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AN ENERGETIC VARIATIONAL APPROACH FOR NONLINEAR DIFFUSION EQUATIONS IN MOVING THIN DOMAINS

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Abstract. This paper concerns the processes of nonlinear diffusion in a moving domain which lies on a moving closed surface. The nonlinear diffusion equations and corresponding energy identities on the moving surface are derived by regarding it as a thin width (thickness) limit of moving thin domains, for which suitable boundary conditions are imposed to insure that there is no exchange of mass between the thin domains and the environments. We also employ an energetic variational approach to derive these nonlinear diffusion equations. Most of all, we show that these nonlinear energetic variational procedures can commute with the passing to the zero width limits.

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1 Introduction

In this paper, we are interested in deriving diffusion equations on a moving surface, by regarding it as a thin width limit of the problem in a moving thin domain around the moving surface.

Let us begin with an equation of the conservation of mass ρ with velocity u in a moving domain $\Omega(t)$, $t \in (0,T)$ in $\mathbb{R}^n$, $n \geq 2$ of the form

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \quad \text{in} \quad \Omega(t),\, t \in (0,T), \tag{1.1}$$

which represents the local conservation of mass. Considering the situation that there is no exchange of mass on the boundary, i.e.

$$u \cdot \nu_\Omega = V_\Omega^N \quad \text{on} \quad \partial\Omega(t),\, t \in (0,T), \tag{1.2}$$

where V_Ω^N is the normal velocity of the boundary $\partial\Omega(t)$ in the direction of the outward normal vector field ν_Ω of $\partial\Omega(t)$. Similar conservation law of mass η with velocity v on a moving surface $\Gamma(t)$ can be derived from the local conservation of mass. It turns out (see Section 3) that, when the normal component of v is equal to the outward normal velocity V_Γ^N of the moving surface $\Gamma(t)$, the resulting equation is of the form:

$$\partial^\circ \eta - V_\Gamma^N H \eta + \operatorname{div}_\Gamma(\eta v^T) = 0 \quad \text{on} \quad \Gamma(t),\, t \in (0,T), \tag{1.3}$$

where $\partial^\circ = \partial_t + V_\Gamma^N \nu_\Gamma \cdot \nabla$ is the normal time derivative, ν_Γ is the outward normal vector field of $\Gamma(t)$, H is the ($n-1$ times) mean curvature of $\Gamma(t)$, $\operatorname{div}_\Gamma$ is the surface divergence operator on $\Gamma(t)$, and v^T is a tangential vector field satisfying $v = V_\Gamma^N \nu_\Gamma + v^T$. Note that this equation is obtained as the zero width limit of the corresponding equation (1.1) in a moving thin domain $\Omega_\varepsilon(t)$ defined as the set of all points in $\mathbb{R}^n$ with distance less than ε from $\Gamma(t)$ (see Remark 4.2).

The conventional diffusion equations, or even the porous-media equations, can be viewed as the combination of incompressible fluids with the damping in the form of Darcy's law. Take the usual Darcy's law for the velocity u in the moving thin domain $\Omega_\varepsilon(t)$:

$$-\rho u = \nabla p(\rho) \quad \text{in} \quad \Omega_\varepsilon(t),\, t \in (0,T), \tag{1.4}$$

where p is the pressure, then we can prove (see Theorem 4.1) that the zero width limit of the diffusion equations (1.1) and (1.4) yields diffusion equations on the moving surface $\Gamma(t)$: (1.3) and Darcy's law

$$-\eta v^T = \nabla_\Gamma p(\eta) \quad \text{on} \quad \Gamma(t),\, t \in (0,T). \tag{1.5}$$

Here v^T is the tangential component of the velocity v and ∇_Γ is the tangential gradient operator on $\Gamma(t)$.

The diffusion equations (1.1), (1.4) and (1.3), (1.5) possess specific energy identities. It can be easily proven (see Section 5) that for ρ and u satisfying (1.1) and (1.4) the energy identity

$$\frac{d}{dt}\int_{\Omega(t)} \omega(\rho)\, dx = -\int_{\Omega(t)} \rho|u|^2\, dx - \int_{\partial\Omega(t)} p(\rho) V_\Omega^N\, d\mathcal{H}^{n-1} \tag{1.6}$$

holds. Here ω is a function satisfying $p(\rho) = \omega'(\rho)\rho - \omega(\rho)$. Similarly, for η and v satisfying (1.3) and (1.5) we have

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\, d\mathcal{H}^{n-1} = -\int_{\Gamma(t)} \eta|v^T|^2\, d\mathcal{H}^{n-1} + \int_{\Gamma(t)} p(\eta) V_\Gamma^N H\, d\mathcal{H}^{n-1}, \tag{1.7}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Fortunately, the energy identity (1.7) on the moving surface can be derived as the zero width limit of the energy identity (1.6) in the moving thin domain (see Theorem 5.3).

With the results from this paper, we can also note that the passing of zero width limit commutes with an energetic variational approach originated from the works of Lord Rayleigh [24] and Onsager [16, 17] and developed by Liu and others [3, 12, 25] (see Section 6). In summary, we show that the diagram below is commutative.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Diffusion equation}\\ \text{in } \Omega_\varepsilon(t)\end{array} \arrow[r, shift left, "\begin{array}{c}\text{integration}\\ \text{by parts}\end{array}"] \arrow[d, "\varepsilon \to 0"'] & \begin{array}{c}\text{Energy identity}\\ \text{in } \Omega_\varepsilon(t)\end{array} \arrow[l, shift left, "\begin{array}{c}\text{energetic}\\ \text{variation}\end{array}"] \arrow[d, "\varepsilon \to 0"] \\
\begin{array}{c}\text{Diffusion equation}\\ \text{on } \Gamma(t)\end{array} \arrow[r, shift left, "\begin{array}{c}\text{integration}\\ \text{by parts}\end{array}"] & \begin{array}{c}\text{Energy identity}\\ \text{on } \Gamma(t)\end{array} \arrow[l, shift left, "\begin{array}{c}\text{energetic}\\ \text{variation}\end{array}"]
\end{tikzcd}$$

A standard approach for finding the limit of a thin domain problem is a rescaling argument: one transforms a partial differential equation in a thin domain into that in a fixed in width reference domain by the change of variables and then gets a limit equation by assuming that a rescaled solution is independent of variables in thin directions. In our case where the thin domain and the surface both move, one may transform the moving thin domain into a fixed in time and width reference domain. However, it yields tedious calculations because of the geometry of the limit moving surface and it is difficult to bring a limit equation obtained on a stationary reference surface back to an equation on the original moving surface. One other method is to rescale the width of the moving thin domain without fixing time, which is used in [15] to find the limit of the Neumann type problem of the heat equation (equations (1.1), (1.2), and (1.4) with $p(\rho) = \rho$) in moving thin domains. However, it is still complicated and requires a questionable assumption that the boundary condition holds in a middle of the moving thin domain. It is also artificial in the sense that we have to make rescaled solutions constant in the thin direction at an "appropriate" point to derive the limit energy identity and if we take a wrong point then we get a wrong limit (see Remarks 5.5).

To derive a thin width limit with more straightforward calculations we consider the Taylor series of a function on $\Omega_\varepsilon(t)$ in powers of the signed distance from $\Gamma(t)$. We assume that $\Omega_\varepsilon(t)$ admits the normal coordinate system around $\Gamma(t)$, i.e. for each $x \in \Omega_\varepsilon(t)$ there exists a unique point $\pi(x,t) \in \Gamma(t)$ such that

$$x = \pi(x,t) + d(x,t)\nu_\Gamma(\pi(x,t),t),$$

where d is the signed distance function from $\Gamma(t)$ increasing in the direction of ν_Γ. Based on the normal coordinate system we consider expansions of

$$\begin{aligned} \rho(x,t) &= \rho(\pi(x,t) + d(x,t)\nu_\Gamma(\pi(x,t),t),t), \\ u(x,t) &= u(\pi(x,t) + d(x,t)\nu_\Gamma(\pi(x,t),t),t) \end{aligned}$$

in powers of the signed distance $d(x,t)$:

$$\begin{aligned} \rho(x,t) &= \eta(\pi(x,t),t) + d(x,t)\eta^1(\pi(x,t),t) + d(x,t)^2\eta^2(\pi(x,t),t) + \cdots, \\ u(x,t) &= v(\pi(x,t),t) + d(x,t)v^1(\pi(x,t),t) + d(x,t)^2v^2(\pi(x,t),t) + \cdots. \end{aligned} \tag{1.8}$$

In these expansions we assume that η, v, and the coefficients of the powers of $d(x,t)$ are functions on $\Gamma(t)$ and independent of ε (note that the functions ρ and u on $\Omega_\varepsilon(t)$ depend on ε). Under this and other suitable assumptions, we obtain the limit equations (1.3) and (1.5) as the zeroth order terms of expansions in powers of $d(x,t)$ (or ε) of the bulk equations (1.1), (1.2), and (1.4) by differentiating (1.8) and substituting them for (1.1), (1.2), and (1.4) (see Section 4). Note that, if we take the average of (1.8) in the normal direction of $\Gamma(t)$, then we get

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \rho(y + r\nu_\Gamma(y,t),t)\,dr = \eta(y,r) + (\text{higher order terms in } \varepsilon), \quad y \in \Gamma(t)$$

and a similar equality for u. Thus, formally speaking, we derive the limit equations (1.3) and (1.5) as equations on $\Gamma(t)$ satisfied by the limit as $\varepsilon \to 0$ of the averages of ρ and u in the thin direction.

The idea mentioned above also applies to derivation of the energy identity (1.7) on the moving surface from that in the moving thin domain (1.6) (see Section 5). To get the limit energy identity we use integral transformation formulas from surface integrals over the level-set surfaces $\{x \in \mathbb{R}^n \mid d(x,t) = r\}$ $(-\varepsilon < r < \varepsilon)$ into that over the zero level-set surface $\Gamma(t)$ (see Lemma 5.4).

