solution of $(\hat{T})$ given by

$$\begin{aligned}\hat{\alpha}_1(t,\xi) &= e^{-t\frac{d}{\nu}}\alpha_{0,1}(\xi) + \int_0^t e^{-(t-\tau)\frac{d}{\nu}}\hat{f}_1(\tau,\xi)d\tau,\\ \hat{\alpha}_2(t,\xi) &= e^{-dt/\nu}\alpha_{0,2}(\xi) + \int_0^t e^{-d(t-\tau)/\nu}(\hat{f}_2(\tau,\xi) - \hat{\alpha}_1(\tau,\xi)(\widehat{\partial_2\phi})(\tau,\xi))d\tau,\end{aligned} \tag{35}$$

for any $t > 0$. Thus, we first obtain an estimate on α_1, $\partial_t\alpha_1$, $\nabla\alpha_1$, $\nabla^2\alpha_1$ and $\partial_t\nabla\alpha_1$ in terms of f and can subsequently use this information to also bound α_2 by using the assumption on ϕ. Therefore, there is a $C > 0$ such that

$$\|\hat{\alpha}\|_{\mathbb{A}} \le C(\|\alpha_0\|_{D(\Delta_p)} + \|\hat{f}\|_{\mathbb{E}}).$$

Similarly, if we look at

$$\begin{cases} \partial_t\hat{\beta} + (\frac{d}{\nu} - \delta)\hat{\beta} &= e^{\delta t}\hat{f}, \quad t \in (0,T) \\ \hat{\beta}(0) &= \alpha_0, \end{cases} \tag{$\hat{T}_\delta$}$$

we obtain a unique solution $\hat{\beta} \in \mathbb{A}$ with $\|\hat{\beta}\|_{\mathbb{A}} \le \tilde{C}(\|\alpha_0\|_{X_1} + \|e^{\delta t}\hat{f}\|_{\mathbb{E}})$. By uniqueness $\hat{\alpha} = e^{-t\delta}\hat{\beta}$, and so

$$\|\hat{\alpha}\|_{\mathbb{A}^\delta} \le \tilde{C}(\|\alpha_0\|_{D(\Delta_p)} + \|\hat{f}\|_{\mathbb{E}^\delta}).$$

Note that $\tilde{C}$ does not depend on δ since $\frac{d}{\nu} - \delta$ is bounded away from 0. In total we obtain a bounded linear solution operator $L : D(\Delta_p) \times \mathbb{E}^\delta \to \mathbb{A}^\delta$, with $L(\alpha_0, \hat{f}) = \hat{\alpha}$ and with bound $C_L := \tilde{C} > 0$.

Step 2: Pull-back to Eulerian coordinates

Let $H \subset \mathbb{A}_\delta \times \mathbb{E}_\delta$ be defined by

$$H := \{(\alpha,u) \in \mathbb{A}_\delta \times \mathbb{E}_\delta : \|\alpha\|_{\mathbb{A}_\delta} \le h := \frac{\nu\delta\sigma}{4(1+C_\Phi)}, \|u\|_{\mathbb{E}_\delta} \le \frac{\delta\sigma}{2}\},$$

where $\sigma > 0$ is chosen as in Proposition 5 and C_Φ is the constant from (31). We consider the mapping $N : H \to H$ defined via

$$N(\alpha,u) := \begin{pmatrix} N_1(u) \\ N_2(\alpha,u)\end{pmatrix} := \begin{pmatrix} (L(\alpha_0, f\circ\Phi_u))\circ\Phi_u^{-1} \\ (b + \frac{1}{\nu}\alpha^\perp)\circ\Phi_u\end{pmatrix}.$$

We would like to use Lemma 1 to obtain a fixed point of this mapping. First, we check that for sufficiently small $\varepsilon > 0$, N is a self-mapping on H. Indeed, since $\|u\|_{\mathbb{E}_\delta} \le \sigma\delta$, we can make use of Proposition 5 to see

$$\begin{aligned}\|N_1(u)\|_{\mathbb{A}_\delta} &\le C_\Phi C_L(\|\alpha_0\|_{D(\Delta_p)} + C_\Phi\|f\|_{\mathbb{E}_\delta}) \le C_\Phi(1+C_\Phi)C_L\varepsilon \le h,\\ \|N_2(\alpha,u)\|_{\mathbb{E}_\delta} &\le C_\Phi\big(\varepsilon + \frac{1}{\nu}\frac{\nu\delta\sigma}{4(1+C_\Phi)}\big) \le \frac{\delta\sigma}{2}.\end{aligned} \tag{36}$$

By Lemma 1, contraction needs to be shown only in the lower order Banach space $\mathbb{F}_\delta \times_\nu \mathbb{F}_\delta$ with norm

$$\|\alpha, u\|_{\mathbb{F}_\delta \times_\nu \mathbb{F}_\delta} := \|\alpha\|_{\mathbb{F}_\delta} + \frac{\nu}{2C_\Phi}\|u\|_{\mathbb{F}_\delta}.$$

We claim that this is true, *i.e.*, there exists a $\rho < 1$ such that for $(\alpha, u), (\beta, v) \in H$

$$\|N(\alpha, u) - N(\beta, v)\|_{\mathbb{F}_\delta \times_\nu \mathbb{F}_\delta} \leq \rho\|(\alpha - \beta, u - v)\|_{\mathbb{F}_\delta \times_\nu \mathbb{F}_\delta}. \tag{37}$$