There is a long history in the study of partial differential equations in thin domains, such as the pioneering work by Hale and Raugel [8, 9], where they investigated damped hyperbolic equations and reaction-diffusion equations in a flat stationary thin domain of the form

$$\Omega_\varepsilon = \{(x', x_n) \in \mathbb{R}^n \mid x' \in \omega,\ 0 < x_n < \varepsilon g(x')\}, \tag{1.9}$$

where ω is an open set in $\mathbb{R}^{n-1}$ and g is a function on ω. There is also a large number of the literature on reaction-diffusion equations in various types of thin domains such as a thin L-shaped domain [10], a moving flat thin domain of the form (1.9) with g time-dependent [18], and flat and curved thin domains with holes [19, 20, 21] (here a curved thin domain is a thin domain degenerating into a lower dimensional manifold). A main subject in the above literature is to compare the dynamics of equations in thin domains with that of limit equations in their degenerate sets rather than to find the limit equations of the original equations in the thin domains, since their degenerate sets are stationary and thus the rescaling argument works well for finding the limit equations. The Navier–Stokes equations in thin domains has been also studied well [11, 13, 23, 26, 27] since fluid flows in thin domains often appear in natural sciences like the flow of water in a large lake, geophysical flows, etc. Researchers are especially interested in the relation between the smallness of the width of thin domains and the large time behavior of solutions to the Navier–Stokes equations in thin domains. We refer to [22] and references therein for other types of thin domains degenerating into stationary sets and mathematical analysis of partial differential equations in such thin domains.

In the case where the degenerate set of a thin domain moves, derivation of the limit of a partial differential equation in the thin domain is more complicated since the geometry of the degenerate set changes as it moves. Such a problem was first considered in [15] where the author derived both formally and rigorously the limit equation of the Neumann type problem of the heat equation (equations (1.1), (1.2), and (1.4) with $p(\rho) = \rho$) in a moving thin domain degenerating into a closed smooth moving surface. He also found that the normal velocity and the mean curvature of the degenerate moving surface affects the limit equation, which is not observed in the case where the degenerate set of a thin domain does not move.

The rest of this paper is organized as follows. In Section 2, we fix notations on various quantities related to the moving surface. In Section 3, we briefly observe that the transport equations in the moving domain and the moving surface are equivalent to the local mass conservation. In Section 4, we derive the limit equations (1.3) and (1.5) on the moving surface from the diffusion equations (1.1), (1.2), and (1.4) on the moving thin domain by means of expansion in terms of the signed distance. In Section 5, we derive the energy identities (1.6) and (1.7) from corresponding diffusion equations and then show that the energy identity (1.7) on the moving surface is the zero width limit of the energy identity (1.6) on the moving thin domain. In Section 6, we apply an energetic variational approach to the energy identities (1.6) and (1.7) to obtain Darcy's laws (1.4) and (1.5).

2 Quantities on a moving surface

We start with several notations for a moving surface. Let $\Gamma(t)$, $t \in [0,T]$ be an $(n-1)$-dimensional closed (that is, compact and without boundary), connected, oriented and smooth moving surface in $\mathbb{R}^n$ with $n \geq 2$. Also, let

$$S_T := \bigcup_{t\in(0,T)} \Gamma(t) \times \{t\} \subset \mathbb{R}^{n+1}$$

be a space-time hypersurface associated with the moving surface $\Gamma(t)$. For each $t \in [0,T]$ we write $\nu_\Gamma(\cdot,t)$, $V_\Gamma^N(\cdot,t)$, and $d(\cdot,t)$ for the unit outward normal vector field of $\Gamma(t)$, the scalar outward normal velocity of $\Gamma(t)$, and the signed distance function from $\Gamma(t)$, respectively. Note that to describe the evolution of a closed surface it is sufficient to give the normal velocity. Since the smooth closed surface $\Gamma(t)$ varies smoothly in time, the principal curvatures $\kappa_1(\cdot,t),\dots,\kappa_{n-1}(\cdot,t)$ of $\Gamma(t)$ are bounded uniformly in $t \in [0,T]$. Then there exists a constant $\delta > 0$ independent of t such that for each $t \in [0,T]$ the tubular neighborhood of $\Gamma(t)$ of the form

$$N(t) := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x,\Gamma(t)) < \delta\}$$

admits the normal coordinate system

$$x = \pi(x,t) + d(x,t)\nu_\Gamma(\pi(x,t),t), \quad x \in N(t), \tag{2.1}$$

where $\pi(x,t)$ is the closest point on $\Gamma(t)$ to x (see [6, Section 14.6] for example). For each $t \in [0,T]$ we suppose that $d(\cdot,t)$ increases along the direction of $\nu_\Gamma(\cdot,t)$. Then we have

$$\begin{aligned} \nabla d(x,t) &= \nu_\Gamma(\pi(x,t),t), \quad && (x,t) \in N(t), \\ \partial_t d(y,t) &= -V_\Gamma^N(y,t), && (y,t) \in \Gamma(t), \end{aligned} \tag{2.2}$$

Moreover, differentiating both sides of

$$d(x,t) = \{x - \pi(x,t)\} \cdot \nabla d(x,t), \quad d(\pi(x,t),t) = 0$$

with respect to t we easily obtain

$$\partial_t d(x,t) = \partial_t d(\pi(x,t),t) = -V_\Gamma^N(\pi(x,t),t), \quad (x,t) \in N_T, \tag{2.3}$$

where $N_T := \bigcup_{t\in(0,T)} N(t) \times \{t\}$.

Next we fix $t \in [0,T]$ and give differential operators on the surface $\Gamma(t)$. We define the orthogonal projection onto the tangent plane of $\Gamma(t)$ by

$$P_\Gamma(y,t) := I_3 - \nu_\Gamma(y,t) \otimes \nu_\Gamma(y,t), \quad y \in \Gamma(t).$$

Here I_n denotes the identity matrix of size n and $a \otimes b = (a_i b_j)_{i,j}$ is the tensor product of two vectors $a = (a_1,\dots,a_n)$ and $b = (b_1,\dots,b_n)$ in $\mathbb{R}^n$. For a function $f\colon \Gamma(t) \to \mathbb{R}$ and a vector field $F\colon \Gamma(t) \to \mathbb{R}^n$ we define the tangential gradient of f and the surface divergence of F as

$$\nabla_\Gamma f(y) := P_\Gamma(y,t)\nabla \tilde{f}(y), \quad \mathrm{div}_\Gamma F(y) := \mathrm{tr}[P_\Gamma(y,t)\nabla \widetilde{F}(y)], \quad y \in \Gamma(t).$$

Here $\tilde{f}$ and $\widetilde{F}$ are extensions of f and F to $N(t)$ satisfying $\tilde{f} = f$ and $\widetilde{F} = F$ on $\Gamma(t)$. Also, $\mathrm{tr}[M]$ denotes the trace of a square matrix M and we use the notation

$$\nabla G = \begin{pmatrix} \partial_1 G_1 & \dots & \partial_1 G_n \\ \vdots & \ddots & \vdots \\ \partial_n G_1 & \dots & \partial_n G_n \end{pmatrix}$$

for the gradient matrix of a vector field $G = (G_1,\dots,G_n)$. Note that the tangential gradient of f and the surface divergence of F do not depend on a choice of extensions (see e.g. [5, Lemma 2.4]). Moreover, for any function f on $\Gamma(t)$ we easily see that

$$\nabla_\Gamma f(y) \cdot \nu_\Gamma(y,t) = 0, \quad y \in \Gamma(t). \tag{2.4}$$

We define the ($n-1$ times) mean curvature H of $\Gamma(t)$ as

$$H(y,t) := -\mathrm{div}_\Gamma \nu_\Gamma(y,t), \quad y \in \Gamma(t). \tag{2.5}$$

Note that the mean curvature is equal to the sum of the principal curvatures:

$$H(y,t) = \sum_{i=1}^{n-1} \kappa_i(y,t), \quad y \in \Gamma(t). \tag{2.6}$$

Finally, for a function f on the space-time hypersurface S_T we define the normal time derivative (the time derivative along the normal velocity) as

$$\partial^\circ f(y,t) := \partial_t \tilde{f}(y,t) + V_\Gamma(y,t)\nu_\Gamma(y,t) \cdot \nabla \tilde{f}(y,t), \quad (y,t) \in S_T.$$

Here $\tilde{f}$ is an extension of f to N_T satisfying $\tilde{f} = f$ on S_T. Note that the value of $\partial^\circ f$ does not depend on a choice of an extension of f and the formula

$$\partial^\circ f(y,t) = \frac{d}{dt}\big(f(\pi(y,t),t)\big), \quad (y,t) \in S_T \tag{2.7}$$

holds (see [2, Section 3.4] for details).

3 Transport equation in a moving domain and on a moving surface

In this section we give the transport equation for a scalar quantity in a moving domain and on a moving surface. We use some of the same terminology and techniques as in [14]. We first consider transportation of a scalar quantity in a bounded moving domain $\Omega(t)$ in $\mathbb{R}^n$. Let $\rho(x,t)$ and $u(x,t)$ be the density and the velocity field of the scalar quantity at $x \in \Omega(t)$, respectively. Our starting point is the local mass conservation

$$\frac{d}{dt}\int_{U(t)} \rho\, dx = 0 \tag{3.1}$$

for any portion $U(t)$ (relatively open set) of $\Omega(t)$ moving with velocity $u(\cdot,t)$ and whose closure (in $\mathbb{R}^n$) is contained in $\Omega(t)$. Since the left-hand side is equal to $\int_{U(t)}\{\partial_t\rho + \mathrm{div}(\rho u)\}\, dx$ by the Reynolds transport theorem [7] and the divergence theorem, the condition (3.1) for any $U(t)$ is equivalent to the transport equation

$$\partial_t\rho + \mathrm{div}(\rho u) = 0 \quad \text{in} \quad Q_T := \bigcup_{t\in(0,T)} \Omega(t)\times\{t\}. \tag{3.2}$$

To make the total mass $\int_{\Omega(t)} \rho\, dx$ conserved, we impose the boundary condition

$$u \cdot \nu_\Omega = V_\Omega^N \quad \text{on} \quad \partial_\ell Q_T := \bigcup_{t\in(0,T)} \partial\Omega(t)\times\{t\}, \tag{3.3}$$

where $\nu_\Omega(\cdot,t)$ and $V_\Omega^N(\cdot,t)$ are the unit outward normal vector field and the scalar outward normal velocity of $\partial\Omega(t)$, respectively. The boundary condition (3.3) physically means that the quantity in $\Omega(t)$ moves along the boundary of $\Omega(t)$ and it does not go into and out of $\Omega(t)$.

Next we give the transport equation for a scalar quantity on a moving surface. Let $\Gamma(t)$ be a closed, connected, oriented moving surface in $\mathbb{R}^n$. As in Section 2, we write $\nu_\Gamma(\cdot,t)$ and $V_\Gamma(\cdot,t)$ for the outward normal vector field and the scalar outward normal velocity of $\Gamma(t)$, respectively. Suppose that for each $t \in (0,T)$ a scalar quantity on $\Gamma(t)$ has the density $\eta(y,t)$ at $y \in \Gamma(t)$ and moves with velocity

$$v(y,t) = V_\Gamma(y,t)\nu_\Gamma(y,t) + v^T(y,t), \quad y \in \Gamma(t),$$

where $v^T(\cdot,t)$ is a given tangential velocity field on $\Gamma(t)$. Then its local mass conservation is expressed as

$$\int_{U(t)} \eta \, d\mathcal{H}^{n-1} = 0 \tag{3.4}$$

for any portion $U(t)$ (relatively open set) of $\Gamma(t)$ moving with velocity $v(\cdot,t)$. The Leibniz formula [4, Lemma 2.2] yields

$$\frac{d}{dt}\int_{U(t)} \eta \, d\mathcal{H}^{n-1} = \int_{U(t)} \{\partial^\circ \eta - V_\Gamma^N H\eta + \mathrm{div}_\Gamma(\eta v^T)\} \, d\mathcal{H}^{n-1}.$$

From this formula, the condition (3.4) for any $U(t)$ is equivalent to

$$\partial^\circ \eta - V_\Gamma^N H\eta + \mathrm{div}_\Gamma(\eta v^T) = 0 \quad \text{on} \quad S_T. \tag{3.5}$$

This is the transport equation on the moving surface $\Gamma(t)$.

4 Zero width limit for nonlinear diffusion equations

Let us consider nonlinear diffusion of a scalar quantity in $\Omega(t)$ with density ρ and velocity u. Suppose that the diffusion process is described by the transport equation (3.2) and Darcy's law $-\rho u = \nabla p(\rho)$, where

$$p(\rho) := \omega'(\rho)\rho - \omega(\rho) \tag{4.1}$$

is the pressure with a given function $\omega(\rho)$, $\rho \in \mathbb{R}$. We impose the boundary condition (3.3). Hence the nonlinear diffusion equations we deal with are

$$\partial_t \rho + \mathrm{div}(\rho u) = 0 \qquad \text{in} \quad Q_T, \tag{4.2}$$

$$-\rho u = \nabla p(\rho) \quad \text{in} \quad Q_T, \tag{4.3}$$

$$u \cdot \nu_\Omega = V_\Omega^N \qquad \text{on} \quad \partial_\ell Q_T. \tag{4.4}$$

We consider these equations in a moving thin domain. For sufficiently small $\varepsilon > 0$, we define a moving thin domain $\Omega_\varepsilon(t)$ as the set of all points in $\mathbb{R}^n$ with distance less than ε from the moving surface $\Gamma(t)$:

$$\Omega_\varepsilon(t) := \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, \Gamma(t)) < \varepsilon\}. \tag{4.5}$$

We write $Q_{\varepsilon,T}$ and $\partial_\ell Q_{\varepsilon,T}$ for Q_T and $\partial_\ell Q_T$ with $\Omega(t) = \Omega_\varepsilon(t)$. Our goal in this section is to find the limit equations of (4.2)–(4.4) in $\Omega_\varepsilon(t)$ as ε goes to zero, that is, the moving thin domain $\Omega_\varepsilon(t)$ degenerates into the moving surface $\Gamma(t)$. According to the normal coordinate system (2.1), we expand ρ and u in powers of the signed distance $d(x,t)$ as

$$\rho(x,t) = \eta(\pi(x,t),t) + d(x,t)\eta^1(\pi(x,t),t) + R(d(x,t)^2), \tag{4.6}$$
$$u(x,t) = v(\pi(x,t),t) + d(x,t)v^1(\pi(x,t),t) + R(d(x,t)^2) \tag{4.7}$$

for $(x,t) \in Q_{\varepsilon,T}$ and assume that η, v, and the coefficients of $d(x,t)^k$ for each $k \in \mathbb{N}$ in (4.6) and (4.7) are independent of ε. Here $R(d(x,t)^k)$ ($k \in \mathbb{N}$) is the sum of the terms of order equal to or higher than k with respect to small $d(x,t)$. In particular, $R(f(x,t))$ for a function $f(x,t)$ can be of the form

$$R(f(x,t)) = f(x,t)g(x,t)$$

with some (bounded) function $g(x,t)$. Note that we can differentiate $R(d(x,t)^k)$ and its j-th order derivative is of the form $R(d(x,t)^{k-j})$ for $j \leq k$ although we cannot differentiate $O(d(x,t)^k)$ since it only represents a quantity whose absolute value is bounded above by $|d(x,t)|^k$. Also, since $d(x,t)$ is of order ε on $Q_{\varepsilon,T}$, we have $R(d(x,t)^k) = O(\varepsilon^k)$ for $(x,t) \in Q_{\varepsilon,T}$ and $k \in \mathbb{N}$.

Under the expansions (4.6) and (4.7), the limit equations of (4.2)–(4.4) in $\Omega_\varepsilon(t)$ as ε goes to zero are given as equations on $\Gamma(t)$ satisfied by η and v.

Theorem 4.1. *Let ρ and u satisfy the equations* (4.2)–(4.4) *in the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$ given by* (4.5)*. Also, let η and v be the zeroth order terms in the expansions* (4.6) *and* (4.7) *of ρ and u, respectively. Then v is of the form*

$$v = V_\Gamma^N \nu_\Gamma + v^T \quad \text{on} \quad S_T \tag{4.8}$$

with some tangential velocity field v^T on $\Gamma(t)$, and η and v satisfy the equations

$$\partial^\circ \eta - V_\Gamma^N H\eta + \mathrm{div}_\Gamma(\eta v^T) = 0 \qquad \text{on} \quad S_T, \tag{4.9}$$
$$-\eta v^T = \nabla_\Gamma p(\eta) \quad \text{on} \quad S_T. \tag{4.10}$$

Proof. For the sake of simplicity, we use the abbreviations

$$f(\pi,t) = f(\pi(x,t),t), \quad R(d^k) = R(d(x,t)^k) \tag{4.11}$$

for a function f on S_T, $(x,t) \in Q_{\varepsilon,T}$, and $k \in \mathbb{N}$. We also abbreviate the product of several functions with the same argument like

$$[u_1 \cdot u_2](x,t) = u_1(x,t) \cdot u_2(x,t) \tag{4.12}$$

for vector fields u_1 and u_2 on $Q_{\varepsilon,T}$. First we show that v is of the form (4.8). By the definition (4.5) of the moving thin domain $\Omega_\varepsilon(t)$, the unit outward normal vector and the outward normal velocity of its boundary are given by

$$\nu_\Omega(x,t) = \pm\nu_\Gamma(\pi,t), \quad V_\Omega^N(x,t) = \pm V_\Gamma^N(\pi,t) \tag{4.13}$$

for $(x,t) \in \partial_\ell Q_{\varepsilon,T}$ with $d(x,t) = \pm\varepsilon$ (double-sign corresponds). Hence the boundary condition (4.4) reads

$$u(x,t)\cdot\nu_\Gamma(\pi,t) = V_\Gamma^N(\pi,t), \quad (x,t)\in\partial_\ell Q_{\varepsilon,T}.$$

We substitute (4.7) for u in the above equality. Then

$$[v\cdot\nu](\pi,t) \pm \varepsilon[v^1\cdot\nu](\pi,t) + O(\varepsilon^2) = V_\Gamma^N(\pi,t).$$

Since v, v^1, ν_Γ, and V_Γ^N are independent of ε, it follows that

$$[v\cdot\nu](\pi(x,t),t) = V_\Gamma(\pi(x,t),t), \quad [v^1\cdot\nu](\pi(x,t),t) = 0$$

for all $(x,t)\in\partial_\ell Q_{\varepsilon,T}$, which imply that

$$[v\cdot\nu](y,t) = V_\Gamma^N(y,t), \quad (y,t)\in S_T, \tag{4.14}$$
$$[v^1\cdot\nu](y,t) = 0, \qquad (y,t)\in S_T. \tag{4.15}$$

Hence v is of the form (4.8) with some tangential velocity field v^T on $\Gamma(t)$.