Indeed, for the N_2 part we note that for a sufficiently regular function g it holds for all $(t, \xi) \in [0, T] \times \Omega$ the estimate

$$\begin{aligned} |g(t,X_u(t,\xi)) - g(t, X_v(t,\xi))| &\leq |X_u(t,\xi) - X_v(t,\xi)| \sup_{x\in\Omega} |\nabla g(t,x)| \\ = |\int_0^t u(s,\xi) - v(s,\xi) \, \mathrm{d}s| \sup_{x\in\Omega} |\nabla g(t,x)| \end{aligned}$$

and so by Lemma 4 and the Sobolev embedding $W^{1,p}(\Omega)^2 \hookrightarrow L^\infty(\Omega)^2$, we have

$$\begin{aligned} \|g \circ \Phi_u - g \circ \Phi_v\|_{\mathbb{F}_\delta} &\leq \frac{1}{(\delta p')^{1/p'}}\|u - v\|_{\mathbb{F}_\delta}\|e^{\delta t}\nabla g\|_{\mathbb{F}^{p,\infty}} \\ &\leq \frac{C}{(\delta p')^{1/p'}}\|u - v\|_{\mathbb{F}_\delta}(\|\nabla g\|_{\mathbb{F}_\delta} + \|\nabla^2 g\|_{\mathbb{F}_\delta}) \\ &\leq \frac{C}{(\delta p')^{1/p'}}\|u - v\|_{\mathbb{F}_\delta}\|g\|_{\mathbb{E}_\delta}. \end{aligned}$$

Choosing $g := b - \frac{1}{\nu}\alpha^\perp$ such that $\|g\|_{\mathbb{E}_\delta} = \|b - \frac{1}{\nu}\alpha^\perp\|_{\mathbb{E}_\delta} \leq \varepsilon + \frac{1}{\nu}\frac{\nu\delta\sigma}{4(1+C_\Phi)} \leq \frac{\delta\sigma}{2}$, we can estimate

$$\begin{aligned} \|N_2(\alpha, u) - N_2(\beta, v)\|_{\nu\mathbb{F}_\delta} &\leq \|N_2(\alpha, u) - N_2(\alpha, v)\|_{\nu\mathbb{F}_\delta} + \|N_2(\alpha, v) - N_2(\beta, v)\|_{\nu\mathbb{F}_\delta} \\ &\leq C\delta^{\frac{1}{p}}\frac{\sigma}{2p'}\|u - v\|_{\nu\mathbb{F}_\delta} + \frac{C_\Phi}{\nu}\|\alpha - \beta\|_{\nu\mathbb{F}_\delta} \\ &= C\delta^{\frac{1}{p}}\frac{\sigma}{2p'}\|u - v\|_{\nu\mathbb{F}_\delta} + \frac{1}{2}\|\alpha - \beta\|_{\mathbb{F}_\delta} \\ &\leq \rho\|(\alpha - \beta, u - v)\|_{\mathbb{F}_\delta \times_\nu \mathbb{F}_\delta}. \end{aligned}$$

For the N_1 part, we observe that $\gamma := N_1(u) - N_1(v)$ solves by construction

$$\begin{cases} \partial_t\gamma + \widetilde{u} \cdot \nabla\gamma + \frac{1}{\nu}K\gamma &= (\tilde{v} - \widetilde{u}) \cdot \nabla N_1(v), \quad t \in (0, T) \\ \gamma(0) &= 0. \end{cases}$$

Therefore, appealing again to the Lagrangian analysis and in particular to formula (35), we see

$$\begin{aligned} \|\gamma\|_{\mathbb{F}_\delta} &\leq C_\Phi\|\gamma \circ \Phi_u\|_{\mathbb{F}_\delta} \leq C_\Phi\|[(\widetilde{u} - \tilde{v}) \cdot \nabla N_1(v)] \circ \Phi_u\|_{\mathbb{F}_\delta} \\ &\leq C_\Phi^2\|u - v\|_{\mathbb{F}_\delta}|||\nabla N_1(v)||| \leq C_\Phi^2\|u - v\|_{\mathbb{F}_\delta}\|N_1(v)\|_{\mathbb{A}_\delta} \\ &\leq C_\Phi^2\frac{\nu\delta\sigma}{4(1 + C_\Phi)}\|u - v\|_{\mathbb{F}_\delta} \leq \rho\|u - v\|_{\nu\mathbb{F}_\delta}, \end{aligned}$$

where we have used $\|N_1(v)\|_{\mathbb{A}_\delta} \le \frac{\nu\delta\sigma}{4(1+C_\Phi)} \le \frac{\delta\sigma}{2}\frac{\nu}{2C_\Phi}$. Hence, (37) follows and by Lemma 1 we obtain a unique fixed point $(\alpha, u) \in H$ of the mapping N. In other words, α is the unique small solution to (T), and as in (36) we learn

$$\|\alpha\|_{\mathbb{A}_\delta} = \|N_1(\alpha, u)\|_{\mathbb{A}_\delta} \le C_\Phi C_L(\|\alpha_0\|_{D(\Delta_p)} + C_\Phi\|f\|_{\mathbb{E}_\delta}),$$

which is (33).