Next we derive the equations (4.9)–(4.10). Let $(x,t)\in Q_{\varepsilon,T}$. We differentiate both sides of (4.6) with respect to t and apply (2.3) and (2.7) to get

$$\partial_t\rho(x,t) = \partial^\circ\eta(\pi,t) - [V_\Gamma^N\eta^1](\pi,t) + R(d). \tag{4.16}$$

Let us compute the divergence of ρu. We differentiate $\pi(x,t) = x - d(x,t)\nu_\Gamma(\pi,t)$ with respect to x and apply (2.2) to get

$$\nabla\pi(x,t) = P_\Gamma(\pi,t) + R(d). \tag{4.17}$$

From the expansions (4.6) and (4.7),

$$[\rho u](x,t) = V(\pi,t) + d(x,t)V^1(\pi,t) + R(d^2), \tag{4.18}$$

where

$$V(\pi,t) := [\eta v](\pi,t), \tag{4.19}$$
$$V^1(\pi,t) := [\eta v^1](\pi,t) + [\eta^1 v](\pi,t). \tag{4.20}$$

We differentiate both sides of (4.18) with respect to x. Then by (2.2) and (4.17),

$$\begin{aligned}[\nabla(\rho u)](x,t) &= \nabla\pi(x,t)\nabla V(\pi,t) + \nabla d(x,t)\otimes V^1(\pi,t) + R(d)\\ &= [P_\Gamma\nabla V](\pi,t) + [\nu_\Gamma\otimes V^1](\pi,t) + R(d).\end{aligned}$$

From this formula and $\mathrm{tr}[\nu_\Gamma\otimes V^1] = \nu_\Gamma\cdot V^1$, the divergence of ρu is

$$[\mathrm{div}(\rho u)](x,t) = \mathrm{div}_\Gamma V(\pi,t) + [\nu_\Gamma\cdot V^1](\pi,t) + R(d).$$

Since v is of the form (4.8) and V is given by (4.19),

$$\begin{aligned}\mathrm{div}_\Gamma V &= \mathrm{div}_\Gamma[\eta(V_\Gamma^N\nu_\Gamma + v^T)] = \nabla_\Gamma(\eta V_\Gamma^N)\cdot\nu_\Gamma + \eta V_\Gamma^N\mathrm{div}_\Gamma\nu_\Gamma + \mathrm{div}_\Gamma(\eta v^T)\\ &= -\eta V_\Gamma^N H + \mathrm{div}_\Gamma(\eta v^T)\end{aligned}$$

on S_T by (2.4) and (2.5). We also have

$$[\nu_\Gamma \cdot V^1](\pi,t) = [\eta^1 V_\Gamma^N](\pi,t)$$

by (4.14), (4.15), and (4.20). Therefore,

$$[\mathrm{div}(\rho u)](x,t) = -[V_\Gamma^N H\eta](\pi,t) + [\mathrm{div}_\Gamma(\eta v^T)](\pi,t) + [\eta^1 V_\Gamma^N](\pi,t) + R(d). \tag{4.21}$$

Substituting (4.16) and (4.21) for (4.2), we obtain

$$\partial^\circ \eta(\pi,t) - [V_\Gamma^N H\eta](\pi,t) + [\mathrm{div}_\Gamma(\eta v^T)](\pi,t) = R(d).$$

Here each term on the left-hand side is independent of $d = d(x,t)$. Hence

$$\partial^\circ \eta(\pi(x,t),t) - [V_\Gamma^N H\eta](\pi(x,t),t) + [\mathrm{div}_\Gamma(\eta v^T)](\pi(x,t),t) = 0$$

for all $(x,t) \in Q_{\varepsilon,T}$, which shows that η and $v = V_\Gamma^N \nu_\Gamma + v^T$ satisfy (4.9) on S_T.

Let us derive (4.10). We expand the pressure $p(\rho)$ in $d(x,t)$ as

$$p(\rho(x,t)) = p^0(\pi,t) + d(x,t)p^1(\pi,t) + R(d^2), \quad (x,t) \in Q_{\varepsilon,T}. \tag{4.22}$$

Then it follows form the expansions (4.6) and (4.22) that

$$p^0(\pi(x,t),t) = p(\eta(\pi(x,t),t))$$

for all $(x,t) \in Q_{\varepsilon,T}$, which implies that

$$p^0(y,t) = p(\eta(y,t)), \quad (y,t) \in S_T. \tag{4.23}$$

Moreover, differentiating (4.22) in x and applying (2.2) and (4.17) we get

$$\begin{aligned} \nabla p(\rho(x,t)) &= \nabla\pi(x,t)\nabla p^0(\pi,t) + p^1(\pi,t)\nabla d(x,t) + R(d) \\ &= \nabla_\Gamma p^0(\pi,t) + [p^1\nu_\Gamma](\pi,t) + R(d). \end{aligned}$$

for $(x,t) \in Q_{\varepsilon,T}$. We substitute this for (4.3) and apply (4.8). Then we have

$$-[\eta v^T](\pi,t) - [\eta V_\Gamma^N \nu_\Gamma](\pi,t) + R(d) = \nabla_\Gamma p^0(\pi,t) + [p^1\nu_\Gamma](\pi,t) + R(d).$$

Since all terms except of $R(d)$ are independent of $d = d(x,t)$ and the vectors v^T and $\nabla_\Gamma p^0$ are tangential to $\Gamma(t)$, it follows that

$$-[\eta v^T](\pi(x,t),t) = \nabla_\Gamma p^0(\pi(x,t),t), \quad -[\eta V_\Gamma^N](\pi(x,t),t) = p^1(\pi(x,t),t)$$

for all $(x,t) \in Q_{\varepsilon,T}$. Therefore, we get

$$-[\eta v^T](y,t) = \nabla_\Gamma p^0(y,t), \quad (y,t) \in S_T, \tag{4.24}$$

$$-[\eta V_\Gamma^N](y,t) = p^1(y,t), \qquad (y,t) \in S_T. \tag{4.25}$$

By (4.23) and (4.24) we conclude that η and v satisfy (4.10) on S_T. □

Remark 4.2. By the proof of Theorem 4.1 we observe that the transport equation (3.5) on the moving surface $\Gamma(t)$ can be derived as the limit of the transport equation (3.2) in the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$ with the boundary condition (3.3) as ε goes to zero.

5 Energy law

The subject in this section is the energy law for nonlinear diffusion equations (4.2)–(4.4) and (4.9)–(4.10). As in Section 4, the pressure $p(\rho)$ is given by (4.1) with a given function $\omega(\rho)$.

Proposition 5.1. *Assume that ρ and u satisfy* (4.2)–(4.4). *Then*

$$\frac{d}{dt}\int_{\Omega(t)} \omega(\rho)\,dx = -\int_{\Omega(t)} \rho|u|^2\,dx - \int_{\partial\Omega(t)} p(\rho)V_\Omega^N\,d\mathcal{H}^{n-1}. \tag{5.1}$$

Proof. By the Reynolds transport theorem,

$$\frac{d}{dt}\int_{\Omega(t)} \omega(\rho)\,d\mathcal{H}^{n-1} = \int_{\Omega(t)} \partial_t\omega(\rho)\,dx + \int_{\partial\Omega(t)} \omega(\rho)V_\Omega^N\,d\mathcal{H}^{n-1}.$$

Since $\partial_t\omega(\rho) = \omega'(\rho)\partial_t\rho$ and the transport equation (4.2) is satisfied,

$$\partial_t\omega(\rho) = -\omega'(\rho)\mathrm{div}(\rho u) = -\mathrm{div}(\omega'(\rho)\rho u) + \nabla\omega'(\rho)\cdot(\rho u).$$

Hence the divergence theorem and (4.4) yield

$$\int_{\Omega(t)} \partial_t\omega(\rho)\,dx = -\int_{\partial\Omega(t)} \omega'(\rho)\rho V_\Omega^N\,d\mathcal{H}^{n-1} + \int_{\Omega(t)} \nabla\omega'(\rho)\cdot(\rho u)\,dx.$$

Using this formula we get

$$\frac{d}{dt}\int_{\Omega(t)} \omega(\rho)\,dx = \int_{\Omega(t)} \nabla\omega'(\rho)\cdot(\rho u)\,dx - \int_{\partial\Omega(t)} \{\omega'(\rho)\rho - \omega(\rho)\}V_\Omega^N\,d\mathcal{H}^{n-1}.$$

The energy law (5.1) follows from this equality, (4.1), and

$$u = -\frac{\nabla p(\rho)}{\rho} = -\nabla\omega'(\rho)$$

by (4.1) and (4.3). $\square$

Proposition 5.2. *Suppose that η and v of the form* (4.8) *satisfy* (4.9) *and* (4.10). *Then*

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\,d\mathcal{H}^{n-1} = -\int_{\Gamma(t)} \eta|v^T|^2\,d\mathcal{H}^{n-1} + \int_{\Gamma(t)} p(\eta)V_\Gamma^N H\,d\mathcal{H}^{n-1}. \tag{5.2}$$

Proof. By the Leibniz formula [4, Lemma 2.2],

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\,d\mathcal{H}^{n-1} = I - \int_{\Gamma(t)} \omega(\eta)V_\Gamma^N H\,d\mathcal{H}^{n-1},$$

where

$$I = \int_{\Gamma(t)} \{\partial^\circ\omega(\eta) + \mathrm{div}_\Gamma(\omega(\eta)v^T)\}\,d\mathcal{H}^{n-1}.$$

Since $\partial^\circ \omega(\eta) = \omega'(\eta)\partial^\circ \eta$, the transport equation (4.9) implies that

$$\begin{aligned}\partial^\circ \omega(\eta) &= \omega'(\eta)\{V_\Gamma^N H\eta - \mathrm{div}_\Gamma(\eta v^T)\} \\ &= \omega'(\eta)V_\Gamma^N H\eta + \nabla_\Gamma \omega'(\eta)\cdot(\eta v^T) - \mathrm{div}_\Gamma(\omega'(\eta)\eta v^T).\end{aligned}$$

Hence

$$I = \int_{\Gamma(t)} \{\omega'(\eta)V_\Gamma^N H\eta + \nabla_\Gamma \omega'(\eta)\cdot(\eta v^T)\}\, d\mathcal{H}^{n-1} + \int_{\Gamma(t)} \mathrm{div}_\Gamma[(\omega(\eta) - \omega'(\eta)\eta)v^T]\, d\mathcal{H}^{n-1}.$$

The second integral on the right-hand side vanishes by the Stokes formula and the fact that v^T is tangential and $\Gamma(t)$ has no boundary. Therefore,

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\, d\mathcal{H}^{n-1} = \int_{\Gamma(t)} \nabla_\Gamma \omega'(\eta)\cdot(\eta v^T)\, d\mathcal{H}^{n-1} + \int_{\Gamma(t)} \{\omega'(\eta)\eta - \omega(\eta)\}V_\Gamma^N H\, d\mathcal{H}^{n-1}.$$

Applying

$$v^T = -\frac{\nabla_\Gamma p(\eta)}{\eta} = -\nabla_\Gamma \omega'(\eta),$$

which follows from (4.1) and (4.10), to the first term on the right-hand side and (4.1) to the second term, we get the energy identity (5.2). □

Next we derive the energy law (5.2) as a limit of the energy law (5.1) with the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$ when ε goes to zero. As in Section 4, we expand ρ and u in powers of the signed distance as (4.6) and (4.7) and determine an equality satisfied by η and v.

Theorem 5.3. *Let ρ and u satisfy the energy law* (5.1) *in the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$ given by* (4.5). *Also, let η and v be the zeroth order terms in the expansions* (4.6) *and* (4.7) *of ρ and u, respectively. Assume that v is of the form* (4.8) *with some tangential velocity field v^T on $\Gamma(t)$ and Darcy's law* (4.3) *holds in $\Omega_\varepsilon(t)$. Then η and v satisfy the energy law* (5.2).

We give change of variables formulas for integrals which we use in the proof of Theorem 5.3. For $y \in \Gamma(t)$ and $\rho \in [-\varepsilon, \varepsilon]$ we set

$$J(y,t,r) := \prod_{i=1}^{n-1}\{1 - r\kappa_i(y,t)\}, \tag{5.3}$$

where $\kappa_1(\cdot,t)$, …, $\kappa_{n-1}(\cdot,t)$ are the principal curvatures of $\Gamma(t)$. It is the Jacobian that appears when we change variables of integrals over a tubular neighborhood $\{x \in \mathbb{R}^n \mid -r < d(x,t) < r\}$ $(r > 0)$ of $\Gamma(t)$ and a level-set surface $\{x \in \mathbb{R}^n \mid d(x,t) = s\}$ $(s \in \mathbb{R})$ in terms of the normal coordinate system around $\Gamma(t)$ (see [6, Section 14.6] for example). The first formula in Lemma 5.4 is often called the co-area formula.

Lemma 5.4. *For functions f on $\Omega_\varepsilon(t)$ and g on $\partial\Omega_\varepsilon(t)$ we have*

$$\int_{\Omega_\varepsilon(t)} f(x)\,dx = \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} f(y + r\nu_\Gamma(y,t)) J(y,t,r)\,dr\,d\mathcal{H}^{n-1}(y) \tag{5.4}$$

and

$$\begin{aligned} \int_{\partial\Omega_\varepsilon(t)} g(x)\,d\mathcal{H}^{n-1}(x) = \int_{\Gamma(t)} & g(y + \varepsilon\nu_\Gamma(y,t)) J(y,t,\varepsilon)\,d\mathcal{H}^{n-1}(y) \\ & + \int_{\Gamma(t)} g(y - \varepsilon\nu_\Gamma(y,t)) J(y,t,-\varepsilon)\,d\mathcal{H}^{n-1}(y). \end{aligned} \tag{5.5}$$

Proof of Theorem 5.3. As in the proof of Theorem 4.1, we use the abbreviations (4.11) and (4.12). Let us calculate each term of (5.1). We expand $\omega(\rho)$ in powers of the signed distance $d(x,t)$ as

$$\omega(\rho(x,t)) = \omega(\eta(\pi,t)) + d(x,t)\omega^1(\pi,t) + R(d^2), \quad (x,t) \in Q_{\varepsilon,T}.$$

Here the zeroth order term is $\omega(\eta(\pi,t))$ since the zeroth order term of $\rho(x,t)$ is $\eta(\pi,t)$. We divide the integral of $\omega(\rho)$ over $\Omega_\varepsilon(t)$ as

$$\int_{\Omega_\varepsilon(t)} \omega(\rho(x,t))\,dx = I_1 + I_2 + I_3,$$

where

$$I_1 := \int_{\Omega_\varepsilon(t)} \omega(\eta(\pi,t))\,dx, \quad I_2 := \int_{\Omega_\varepsilon(t)} d(x,t)\omega^1(\pi,t)\,dx, \quad I_3 := \int_{\Omega_\varepsilon(t)} R(d(x,t)^2)\,dx.$$

By the co-area formula (5.4) and the fact that $J(y,t,r)$ is a polynomial in r with $J(y,t,0) = 1$ whose coefficients are polynomials in the principal curvatures, we have

$$\begin{aligned} I_1 &= \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} \omega(\eta(y,t)) J(y,t,r)\,dr\,d\mathcal{H}^{n-1}(y) \\ &= 2\varepsilon \int_{\Gamma(t)} \omega(\eta(y,t))\,d\mathcal{H}^{n-1}(y) + \varepsilon^2 f_1(\varepsilon,t), \end{aligned}$$

where $f_1(\varepsilon,t)$ is a polynomial in ε with time-dependent coefficients. Therefore,

$$\frac{dI_1}{dt} = 2\varepsilon \frac{d}{dt} \int_{\Gamma(t)} \omega(y,t)\,d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \tag{5.6}$$

Similarly we have

$$I_2 = \int_{\Gamma(t)} \int_{-\varepsilon}^{\varepsilon} r\omega^1(y,t) J(y,t,r)\,dr\,d\mathcal{H}^{n-1}(y) = \varepsilon^2 f_2(\varepsilon,t)$$

with a polynomial $f_2(\varepsilon,t)$ in ε with time-dependent coefficients and thus

$$\frac{dI_2}{dt} = O(\varepsilon^2). \tag{5.7}$$

We apply the Reynolds transport theorem to the time derivative of I_3. Then, since the time derivative of $R(d(x,t)^2)$ is $R(d(x,t))$, we have

$$\frac{dI_3}{dt} = \int_{\Omega_\varepsilon(t)} R(d(x,t))\,dx + \int_{\partial\Omega_\varepsilon(t)} R(d(x,t)^2)V_\Omega^N(x,t)\,d\mathcal{H}^{n-1}(x).$$

Since $J(y,t,r)$ is bounded independently of ε, the co-area formula (5.4) yields

$$\int_{\Omega_\varepsilon(t)} R(d(x,t))\,dx = \int_{\Gamma(t)}\int_{-\varepsilon}^{\varepsilon} R(r)J(y,t,r)\,dr\,d\mathcal{H}^{n-1}(y) = O(\varepsilon^2).$$

Moreover, applying (4.13) and (5.5) to the integral of $R(d(x,t)^2)V_\Omega^N(x,t)$ over $\partial\Omega_\varepsilon(t)$ and observing that $R(d(x,t)^2) = R(\varepsilon^2)$ holds for $x \in \partial\Omega_\varepsilon(t)$ we have

$$\int_{\partial\Omega_\varepsilon(t)} R(d(x,t)^2)V_\Omega^N(x,t)\,d\mathcal{H}^{n-1}(x) = O(\varepsilon^2).$$

Thus, we get the estimate

$$\frac{dI_3}{dt} = O(\varepsilon^2). \tag{5.8}$$

Since the integral of $\omega(\rho)$ over $\Omega_\varepsilon(t)$ is the sum of I_1, I_2, and I_3, it follows from (5.6), (5.7), and (5.8) that

$$\frac{d}{dt}\int_{\Omega_\varepsilon(t)} \omega(\rho(x,t))\,dx = 2\varepsilon\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta(y,t))\,d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \tag{5.9}$$

Next we calculate the first term on the right-hand side of (5.1). From the expansions (4.6) and (4.7), the product $\rho|u|^2$ is of the form

$$[\rho|u|^2](x,t) = [\eta|v|^2](\pi,t) + R(d), \quad (x,t) \in Q_{\varepsilon,T}.$$

Hence, by (5.4),

$$\begin{aligned}\int_{\Omega_\varepsilon(t)} [\rho|u|^2](x,t)\,dx &= \int_{\Gamma(t)}\int_{-\varepsilon}^{\varepsilon} \{[\eta|v|^2](y,t) + R(r)\}J(y,t,r)\,dr\,d\mathcal{H}^{n-1}(y) \\ &= 2\varepsilon\int_{\Gamma(t)} [\eta|v|^2](y,t)\,d\mathcal{H}^{n-1}(y) + O(\varepsilon^2).\end{aligned} \tag{5.10}$$

Let us compute the last term on the right-hand side of (5.1). We expand the pressure $p(\rho)$ in $d(x,t)$ as (4.22). Then, by the assumption that v is of the form (4.8) and Darcy's law (4.3) holds, we get (4.23) and (4.25) as in the proof of Theorem 4.1 and thus we can write

$$p(\rho(x,t)) = p(\eta(\pi,t)) - d(x,t)[\eta V_\Gamma^N](\pi,t) + R(d^2), \quad (x,t) \in Q_{\varepsilon,T}.$$

Therefore, by (4.13) and (5.5),

$$\int_{\partial\Omega_\varepsilon(t)} [p(\rho)V_\Omega^N](x,t)\,d\mathcal{H}^{n-1}(x) = J_1 + J_2 + O(\varepsilon^2),$$

where

$$J_1 := \int_{\Gamma(t)} [p(\eta)V_\Gamma^N](y,t)\{J(y,t,\varepsilon) - J(y,t,-\varepsilon)\}\, d\mathcal{H}^{n-1}(y),$$
$$J_2 := -\varepsilon \int_{\Gamma(t)} [\eta|V_\Gamma^N|^2](y,t)\{J(y,t,\varepsilon) + J(y,t,-\varepsilon)\}\, d\mathcal{H}^{n-1}(y).$$

By (5.3) and (2.6) we have

$$J(y,t,\varepsilon) - J(y,t,-\varepsilon) = -2\varepsilon H(y,t) + O(\varepsilon^2),$$
$$J(y,t,\varepsilon) + J(y,t,-\varepsilon) = 2 + O(\varepsilon).$$

Hence it follows that

$$J_1 = -2\varepsilon \int_{\Gamma(t)} [p(\eta)V_\Gamma^N H](y,t)\, d\mathcal{H}^{n-1}(y) + O(\varepsilon^2),$$
$$J_2 = -2\varepsilon \int_{\Gamma(t)} [\eta|V_\Gamma^N|^2](y,t)\, d\mathcal{H}^{n-1}(y) + O(\varepsilon^2)$$

and the integral of $p(\rho)V_\Omega^N$ over $\partial\Omega_\varepsilon(t)$ becomes

$$\int_{\partial\Omega_\varepsilon(t)} [p(\rho)V_\Omega^N](x,t)\, d\mathcal{H}^{n-1}(x) = -2\varepsilon \int_{\Gamma(t)} [p(\eta)V_\Gamma^N H](y,t)\, d\mathcal{H}^{n-1}(y) - 2\varepsilon \int_{\Gamma(t)} [\eta|V_\Gamma^N|^2](y,t)\, d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \quad (5.11)$$