It is left to show the Lipschitz assertion (34). We define $\gamma := \alpha - \bar{\alpha}$ and observe that it solves

$$\begin{cases} \partial_t\gamma + (b + \frac{1}{\nu}\alpha^\perp)\cdot\nabla\gamma + \frac{1}{\nu}K\gamma &= f - \bar{f} + (b - \bar{b} + \frac{1}{\nu}\gamma^\perp)\cdot\nabla\bar{\alpha}, \\ \gamma(0) &= \alpha_0 - \bar{\alpha}_0. \end{cases}$$

Let $u \in \mathbb{E}_\delta$ be the drift associated to α, *i.e.*, such that $(\alpha, u) \in H$ and $N(\alpha, u) = (\alpha, u)$. Then using once again formula (35), we can leverage upon the dampening term to change the time integrability and obtain

$$\begin{aligned} \|\gamma\|_{\mathbb{F}^{q,p}} &\le c\|\gamma\circ\Phi_u\|_{\mathbb{F}^{q,p}} \\ &\le c\big(\|\alpha_0 - \bar{\alpha}_0\|_{L^p(\Omega)^2} + \|[f - \bar{f}]\circ\Phi_u\|_{\mathbb{F}^{r,p}} \\ &\qquad + \|[(b - \bar{b})\cdot\nabla\bar{\alpha}]\circ\Phi_u\|_{\mathbb{F}^{r,p}} + \|[\frac{1}{\nu}\gamma^\perp\cdot\nabla\bar{\alpha}]\circ\Phi_u\|_{\mathbb{F}^{q,p}}\big) \\ &\le c\big(\|\alpha_0 - \bar{\alpha}_0\|_{L^p(\Omega)^2} + \|f - \bar{f}\|_{\mathbb{F}^{r,p}} + \big(\|b - \bar{b}\|_{\mathbb{F}^{r,p}} + \frac{1}{\nu}\|\gamma\|_{\mathbb{F}^{q,p}}\big)|||\nabla\bar{\alpha}|||. \end{aligned}$$

It follows with the embedding in (18)

$$\|\gamma\|_{\mathbb{F}^{q,p}} \le c(\big(\|\alpha_0 - \bar{\alpha}_0\|_{L^p(\Omega)^2} + \|f - \bar{f}\|_{\mathbb{F}^{r,p}} + \big(\|b - \bar{b}\|_{\mathbb{F}^{r,p}} + \frac{1}{\nu}\|\gamma\|_{\mathbb{F}^{q,p}}\big)\|\bar{\alpha}\|_{\mathbb{A}_\delta}\big),$$

and absorbing the term $\frac{c}{\nu}\|\gamma\|_{\mathbb{F}^{q,p}}\|\bar{\alpha}\|_{\mathbb{A}_\delta} \ll \|\gamma\|_{\mathbb{F}^{q,p}} < \infty$ on the right-hand side into the left-hand side, we obtain the assertion. □

5 Nonlinear Estimates

Lemma 7. *Let $T \in (0, \infty]$ and $p \in (4, \infty)$. There is a $C > 0$ such that for $\delta \ge 0$ and $\alpha, \bar{\alpha} \in \mathbb{A}_\delta$, we have for $l, k, i, j = 1, 2$,*

$$\|\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{\mathbb{G}_\delta} \le C\|\alpha\|_{\mathbb{A}_\delta}\|\bar{\alpha}\|_{\mathbb{A}_\delta} \tag{38}$$

and

$$\|\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{L^{p/2}(0,T;W_0^{-1,p}(\Omega)^2)} \le c\|\alpha\|_{\mathbb{F}}\|\bar{\alpha}\|_{\mathbb{A}_\delta}. \tag{39}$$

Proof. We start with estimate (38). We have

$$\begin{aligned} \|\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{\mathbb{G}_\delta} &\le \|e^{t\delta}\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{\mathbb{F}} + \|e^{t\delta}\nabla\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{\mathbb{F}} \\ &\quad + \|e^{t\delta}\partial_k\alpha_i\nabla\partial_l\bar{\alpha}_j\|_{\mathbb{F}} + (1+\delta)\|e^{t\delta}\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{L^p(0,T;W_0^{-1,p}(\Omega))} \\ &\quad + \|e^{t\delta}\partial_t\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{L^p(0,T;W_0^{-1,p}(\Omega))} + \|e^{t\delta}\partial_k\alpha_i\partial_t\partial_l\bar{\alpha}_j\|_{L^p(0,T;W_0^{-1,p}(\Omega))}. \end{aligned}$$

For the first term, one observes

$$\|e^{t\delta}\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{\mathbb{F}} \le |||\partial_l\bar{\alpha}||| \|e^{t\delta}\alpha\|_{\mathbb{A}} \le \|\bar{\alpha}\|_{\mathbb{A}_\delta}\|\alpha\|_{\mathbb{A}_\delta},$$

and similarly, for the second and third term

$$\|e^{t\delta}\nabla\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{\mathbb{F}} \le |||\partial_l\bar{\alpha}||| \|e^{t\delta}\alpha\|_{\mathbb{A}} \le \|\bar{\alpha}\|_{\mathbb{A}_\delta}\|\alpha\|_{\mathbb{A}_\delta}.$$