Finally, substituting (5.9), (5.10), and (5.11) for (5.1), dividing both sides by 2ε, and observing that $|v|^2 = |V_\Gamma^N|^2 + |v^T|^2$ we obtain

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta(y,t))\, d\mathcal{H}^{n-1}(y) = -\int_{\Gamma(t)} [\eta|v^T|^2](y,t)\, d\mathcal{H}^{n-1}(y) + \int_{\Gamma(t)} [p(\eta)V_\Gamma^N H](y,t)\, d\mathcal{H}^{n-1}(y) + O(\varepsilon).$$

In the above equality all terms except for $O(\varepsilon)$ are independent of ε. Hence we conclude that η and v satisfy the energy law (5.2). □

Remark 5.5 (A failure of a simple rescaling argument for a moving surface)**.** It is possible to derive the limit energy identity by a rescaling argument. However, derivation by a rescaling argument is somewhat misleading. Let ρ and u satisfy the energy identity (5.1) in the moving thin domain $\Omega(t) = \Omega_\varepsilon(t)$. We set

$$\eta(y,t,r) := \rho(y + \varepsilon r \nu_\Gamma(y,t), t), \quad v(y,t,r) := u(y + \varepsilon r \nu_\Gamma(y,t), t)$$

for $(y,t) \in S_T$ and $r \in (-1,1)$. Then by (4.13) and the integral transformation formulas (5.4) and (5.5) we can write (5.1) in terms of η and v as

$$\begin{aligned}
&\varepsilon\frac{d}{dt}\int_{\Gamma(t)}\int_{-1}^{1}\omega(\eta(y,r))J(y,\varepsilon r)\,dr\,d\mathcal{H}^{n-1}(y)\\
&= -\varepsilon\int_{\Gamma(t)}\int_{-1}^{1}[\eta|v|^2](y,r)J(y,\varepsilon r)\,dr\,d\mathcal{H}^{n-1}(y) - \int_{\Gamma(t)} p(\eta(y,1))V_\Gamma^N(y)J(y,\varepsilon)\,d\mathcal{H}^{n-1}(y)\\
&\qquad + \int_{\Gamma(t)} p(\eta(y,-1))V_\Gamma^N(y)J(y,-\varepsilon)\,d\mathcal{H}^{n-1}(y). \quad (5.12)
\end{aligned}$$

Here we used the abbreviation (4.12) and suppressed the argument t of functions.

In formal derivation of a thin width limit by a rescaling argument we usually assume that rescaled functions are independent of the thin direction to get limit relations on a degenerate set. However, making the assumption at an inappropriate point may result in a wrong limit. To see this, let us assume that η and v are independent of the variable r in (5.12). Then since

$$J(y,\varepsilon) - J(y,-\varepsilon) = -2\varepsilon H(y) + O(\varepsilon^2)$$

by (5.3) and (2.6), it follows that

$$\begin{aligned}
2\varepsilon\frac{d}{dt}\int_{\Gamma(t)}\omega(\eta(y))\,d\mathcal{H}^{n-1}(y) &= -2\varepsilon\int_{\Gamma(t)}[\eta|v|^2](y)\,d\mathcal{H}^{n-1}(y)\\
&\qquad + 2\varepsilon\int_{\Gamma(t)}[p(\eta)V_\Gamma^N H](y)\,d\mathcal{H}^{n-1}(y) + O(\varepsilon^2).
\end{aligned}$$

Dividing both sides by 2ε and taking the principal term we obtain

$$\frac{d}{dt}\int_{\Gamma(t)}\omega(\eta)\,d\mathcal{H}^{n-1} = -\int_{\Gamma(t)}\eta|v|^2\,d\mathcal{H}^{n-1} + \int_{\Gamma(t)}p(\eta)V_\Gamma^N H\,d\mathcal{H}^{n-1}.$$

In this equality v should be of the form $v = V_\Gamma^N\nu_\Gamma + v^T$ with some tangential velocity field v^T, since it is the velocity of a substance on the moving surface $\Gamma(t)$ with normal velocity V_Γ^N. Hence we get

$$\frac{d}{dt}\int_{\Gamma(t)}\omega(\eta)\,d\mathcal{H}^{n-1} = -\int_{\Gamma(t)}\eta(|V_\Gamma^N|^2 + |v^T|^2)\,d\mathcal{H}^{n-1} + \int_{\Gamma(t)}p(\eta)V_\Gamma^N H\,d\mathcal{H}^{n-1},$$

which includes an additional term $\int_{\Gamma(t)}\eta|V_\Gamma^N|^2\,d\mathcal{H}^{n-1}$ compared to the limit energy identity (5.2). This improper term appears because we ignore the difference between $p(\eta(y,t,1))$ and $p(\eta(y,t,-1))$ in (5.12) by assuming that η is independent of the variable r. Of course it vanishes if the shape of the surface does not change, i.e. $V_\Gamma^N = 0$. This is the reason why this simple rescaling argument is popular to derive a thin width limit problem in a formal level when the degenerate set of a thin domain does not change its shape.

Remark 5.6 (Corrected rescaling argument)**.** To obtain the correct limit (5.2) we should take into account the difference between $p(\eta(y,t,1))$ and $p(\eta(y,t,-1))$ in (5.12). Let us rewrite the sum of the last two terms in the right-hand side of (5.12) into the sum of

$$I_1 = -\int_{\Gamma(t)} \{p(\eta(y,1)) - p(\eta(y,-1))\} V_\Gamma^N(y)\, d\mathcal{H}^{n-1}(y),$$
$$I_2 = \varepsilon \int_{\Gamma(t)} \{p(\eta(y,1)) + p(\eta(y,-1))\} [V_\Gamma^N H](y)\, d\mathcal{H}^{n-1}(y),$$

and a residual term $O(\varepsilon^2)$ and calculate them properly (here we again suppressed the argument t of functions). For I_2 we merely assume that η is independent of r to get

$$I_2 = 2\varepsilon \int_{\Gamma(t)} [p(\eta) V_\Gamma^N H](y)\, d\mathcal{H}^{n-1}(y). \tag{5.13}$$

For a proper calculation of I_1 we need to impose Darcy's law (4.3) in $\Omega_\varepsilon(t)$ and describe it in terms of the rescaled functions. By the definition of η,

$$p(\rho(x)) = p(\eta(\pi(x), \varepsilon^{-1}d(x))), \quad x \in \Omega_\varepsilon(t).$$

We differentiate both sides in x and use (2.2) and (4.17). Then

$$\nabla p(\rho(x)) = \nabla_\Gamma p(\eta(\pi, \varepsilon^{-1}d)) + \varepsilon^{-1}\partial_r p(\eta(\pi, \varepsilon^{-1}d))\nu_\Gamma(\pi) + O(\varepsilon),$$

where we abbreviate $\pi(x)$ and $d(x)$ to π and d in the right-hand side. Substituting this for (4.3) and taking the normal component of the resulting equation we obtain

$$\partial_r p(\eta(y,r)) = -\varepsilon[\eta v](y,r) \cdot \nu_\Gamma(y) + O(\varepsilon^2) \tag{5.14}$$

for $y \in \Gamma(t)$ and $r \in (-1,1)$. We apply the mean value theorem and (5.14) to the difference between $p(\eta(y,1))$ and $p(\eta(y,-1))$. Then

$$p(\eta(y,1)) - p(\eta(y,-1)) = 2\partial_r p(\eta(y,\theta)) = -2\varepsilon[\eta v](y,\theta) \cdot \nu_\Gamma(y) + O(\varepsilon^2)$$

with some $\theta = \theta(y,t) \in (-1,1)$. Hence I_1 is expressed as

$$I_1 = 2\varepsilon \int_{\Gamma(t)} [\eta v](y,\theta) \cdot \nu_\Gamma(y) V_\Gamma^N(y)\, d\mathcal{H}^{n-1}(y) + O(\varepsilon^2).$$

Now we assume that η and v are independent of the argument r. Then we have

$$I_1 = 2\varepsilon \int_\Gamma [\eta(v \cdot \nu_\Gamma) V_\Gamma^N](y)\, d\mathcal{H}^{n-1}(y) + O(\varepsilon^2). \tag{5.15}$$

We substitute (5.13) and (5.15) for (5.12), assume that the rescaled functions are constant in the variable r for the left-hand side and the first term on the right-hand side, and divide both sides by 2ε after calculations. Then the principal term on the resulting equation is

$$\frac{d}{dt}\int_{\Gamma(t)} \omega(\eta)\, d\mathcal{H}^{n-1} = -\int_{\Gamma(t)} \eta|v|^2\, d\mathcal{H}^{n-1} + \int_{\Gamma(t)} \eta(v \cdot \nu_\Gamma) V_\Gamma^N\, d\mathcal{H}^{n-1}$$
$$+ \int_{\Gamma(t)} p(\eta) V_\Gamma^N H\, d\mathcal{H}^{n-1}.$$

Finally we suppose that v is of the form $v = V_\Gamma^N \nu_\Gamma + v^T$ with some tangential velocity v^T, which is natural since it is the velocity of a substance on the moving surface $\Gamma(t)$ with normal velocity V_Γ^N as we mentioned in Remark 5.5. Then we obtain the proper limit energy identity (5.2) from the above equality.