For the fourth term, we obtain with $1/p' + 1/p = 1$,

$$\begin{aligned}\|e^{t\delta}\partial_k\alpha_i\partial_l\bar{\alpha}_j\|^p_{L^p(0,T;W_0^{-1,p}(\Omega))} &= \int_0^T \|e^{t\delta}\partial_k\alpha_i\partial_l\bar{\alpha}_j\|^p_{W_0^{-1,p}(\Omega)}\,\mathrm{d}t \\ &= \int_0^T \left| \sup_{\|\varphi\|_{W^{1,p'}(\Omega)}=1} \int_\Omega e^{t\delta}\partial_k\alpha_i(t,\cdot)\partial_l\bar{\alpha}_j(t,\cdot)\varphi\,dx\right|^p \mathrm{d}t.\end{aligned}$$

But for $\varphi \in W^{1,p'}(\Omega)^2$ with $\|\varphi\|_{W^{1,p'}(\Omega)} = 1$ we have for almost all $t \in [0,T]$

$$|\int_\Omega e^{t\delta}\partial_k\alpha_i(t,\cdot)\partial_l\bar{\alpha}_j(t,\cdot)\varphi\,dx| \le |||\nabla\bar{\alpha}||| \|e^{t\delta}\nabla\alpha(t,\cdot)\|_{L^p(\Omega)}.$$

Hence, in total, we have for the fourth term

$$\|e^{t\delta}\partial_k\alpha_i\partial_l\bar{\alpha}_j\|_{L^p(0,T;W_0^{-1,p}(\Omega))} \le \|\alpha\|_{\mathbb{A}_\delta}\|\bar{\alpha}\|_{\mathbb{A}_\delta}$$

Similarly, for the fifth and sixth term, we have for almost all $t \in [0,T]$

$$\begin{aligned}&|\int_\Omega e^{t\delta}\partial_t\partial_k\alpha_i(t,\cdot)\partial_l\bar{\alpha}_j(t,\cdot)\varphi dx| \\ &\quad\le (\|\partial_t\nabla(e^{t\delta}\alpha(t,\cdot))\|_{L^p(\Omega)} + \|\delta e^{t\delta}\nabla\alpha(t,\cdot)\|_{L^p(\Omega)})|||\nabla\bar{\alpha}||| \|\varphi\|_{L^{p'}(\Omega)} \\ &\quad\le (1+\delta)(\|\partial_t\nabla(e^{t\delta}\alpha(t,\cdot))\|_{L^p(\Omega)} + \|e^{t\delta}\nabla\alpha(t,\cdot)\|_{L^p(\Omega)})|||e^{t\delta}\nabla\bar{\alpha}|||.\end{aligned}$$

Hence in total, the fifth and sixth term can be estimated by $\|\alpha\|_{\mathbb{A}_\delta}\|\bar{\alpha}\|_{\mathbb{A}_\delta}$ as well. This establishes (38).

For (39), we observe that by Sobolev's embedding theorem $W^{1,p'}(\Omega) \hookrightarrow L^{r'}(\Omega)$ for $r \in (1,\infty)$ with $\frac{1}{2} + \frac{1}{p} \ge \frac{1}{r}$. In particular, since $p > 2$, we can choose $r = p/2$. Thus, for almost all $t \in [0,T]$ we see by integration by parts

$$\begin{aligned}&|\int_\Omega \partial_k\alpha_i(t,\cdot)\partial_l\bar{\alpha}_j(t,\cdot)\varphi\,dx| \\ &\le \|\alpha_i(t,\cdot)\partial_k\partial_l\bar{\alpha}_j(t,\cdot)\varphi\|_{L^1(\Omega)} + \|\alpha_i(t,\cdot)\partial_l\bar{\alpha}_j(t,\cdot)\partial_k\varphi\|_{L^1(\Omega)} \\ &\le \|\alpha(t,\cdot)\|_{L^p(\Omega)}\|\nabla^2\bar{\alpha}(t,\cdot)\|_{L^p(\Omega)}\|\varphi\|_{L^{r'}(\Omega)} + |||\nabla\bar{\alpha}||| \|\alpha(t,\cdot)\|_{L^p(\Omega)}\|\nabla\varphi\|_{L^{p'}(\Omega)} \\ &\le \|\alpha(t,\cdot)\|_{L^p(\Omega)}\|\nabla^2\bar{\alpha}(t,\cdot)\|_{L^p(\Omega)}\|\varphi\|_{W^{1,p'}(\Omega)},\end{aligned}$$

whence (39) follows by integrating over time and using once more Hölder's inequality with $\frac{1}{p} + \frac{1}{p} = \frac{2}{p}$. □

Next, we investigate the right-hand sides of (P). We formally set

$$\begin{aligned} F(\alpha_0, v) &:= -\frac{1}{\nu}\alpha \cdot \nabla\bar{u} - \frac{1}{d^2}\Delta\alpha_k\nabla\alpha_k - \frac{1}{\nu^2}\alpha_1\nabla\phi^\perp \\ &\quad + \frac{1}{d^2}\partial_2^2\phi\nabla\alpha_2 - \frac{d}{\nu^2}\alpha^\perp + \frac{1}{d^2}\nabla\phi\Delta\alpha_2 \\ &\quad - \bar{u}\cdot\nabla v + \frac{1}{\nu}v_2\nabla\phi^\perp + \frac{d}{\nu}v - v\cdot\nabla v \\ &\quad + \frac{1}{\nu}\alpha^\perp\cdot\nabla v + v\cdot\nabla\bar{u} \\ G(\alpha_0, v) &:= (d\nu)^{-1}(\partial_1\alpha_1\partial_2\alpha_2 - \partial_2\alpha_1\partial_1\alpha_2 - \partial_2\phi\partial_1\alpha_1), \end{aligned} \tag{40}$$

where $\alpha = \alpha(\alpha_0, v)$ is obtained in Proposition 6 with data $f = dv^\perp - \nabla\phi\cdot v$, $b = \bar{u} + v$ and α_0. Let $(\bar{u}, \bar{\zeta}, \bar{\pi}) \in BUC^1(BUC^3(\Omega))$ and assume that there is $\kappa > 0$ such that

$$\|\bar{u}\|_{\mathbb{E}_\delta} + \|\nabla\phi, \nabla^2\phi, \nabla^3\phi\| \le \kappa \tag{41}$$

for all $\delta \in (0, \delta_0)$ for some $\delta_0 > 0$.

Lemma 8. *Let $d, \nu > 0$, $p \in (4, \infty)$, $T \in (0, \infty]$. Assume that $\delta, s, \kappa, \tau > 0$ are sufficiently small, where κ is the bound in (41), and where $d/\nu \le \tau$. There is a $C > 0$ such that for $\alpha_0, \bar{\alpha}_0 \in D(\Delta_p)$, $v, \bar{v} \in \mathbb{E}_\delta$*

$$\|\alpha_0, \bar{\alpha}_0\|_{D(\Delta_p)} \le \varepsilon, \quad \|v, \bar{v}\|_{\mathbb{E}_\delta} \le s,$$

where $\varepsilon > 0$ as in Proposition 6, we have

1. $\|F(\alpha_0, v)\|_{\mathbb{F}_\delta} \le C(s + \varepsilon + \kappa + \tau)(s + \varepsilon + \kappa)$ *and the very weak Lipschitz estimate*

$$\begin{aligned} &\|F(\alpha_0, v) - F(\bar{\alpha}_0, \bar{v})\|_{L^{\frac{p}{2}}(0,T;D(\Delta_{p'})')} \\ &\qquad \le C(s + \varepsilon + \kappa + \tau)(\|\alpha_0 - \bar{\alpha}_0\|_{L^p(\Omega)^2} + \|v - \bar{v}\|_{\mathbb{F}^{p/2,p}}). \end{aligned}$$

2. $\|G(\alpha_0, v)\|_{\mathbb{G}_\delta} \le C(s + \varepsilon + \kappa)^2$ *and the very weak Lipschitz estimate*

$$\begin{aligned} &\|G(\alpha_0, v) - G(\bar{\alpha}_0, \bar{v})\|_{L^{\frac{p}{2}}(0,T;W_0^{-1,p}(\Omega)^2)} \\ &\qquad \le C(s + \varepsilon + \kappa)(\|\alpha_0 - \bar{\alpha}_0\|_{L^p(\Omega)^2} + \|v - \bar{v}\|_{\mathbb{F}^{p/2,p}}). \end{aligned}$$

The right-hand sides are finite due to $D(\Delta_p) \hookrightarrow L^p(\Omega)^2$ and $\mathbb{E}_\delta \hookrightarrow \mathbb{F}^{p/2,p}$.

Proof. Apply Proposition 6 with $f = dv^\perp - \nabla\phi\cdot v$, $b = \bar{u} + v$ and α_0 to obtain the unique solution $\alpha = \alpha(\alpha_0, v)$, and similarly $\bar{\alpha} = \alpha(\bar{\alpha}_0, \bar{v}) \in \mathbb{A}_\delta$ to (T) which then satisfies by (33)

$$\|\alpha\|_{\mathbb{A}_\delta} \le c(\|\alpha_0\|_{X_1} + (d + \kappa)\|v\|_{\mathbb{E}_\delta}) \le c_d(\varepsilon + s), \tag{42}$$

and with both choices for q and r

$$(q, r) \in \{(\frac{p}{2}, \frac{p}{2}), (p, \frac{p}{2})\}$$

by (34)

$$\begin{aligned}\|\alpha-\bar\alpha\|_{\mathbb{F}^{q,p}} &\le c(\|\alpha_0-\bar\alpha_0\|_{L^p(\Omega)^2}+(d+\kappa+1)\|v-\bar v\|_{\mathbb{F}^{r,p}})\\ &\le \tilde c(\|\alpha_0-\bar\alpha_0\|_{L^p(\Omega)^2}+\|v-\bar v\|_{\mathbb{F}^{r,p}}).\end{aligned}\tag{43}$$