6 Energetic variation for derivation of Darcy's law

In this section we discuss the energetic variational approach [3, 12, 25] for nonlinear diffusion equations in a moving domain and on a moving surface. For a general non-equilibrium thermodynamic system, if the system is isothermal, then the combination of the first and second laws of thermodynamics yields

$$\frac{d}{dt}E^{\text{total}} = \dot{W} - \Delta,$$

where $E^{\text{total}} = \mathcal{K} + \mathcal{F}$ is the sum of the kinetic energy $\mathcal{K}$ and the Helmholtz free energy $\mathcal{F}$, Δ is the entropy production, and $\dot{W}$ is the rate of change of work done by the external environment. If the system is closed, i.e. $\dot{W} = 0$, we further get the energy dissipation law

$$\frac{d}{dt}E^{\text{total}} = -2\mathcal{D},$$

where $\mathcal{D} = \Delta/2$ is sometimes called the energy dissipation. For a conservative system ($\Delta = 0$), the principle of least action (LAP) [1] states that the variation of the kinetic and the free energies with respect to the flow map in Lagrangian coordinates yield the internal force F_i and the conservative force F_c. Formally it can be written as

$$\delta\left(\int_0^T \mathcal{K}\,dt\right) = \int_0^T \int (F_i \cdot \delta x)\,dx\,dt,$$
$$\delta\left(\int_0^T \mathcal{F}\,dt\right) = \int_0^T \int (F_c \cdot \delta x)\,dx\,dt,$$

where δ represents the procedure of variation. Sometimes such a calculation is also referred to as the principle of virtual work. Based on the LAP, the equation of motion for a conservative system is described by balance of forces:

$$F_i = F_c.$$

For a dissipative system, we use the maximum dissipation principle (MDP) [16, 17] to the dissipative force F_d: by taking the variation of the dissipation with respect to the velocity in Eulerian coordinates, we have

$$\delta\mathcal{D} = F_d \cdot \delta u.$$

When all forces are derived, the equation of motion for a dissipative system is formulated as balance of forces (Newton's third law):

$$F_i = F_c + F_d.$$

Let us apply the above energetic variational framework to the energy laws (5.1) and (5.2). For (5.1) we have

$$\mathcal{K} = 0, \quad \mathcal{F} = \int_{\Omega(t)} \omega(\rho)\, dx,$$
$$\mathcal{D} = \frac{1}{2}\int_{\Omega(t)} \rho|u|^2\, dx, \quad \dot{W} = -\int_{\partial\Omega(t)} p(\rho) V_\Omega^N\, d\mathcal{H}^{n-1}.$$

Let $x\colon \Omega(0) \times [0,T] \to \mathbb{R}^n$ be the flow map of the velocity field u, i.e. for each $t \in [0,T]$ the mapping $x(\cdot,t)$ is a diffeomorphism from $\Omega(0)$ onto $\Omega(t)$ and

$$x(X,0) = X, \quad \frac{\partial}{\partial t} x(X,t) = u(x(X,t),t), \quad (X,t) \in \Omega(0) \times (0,T).$$

We write F for the deformation matrix of x:

$$F(X,t) = \frac{\partial x}{\partial X}(X,t), \quad (X,t) \in \Omega(0) \times (0,T).$$

The MDP gives the dissipative force

$$\frac{\delta \mathcal{D}}{\delta u} = \rho u. \tag{6.1}$$

On the other hand, the LAP shows that the conservative force is given by the gradient of the pressure.

Lemma 6.1. *Suppose that ρ and u satisfy the transport equation* (3.2). *Then*

$$\frac{\delta \mathcal{F}}{\delta x} = \nabla p(\rho), \tag{6.2}$$

where $p(\rho)$ is given by (4.1).

Proof. Throughout the proof we use the notation

$$f^\sharp(X,t) = f(x(X,t),t), \quad (X,t) \in \Omega(0) \times (0,T)$$

for a function f on Q_T. Since the transport equation (3.2) is satisfied, the density ρ is given by

$$\rho(x(X,t),t) = \frac{\rho_0(X)}{\det F(X,t)}, \quad (X,t) \in \Omega(0) \times (0,T) \tag{6.3}$$

with the initial density ρ_0 and thus the free energy $\mathcal{F}(x) = \mathcal{F}(x(\cdot,t))$ is of the form

$$\mathcal{F}(x) = \int_{\Omega(0)} \omega\left(\frac{\rho_0(X)}{\det F(X,t)}\right) \det F(X,t)\, dX, \quad t \in (0,T). \tag{6.4}$$

Let $\{x^\varepsilon\}_\varepsilon$ be a family of flow maps and $u^\varepsilon = \partial x^\varepsilon/\partial t$ such that

$$x^\varepsilon(\cdot,0) = x(\cdot,0), \quad x^\varepsilon(\cdot,T) = x(\cdot,T) \quad \text{for all} \quad \varepsilon,$$
$$x^\varepsilon(\cdot,t)|_{\varepsilon=0} = x(\cdot,t), \quad u^\varepsilon(\cdot,t)|_{\varepsilon=0} = u(\cdot,t), \quad \left.\frac{d}{d\varepsilon}x^\varepsilon(\cdot,t)\right|_{\varepsilon=0} = w(x(\cdot,t),t)$$

with any given vector field $w\colon Q_T \to \mathbb{R}^n$. We write F^ε for the deformation matrix of x^ε. Suppose that ρ^ε and u^ε satisfy the transport equation (3.2) with the same initial density ρ_0. Then the relation (6.3) with ρ, x, and F replaced by ρ^ε, x^ε, and F^ε holds and by (6.4) the free energy $\mathcal{F}$ with respect to the perturbed flow map x^ε is given by

$$\mathcal{F}(x^\varepsilon) = \int_{\Omega(0)} \omega\left(\frac{\rho_0}{\det F^\varepsilon}\right) \det F^\varepsilon \, dX. \tag{6.5}$$

Note that the argument $t \in (0,T)$ is suppressed in the above equality. We differentiate $\int_0^T \mathcal{F}(x^\varepsilon)\, dt$ with respect to ε at $\varepsilon = 0$. Since $F^\varepsilon|_{\varepsilon=0} = F$ and

$$\left.\frac{dF^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial}{\partial X}\frac{dx^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\partial w^\sharp}{\partial X},$$

the derivative of the determinant of F^ε with respect to ε at $\varepsilon = 0$ is

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\det F^\varepsilon\right|_{\varepsilon=0} &= \operatorname{tr}\left((F^\varepsilon)^{-1}\frac{dF^\varepsilon}{d\varepsilon}\right)\det F^\varepsilon\Big|_{\varepsilon=0} \\ &= \operatorname{tr}\left(F^{-1}\frac{\partial w^\sharp}{\partial X}\right)\det F = (\operatorname{div} w)^\sharp \det F,\end{aligned} \tag{6.6}$$

where F^{-1} and $(F^\varepsilon)^{-1}$ are the inverse matrix of F and F^ε, respectively. We differentiate the integrand of (6.5) at $\varepsilon = 0$ and apply (6.3), (6.6), and $F^\varepsilon|_{\varepsilon=0} = F$ to obtain

$$\left.\frac{d}{d\varepsilon}\left(\omega\left(\frac{\rho_0}{\det F^\varepsilon}\right)\det F^\varepsilon\right)\right|_{\varepsilon=0} = \{-\omega'(\rho^\sharp)\rho^\sharp + \omega(\rho^\sharp)\}(\operatorname{div} w)^\sharp \det F.$$

Therefore,

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\int_0^T \mathcal{F}(x^\varepsilon)\, dt\right|_{\varepsilon=0} &= \int_0^T\int_{\Omega(0)} \{-\omega'(\rho^\sharp)\rho^\sharp + \omega(\rho^\sharp)\}(\operatorname{div} w)^\sharp \det F \, dX\, dt \\ &= \int_0^T\int_{\Omega(t)} \{-\omega'(\rho)\rho + \omega(\rho)\}\operatorname{div} w \, dx\, dt \\ &= \int_0^T\int_{\Omega(t)} \nabla[\omega'(\rho)\rho - \omega(\rho)]\cdot w \, dx\, dt\end{aligned}$$

and (6.2) follows. □

By (6.1), (6.2), and $\mathcal{K} = 0$, the balance of forces

$$\frac{\delta\mathcal{K}}{\delta x} = \frac{\delta\mathcal{F}}{\delta x} + \frac{\delta\mathcal{D}}{\delta u}$$

is of the form

$$0 = \nabla p(\rho) + \rho u, \quad \text{i.e.} \quad -\rho u = \nabla p(\rho),$$

which is exactly Darcy's law in a moving domain. Combining this with the transport equation (3.2), we obtain the nonlinear diffusion equations

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad -\rho u = \nabla p(\rho)$$

in the moving domain $\Omega(t)$, where $p(\rho)$ is given by (4.1).

From the above discussion, we expect that the energetic variational approach for (5.2) yields Darcy' law (4.10) on a moving surface. For (5.2) we have

$$\mathcal{K} = 0, \quad \mathcal{F} = \int_{\Gamma(t)} \omega(\eta)\, d\mathcal{H}^{n-1},$$
$$\mathcal{D} = \frac{1}{2}\int_{\Gamma(t)} \eta |v^T|^2\, d\mathcal{H}^{n-1}, \quad \dot{W} = \int_{\Gamma(t)} p(\eta) H V_\Gamma^N\, d\mathcal{H}^{n-1}.$$

The variation of $\mathcal{D}$ with respect to the total velocity $v = V_\Gamma^N \nu + v^T$ gives

$$\frac{\delta \mathcal{D}}{\delta v} = \eta v^T, \tag{6.7}$$

since $v^T = P_\Gamma v$. Let us apply the LAP to the free energy $\mathcal{F}$.

Lemma 6.2. *Suppose that η and v of the form* (4.8) *satisfy the transport equation* (3.5). *Then*

$$\frac{\delta \mathcal{F}}{\delta y} = \nabla_\Gamma p(\eta), \tag{6.8}$$

where $p(\eta)$ is given by (4.1).