For the first claim it holds with the embedding in (18) and estimate (42)

$$\begin{aligned}\|F(\alpha_0,v)\|_{\mathbb{F}_\delta} \le{}& \frac{1}{\nu}\|\alpha\cdot\nabla\bar u\|_{\mathbb{F}_\delta}+\frac{1}{d^2}\|\Delta\alpha_k\nabla\alpha_k\|_{\mathbb{F}_\delta}\\ &+\frac{1}{\nu^2}\|\alpha_1\nabla\phi^\perp\|_{\mathbb{F}_\delta}+\frac{1}{d^2}\|\partial_2^2\phi\nabla\alpha_2\|_{\mathbb{F}_\delta}+\frac{1}{d^2}\|\nabla\phi\Delta\alpha_2\|_{\mathbb{F}_\delta}\\ &+\frac{d}{\nu^2}\|\alpha^\perp\|_{\mathbb{F}_\delta}+\frac{1}{d^2}\|\nabla\phi\Delta\alpha_2\|_{\mathbb{F}_\delta}+\|\bar u\cdot\nabla v\|_{\mathbb{F}_\delta}+\frac{1}{\nu}\|v_2\nabla\phi^\perp\|_{\mathbb{F}_\delta}\\ &+\frac{d}{\nu}\|v\|_{\mathbb{F}_\delta}+\|v\cdot\nabla v\|_{\mathbb{F}_\delta}+\frac{1}{\nu}\|\alpha^\perp\cdot\nabla v\|_{\mathbb{F}_\delta}+\|v\cdot\nabla\bar u\|_{\mathbb{F}_\delta}\\ \le{}& \frac{\kappa}{\nu}\|\alpha\|_{\mathbb{F}_\delta}+\frac{1}{d^2}|||\nabla\alpha|||\|\Delta\alpha\|_{\mathbb{F}_\delta}\\ &+\frac{\kappa}{\nu^2}\|\alpha\|_{\mathbb{F}_\delta}+\frac{\kappa}{d^2}\|\nabla\alpha\|_{\mathbb{F}_\delta}+\frac{\kappa}{d^2}\|\Delta\alpha\|_{\mathbb{F}_\delta}\\ &+\frac{d}{\nu^2}\|\alpha^\perp\|_{\mathbb{F}_\delta}+\frac{\kappa}{d^2}\|\Delta\alpha_2\|_{\mathbb{F}_\delta}+\kappa\|\nabla v\|_{\mathbb{F}_\delta}+\frac{\kappa}{\nu}\|v\|_{\mathbb{F}_\delta}\\ &+\frac{d}{\nu}\|v\|_{\mathbb{F}_\delta}+|||v|||\|\nabla v\|_{\mathbb{F}_\delta}+\frac{1}{\nu}|||\alpha|||\|\nabla v\|_{\mathbb{F}_\delta}+\kappa\|v\|_{\mathbb{F}_\delta}\\ \le{}& C(\kappa+\tau+\|\alpha\|_{\mathbb{A}_\delta}+\|v\|_{\mathbb{E}_\delta})(\|\alpha\|_{\mathbb{A}_\delta}+\|v\|_{\mathbb{E}_\delta})\\ \le{}& C(\kappa+\tau+c_d(s+\varepsilon)+s)(c_d(s+\varepsilon)+s).\end{aligned}$$

For the Lipschitz estimate of F, we see that all terms in the expression $F(\alpha_0,v)-F(\bar\alpha_0,\bar v)$ that do not involve second order derivatives are estimated using the trivial embedding $L^p(\Omega)\hookrightarrow W^{-1,p}(\Omega)$ and estimate (43) with $(q,r)=(\frac{p}{2},\frac{p}{2})$. The remaining terms are up to a constant $\nabla\phi\Delta(\alpha_2-\bar\alpha_2)$, and $\Delta\alpha_k\nabla\alpha_k-\Delta\bar\alpha_k\nabla\bar\alpha_k$. The first term is estimated testing with $\varphi\in D(\Delta_{p'})$ subject to $\|\varphi\|_{D(\Delta_{p'})}=1$, using integration by parts via

$$\begin{aligned}\Big|\int_\Omega\nabla\phi\Delta(\alpha_2(t,\cdot)-\bar\alpha_2(t,\cdot))\varphi\,dx\Big| &\le |||\nabla\phi,\nabla^2\phi,\nabla^3\phi|||\|\alpha-\bar\alpha\|_{L^p(\Omega)}\|\varphi\|_{D(\Delta_{p'})}\\ &\le\kappa\|\alpha-\bar\alpha\|_{L^p(\Omega)}.\end{aligned}$$

Taking the $p/2$-th power and integrating over time, we obtain the correct estimate by appealing to (43) with $(q,r)=(\frac{p}{2},\frac{p}{2})$. For the second term, we observe

$$\Delta\alpha_k\nabla\alpha_k=\operatorname{div}\big[\nabla\alpha_k\otimes\nabla\alpha_k-\frac{1}{2}\operatorname{id}|\nabla\alpha_k|^2\big]$$

and hence

$$\begin{aligned}\Delta\alpha_k\nabla\alpha_k-\Delta\bar\alpha_k\nabla\bar\alpha_k={}&\operatorname{div}\big[\nabla(\alpha_k-\bar\alpha_k)\otimes\nabla\alpha_k-\frac{1}{2}\operatorname{id}\big(\partial_j(\alpha_k-\bar\alpha_k)\partial_j\alpha_k\big]\\ &+\operatorname{div}\big[\nabla\bar\alpha_k\otimes\nabla(\alpha_k-\bar\alpha_k)-\frac{1}{2}\operatorname{id}\big(\partial_j\bar\alpha_k\partial_j(\alpha_k-\bar\alpha_k)\big)\big]\end{aligned}$$