We localize integrals over $\Gamma(t)$ with a partition of unity of $\Gamma(t)$ as in [14, Section 2.4] and take the variation of $\mathcal{F}$ with respect to a flow map in "local Lagrangian coordinates." Let U be an open set in $\mathbb{R}^{n-1}$. We call a mapping $y\colon U \times [0,T] \to \mathbb{R}^n$ the flow map of the velocity $v = V_\Gamma \nu_\Gamma + v^T$ in local Lagrangian coordinates if $y(\cdot,t)\colon U \to \Gamma(t)$ is a smooth local parametrization of $\Gamma(t)$ for each $t \in [0,T]$ and

$$y(Y,0) \in \Gamma(0), \quad \frac{\partial}{\partial t} y(Y,t) = v(y(Y,t),t), \quad (Y,t) \in U \times (0,T). \tag{6.9}$$

We consider a localized surface integral

$$\mathcal{F}(y) = \mathcal{F}(y(\cdot,t)) = \int_{y(U,t)} \omega(\eta)\, d\mathcal{H}^{n-1} \tag{6.10}$$

and take its variation with respect to y. Let $\{y^\varepsilon\}_\varepsilon$ be a family of flow maps in local Lagrangian coordinates and $v^\varepsilon = \partial y^\varepsilon / \partial t$ such that

$$\begin{gathered} y^\varepsilon(\cdot,0) = y(\cdot,0), \quad y^\varepsilon(\cdot,T) = y(\cdot,T) \quad \text{for all} \quad \varepsilon, \\ y^\varepsilon(\cdot,t)|_{\varepsilon=0} = y(\cdot,t), \quad v^\varepsilon(\cdot,t)|_{\varepsilon=0} = v(\cdot,t), \quad \frac{d}{d\varepsilon} y^\varepsilon(\cdot,t)\Big|_{\varepsilon=0} = w(y(\cdot,t),t) \end{gathered} \tag{6.11}$$

with any given vector field $w\colon S_T \to \mathbb{R}^n$ such that $w(\cdot,t)$ is tangential on $\Gamma(t)$ for each $t \in (0,T)$. For a function f on S_T we use the notation

$$f^\sharp(Y,t) = f(y(Y,t),t), \quad (Y,t) \in U \times (0,T). \tag{6.12}$$

Lemma 6.3. *Let $g = (g_{ij})_{i,j}$ be a matrix given by*

$$g_{ij} = \frac{\partial y}{\partial Y_i} \cdot \frac{\partial y}{\partial Y_j}, \quad i,j = 1,\dots,n-1 \tag{6.13}$$

and $g^\varepsilon = (g^\varepsilon_{ij})_{i,j}$ be a matrix given as above with y replaced by y^ε. Then

$$\frac{d}{d\varepsilon}\sqrt{\det g^\varepsilon}\bigg|_{\varepsilon=0} = (\mathrm{div}_\Gamma w)^\sharp \sqrt{\det g}. \tag{6.14}$$

Proof. Since $g^\varepsilon|_{\varepsilon=0} = g$ and

$$\frac{d}{d\varepsilon}\det g^\varepsilon = \mathrm{tr}\left((g^\varepsilon)^{-1}\frac{dg^\varepsilon}{d\varepsilon}\right)\det g^\varepsilon,$$

where $(g^\varepsilon)^{-1}$ is the inverse matrix of g^ε, we have

$$\frac{d}{d\varepsilon}\sqrt{\det g^\varepsilon}\bigg|_{\varepsilon=0} = \frac{1}{2}\mathrm{tr}\left(g^{-1}\frac{dg^\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0}\right)\sqrt{\det g}, \tag{6.15}$$

where $g^{-1} = (g^{ij})_{i,j}$ is the inverse matrix of g. Moreover, since

$$\begin{aligned}\frac{dg^\varepsilon_{ij}}{d\varepsilon}\bigg|_{\varepsilon=0} &= \left(\frac{\partial}{\partial Y_i}\frac{dy^\varepsilon}{d\varepsilon}\cdot\frac{\partial y^\varepsilon}{\partial Y_j} + \frac{\partial y^\varepsilon}{\partial Y_j}\cdot\frac{\partial}{\partial Y_j}\frac{dy^\varepsilon}{d\varepsilon}\right)\bigg|_{\varepsilon=0} \\ &= \frac{\partial w^\sharp}{\partial Y_i}\cdot\frac{\partial y}{\partial Y_j} + \frac{\partial y}{\partial Y_i}\cdot\frac{\partial w^\sharp}{\partial Y_j}\end{aligned}$$

for each $i,j = 1,\dots,n-1$, where we used the notation (6.12), and g^{-1} is symmetric,

$$\begin{aligned}\mathrm{tr}\left(g^{-1}\frac{dg^\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0}\right) &= \sum_{i,j=1}^{n-1} g^{ij}\left(\frac{\partial w^\sharp}{\partial Y_i}\cdot\frac{\partial y}{\partial Y_j} + \frac{\partial y}{\partial Y_i}\cdot\frac{\partial w^\sharp}{\partial Y_j}\right) \\ &= 2\sum_{i,j=1}^{n-1} g^{ij}\frac{\partial w^\sharp}{\partial Y_i}\cdot\frac{\partial y}{\partial Y_j} = 2(\mathrm{div}_\Gamma w)^\sharp.\end{aligned}$$

Substituting this for (6.15), we get (6.14). □

Proof of Lemma 6.2. We first express the free energy $\mathcal{F}$ in "local Lagrangian coordinates." Let U be an open set in $\mathbb{R}^{n-1}$ and $y\colon U \times [0,T] \to \mathbb{R}^n$ be the flow map of the velocity $v = V_\Gamma \nu_\Gamma + v^T$ in local Lagrangian coordinates. Also, let $g = (g_{ij})_{i,j}$ be the matrix given by (6.13). For every open subset U' of U the integral

$$\int_{U'(t)} \eta(y,t)\,d\mathcal{H}^{n-1}(y) = \int_{U'} \eta(y(Y,t),t)\sqrt{\det g(Y,t)}\,dY \quad (U'(t) := y(U',t))$$

is constant in t, since η and v satisfy the transport equation (3.5) and $U'(t)$ moves with velocity v. Hence for $(Y,t) \in U \times (0,T)$ we have

$$\eta(y(Y,t),t)\sqrt{\det g(Y,t)} = \eta(y(Y,0),0)\sqrt{\det g(Y,0)} \tag{6.16}$$

and the localized surface integral (6.10) is expressed as

$$\begin{aligned} \mathcal{F}(y) &= \int_U \omega(\eta(y(Y,t),t))\sqrt{\det g(Y,t)}\,dY \\ &= \int_U \omega\left(\frac{\eta_0(Y)}{\sqrt{\det g(Y,t)}}\right)\sqrt{\det g(Y,t)}\,dY, \end{aligned} \tag{6.17}$$

where $\eta_0(Y)$ is given by the right-hand side of (6.16).

Next we take a variation of $\mathcal{F}$ with respect to the flow map y. Let $\{y^\varepsilon\}_\varepsilon$ be a family of flow maps in local Lagrangian coordinates satisfying (6.11) with $v^\varepsilon = \partial y^\varepsilon/\partial t$. Also, let $g^\varepsilon = (g^\varepsilon_{ij})_{i,j}$ be given by (6.13) with y replaced by y^ε. Suppose that η^ε and v^ε satisfy the transport equation (3.5) and $\eta^\varepsilon|_{t=0} = \eta|_{t=0}$ holds on $y^\varepsilon(U,0) = y(U,0)$. Then the relation (6.16) with η^ε, y^ε, and g^ε holds and by (6.17) the free energy $\mathcal{F}$ with respect to the perturbed flow map y^ε is given by

$$\mathcal{F}(y^\varepsilon) = \int_U \omega\left(\frac{\eta_0}{\sqrt{\det g^\varepsilon}}\right)\sqrt{\det g^\varepsilon}\,dY.$$

Here the the argument $t \in (0,T)$ is suppressed. Note that the right-hand side of (6.16) with η^ε, y^ε, and g^ε is equal to $\eta_0(Y)$ since $\eta^\varepsilon|_{t=0} = \eta|_{t=0}$, $y^\varepsilon|_{t=0} = y|_{t=0}$, and $g^\varepsilon|_{t=0} = g|_{t=0}$. We differentiate the integrand of the right-hand side with respect to ε at $\varepsilon = 0$. Then by (6.14), (6.17), and $g^\varepsilon|_{\varepsilon=0} = g$ we get

$$\frac{d}{d\varepsilon}\left(\omega\left(\frac{\eta_0}{\sqrt{\det g^\varepsilon}}\right)\sqrt{\det g^\varepsilon}\right)\Bigg|_{\varepsilon=0} = \{-\omega'(\eta^\sharp)\eta^\sharp + \omega(\eta^\sharp)\}(\mathrm{div}_\Gamma w)^\sharp\sqrt{\det g}.$$

Here we used the notation (6.12). Hence

$$\begin{aligned} \frac{d}{d\varepsilon}\int_0^T \mathcal{F}(y^\varepsilon)\,dt\Bigg|_{\varepsilon=0} &= \int_0^T\int_U \{-\omega'(\eta^\sharp)\eta^\sharp + \omega(\eta^\sharp)\}(\mathrm{div}_\Gamma w)^\sharp\sqrt{\det g}\,dY\,dt \\ &= \int_0^T\int_{U(t)} \{-\omega'(\eta)\eta + \omega(\eta)\}\mathrm{div}_\Gamma w\,d\mathcal{H}^{n-1}\,dt \\ &= \int_0^T\int_{U(t)} \nabla_\Gamma[\omega'(\eta)\eta - \omega(\eta)]\cdot w\,d\mathcal{H}^{n-1}\,dt, \end{aligned}$$

where $U(t) = y(U,t)$ and the last equality follows from the Stokes theorem and the fact that the vector field w is tangential on $\Gamma(t)$. (Note that we may assume that $\omega'(\eta)\eta - \omega(\eta)$ has a compact support in $U(t)$ since we localize the surface integral by using a partition of unity of $\Gamma(t)$.) Since w is an arbitrary tangential vector field on $\Gamma(t)$, we conclude from the above equality that (6.8) holds. □

By (6.7), (6.8), and $\mathcal{K}=0$, the balance of forces

$$\frac{\delta\mathcal{K}}{\delta y}=\frac{\delta\mathcal{F}}{\delta y}+\frac{\delta\mathcal{D}}{\delta v}$$

is of the form

$$0=\nabla_\Gamma p(\eta)+\eta v^T, \quad \text{i.e.} \quad -\eta v^T=\nabla_\Gamma p(\eta),$$

which is Darcy's law on a moving surface as we expected. Finally, combining this with the transport equation (3.5) we obtain the nonlinear diffusion equations

$$\partial^\circ\eta-V_\Gamma^N H\eta+\mathrm{div}_\Gamma(\eta v^T)=0, \quad -\eta v^T=\nabla_\Gamma p(\eta)$$

on the moving surface $\Gamma(t)$, where $p(\eta)$ is given by (4.1).

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