By symmetry, it suffices to estimate the second term on the right-hand side. For this, it follows as in the proof of estimate (39) (where we replace α by $\alpha - \bar{\alpha}$ and the test function φ by $\nabla\varphi$) that

$$\|\mathrm{div}\left[\nabla\bar{\alpha}_k \otimes \nabla(\alpha_k - \bar{\alpha}_k) - \frac{1}{2}\,\mathrm{id}\left(\partial_j\bar{\alpha}_k\partial_j(\alpha_k - \bar{\alpha}_k)\right)\right]\|_{L^{\frac{p}{2}}(0,T;D(\Delta_{p'})')} \\ \leq c\|\alpha_k - \bar{\alpha}_k\|_{\mathbb{F}}\|\bar{\alpha}_k\|_{\mathbb{A}_\delta},$$

which by (42) and (43) with $(q,r) = (p, \frac{p}{2})$ gives the desired estimate.

For the second claim, we have with the compatibility of $\mathbb{A}_\delta$ and $\mathbb{G}_\delta$ in (38),

$$\|G(\alpha_0, v)\|_{\mathbb{G}_\delta} \leq \frac{1}{d\nu}(\|\partial_1\alpha_1\partial_2\alpha_2\|_{\mathbb{G}_\delta} + \|\partial_2\alpha_1\partial_1\alpha_2\|_{\mathbb{G}_\delta} + \|\partial_2\phi\partial_1\alpha_1\|_{\mathbb{G}_\delta}) \\ \leq C(\|\alpha\|_{\mathbb{A}_\delta} + \kappa)\|\alpha\|_{\mathbb{A}_\delta} \leq C(c_d+1)^2(s+\varepsilon+\kappa)^2.$$

For the Lipschitz assertion for G, let us for simplicity abbreviate the norm of $L^{\frac{p}{2}}(0,T;W_0^{-1,p}(\Omega)^2)$ by $\|\cdot\|$. Then we observe

$$\|G(\alpha_0, v) - G(\bar{\alpha}_0, \bar{v})\| \leq (d\nu)^{-1}\big(\|\partial_1\alpha_1\partial_2(\alpha_2 - \bar{\alpha}_2)\| \\ + \|\partial_1(\alpha_1 - \bar{\alpha}_1)\partial_2\bar{\alpha}_2\| + \|\partial_2\alpha_1\partial_1(\alpha_2 - \bar{\alpha}_2)\| \\ + \|\partial_2(\alpha_1 - \bar{\alpha}_1)\partial_1\bar{\alpha}_2\| + \|\partial_2\phi\partial_1(\alpha_1 - \bar{\alpha}_1)\|\big),$$

and the bound follows via (39) and (43) with $(q,r) = (p, \frac{p}{2})$ for all but the last term, for which we obtain for almost all $t \in [0,T]$ by integration by parts and using $\partial_1\phi = 0$

$$|\int_\Omega \partial_2\phi(t,\cdot)\partial_1(\alpha_1(t,\cdot) - \bar{\alpha}_1(t,\cdot))\varphi\,dx| \leq \|\partial_2\phi(t,\cdot)(\alpha_1(t,\cdot) - \bar{\alpha}_1(t,\cdot))\partial_1\varphi\|_{L^1(\Omega)} \\ \leq \kappa\|\alpha(t,\cdot) - \bar{\alpha}(t,\cdot)\|_{L^p(\Omega)}\|\nabla\varphi\|_{L^{p'}(\Omega)},$$

and integrating over time yields

$$\|\partial_2\phi(\partial_1\alpha_1 - \partial_1\bar{\alpha}_1)\| \leq \kappa\|\alpha - \bar{\alpha}\|_{\mathbb{F}^{p/2,p}},$$

so that (43) with $(q,r) = (\frac{p}{2}, \frac{p}{2})$ gives the assertion. □

6 Proof of Main Theorem

We divide the proof into several steps.

Step 1: Preliminary definitions and embeddings

Proposition 2 yields that the generalized Stokes problem (S) is uniquely solvable in the maximal regularity class $\mathbb{E}_\delta$ for some $\delta > 0$. Fix $\tau \in (0,1)$ and $s = s(\tau) \in (0,\tau)$, $\varepsilon = \varepsilon(s), \kappa = \kappa(s) \in (0,s)$ small enough such that $\tau < [12CM]^{-1}$ and $\varepsilon < s[3M]^{-1}$, where $C > 0$ is chosen as in Lemma 8 and $M > 0$ is chosen as in Proposition 2.

Let $(\alpha_0, w_0) \in \bar{B}_\varepsilon(0,0) \subseteq D(\Delta_p) \times \mathsf{Tr}_{\mathbb{E}}$. Our goal is to use the fixed point assertion in Lemma 1 on the set

$$\mathbb{B} := \mathbb{B}_{w_0} := \{v \in \mathbb{E}_\delta : v(0) = w_0, \|v\|_{\mathbb{E}_\delta} \leq s\}.$$

Note that $\mathbb{B} \subset \mathbb{E}_\delta$ is nonempty, convex, closed and bounded.

Step 2: Set-up for the fixed point argument

Given $v \in \mathbb{B}$, consider the linear problem,

$$\begin{cases} \partial_t u - \Delta u + \nabla q &= F(\alpha_0, v), \\ \operatorname{div} u &= G(\alpha_0, v), \\ u(0) &= w_0, \end{cases} \tag{44}$$

where F and G are defined as in (40). Note that this problem has a unique solution $u \in \mathbb{E}_\delta$ with corresponding pressure $\nabla q \in \mathbb{F}_\delta$ due to Proposition 2 applied with $r = q = p$. Define the solution operator of $W : \mathbb{B} \to \mathbb{E}^\delta$ via $Wv := u$, where u is the solution of (44). We would now like to show that $W(\mathbb{B}) \subseteq \mathbb{B}$ and that W is a contraction in a weak norm, namely in $\|\cdot\|_{\mathbb{F}^{p/2,p}}$. Then by Lemma 1 there is a unique fixed point $w \in \mathbb{B}$ of W, which yields the desired solution to problem (P).

Step 3: W is a self-mapping on $\mathbb{B}$

By definition $u(0) = w_0$, hence we need to verify $\|u\|_{\mathbb{E}_\delta} \leq s$.
From Proposition 2 we learn

$$\|u\|_{\mathbb{E}_\delta} + \|\nabla q\|_{\mathbb{F}_\delta} \leq M(\|F(\cdot;\alpha_0, v)\|_{\mathbb{F}_\delta} + \|G(\cdot;\alpha_0, v)\|_{\mathbb{G}_\delta} + \|w_0\|_{\mathsf{Tr}_\mathbb{E}}).$$

For F and G, we have by Lemma 8,

$$\|F(\alpha_0, \cdot, v)\|_{\mathbb{F}_\delta} + \|G(\alpha_0, \cdot, v)\|_{\mathbb{G}_\delta} \leq 2C(s + \varepsilon + \kappa + \tau)(s + \varepsilon + \kappa) \leq \frac{2}{3M} s$$

Clearly, $\|w_0\|_{\mathsf{Tr}_\mathbb{E}} \leq \varepsilon \leq \frac{s}{3M}$. Thus, in total we have $\|u\|_{\mathbb{E}_\delta} \leq s$ and the assertion of this step follows.

Step 4: W is a strict contraction on $\mathbb{B}$ with respect to a weak norm

Let $v, \bar{v} \in \mathbb{B}$, and set $Wv =: u$ and $W\bar{v} =: \bar{u}$. Then subtracting the equations (44) for $\bar{v}$ and v, respectively, and noting that $(\bar{v} - v)(0) = 0$, we obtain from the very weak *a priori* estimate in Proposition 3 and Lemma 8

$$\begin{aligned} \|u - \bar{u}\|_{\mathbb{F}^{p/2,2}} \leq M(&\|F(\cdot;\alpha_0, v) - F(\cdot;\alpha_0, \bar{v})\|_{L^{\frac{p}{2}}(0,T;D(\Delta_{p'})')} \\ &+ \|G(\cdot;\alpha_0, \bar{v}) - G(\cdot;\alpha_0, v)\|_{L^{\frac{p}{2}}(0,T;W_0^{-1,p}(\Omega)^2)}) \\ \leq\ & 2CM(s + \varepsilon + \kappa + \tau)\|v - \bar{v}\|_{\mathbb{F}^{p/2,2}} < \frac{2}{3}\|v - \bar{v}\|_{\mathbb{F}^{p/2,2}}. \end{aligned}$$

Therefore, Lemma 1 yields a unique fixed point $u = v \in \mathbb{B}$.

Step 5: Proof of the Lipschitz continuity assertion

Additionally to $\mathbb{B}_{w_0}$, consider now $\mathbb{B}_{\bar{w}_0}$. For $v \in \mathbb{B}_{w_0}$ and $\bar{v} \in \mathbb{B}_{\bar{w}_0}$, let $u := W_{w_0}v$ and $\bar{u} := W_{\bar{w}_0}\bar{v}$. Then, similarly to the estimates in Step 4,

$$\begin{aligned}
&\|u - \bar{u}\|_{\mathbb{F}^{p/2,2}} \\
&\leq M(\|w_0 - \bar{w}_0\|_{\left(\mathsf{Tr}_{\mathbb{E}}^{(p/2)',p'}\right)'} + \|F(\cdot;\alpha_0, v) - F(\cdot;\bar{\alpha}_0,\bar{v})\|_{L^{\frac{p}{2}}(0,T;D(\Delta_{p'})')} \\
&\qquad + \|G(\cdot;\alpha_0, v) - G(\cdot;\bar{\alpha}_0,\bar{v})\|_{L^{\frac{p}{2}}(0,T;W_0^{-1,p}(\Omega)^2)}) \\
&\leq M\|w_0 - \bar{w}_0\|_{\left(\mathsf{Tr}_{\mathbb{E}}^{(p/2)',p'}\right)'} + \frac{2}{3}(\|v-\bar{v}\|_{\mathbb{F}^{p/2,2}} + \|\alpha_0 - \bar{\alpha}_0\|_{L^p(\Omega)^2}).
\end{aligned}$$

If we now choose v and $\bar{v}$ to be the fixed points of the map W_{w_0} and $W_{\bar{w}_0}$, respectively, we may absorb the second term of the right hand side into the left-hand side. This concludes the proof.

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