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SINGULAR OPTIMAL CONTROL PROBLEMS FOR DOUBLY NONLINEAR AND QUASI-VARIATIONAL EVOLUTION EQUATIONS

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Abstract. Doubly nonlinear and quasi-variational evolution equations governed by double time-dependent subdifferentials are treated in uniformly convex Banach spaces. We establish some abstract results on the existence–uniqueness of solutions together with related optimal control problems in cases when, in general, the state equations have multiple solutions. In this paper, we propose a general class of singular optimal control problems that are set up for non-well-posed state systems. Moreover, we establish an approximation procedure for such singular optimal control problems and discuss some applications.

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1 Introduction

The present paper is devoted to a systematic study of a class of singular optimal control problems whose state systems are governed by doubly nonlinear variational evolution equations generated by time-dependent subdifferentials of convex functionals.

Let V be a (real) uniformly convex Banach space with uniformly convex dual space V^* and let H be a real Hilbert space such that

$$V \hookrightarrow H \hookrightarrow V^* \text{ with dense and compact embeddings.} \tag{1.1}$$

Recently, in [22] we introduced the following type of doubly nonlinear evolution equations governed by time-dependent subdifferentials in V^*:

$$\text{(P)} \begin{cases} \partial_* \psi^t(u'(t)) + \partial_* \varphi^t(u(t)) + g(t, u(t)) \ni f(t) \ \text{ in } V^* \ \text{ for a.a. } t \in (0,T), \\ u(0) = u_0 \ \text{ in } V, \end{cases} \tag{1.2}$$

where $0 < T < \infty$, $u' = du/dt$ in V, $\psi^t : V \to \mathbb{R} \cup \{\infty\}$ is a time-dependent proper, lower semi-continuous (l.s.c.), and convex function for each $t \in [0,T]$, $\varphi^t : V \to \mathbb{R}$ is a time-dependent, non-negative, continuous convex function for each $t \in [0,T]$, the subdifferential $\partial_* \psi^t$ of ψ^t is a multivalued operator from V into V^*, the subdifferential $\partial_* \varphi^t$ of φ^t is single-valued and linear from V into V^*, $g(t, \cdot)$ is a single-valued Lipschitz operator from V into V^*, f is a given V^*-valued function on $[0,T]$, and $u_0 \in V$ is a given initial datum. In [22, Theorem 1], we established the abstract existence of solutions to (P). Additionally, we showed the non-uniqueness of solutions, giving an example in [22, Section 4]. Moreover, in [22, Theorem 2], we showed the uniqueness of solutions to (P) under the assumption that $\partial_* \psi^t$ is strongly monotone from V into V^*.

As shown in [22, Section 4], problem (P) has multiple solutions, in general, and therefore the optimal control problem associated with state equation (P) is a singular optimal control problem formulated for non-well-posed state systems. Indeed, for a control space $\mathcal{F}_M$ with constant $M > 0$, defined by:

$$\mathcal{F}_M := \left\{ f \in W^{1,2}(0,T;V^*) \cap L^2(0,T;H) \ ; \ \begin{array}{c} |f|_{W^{1,2}(0,T;V^*)} \le M, \\ |f|_{L^2(0,T;H)} \le M \end{array} \right\}, \tag{1.3}$$

where $|\cdot|_{W^{1,2}(0,T;V^*)}$ (resp. $|\cdot|_{L^2(0,T;H)}$) is the norm of $W^{1,2}(0,T;V^*)$ (resp. $L^2(0,T;H)$), we consider the following optimal control problem for (P):

Problem (OP): Find a control $f^* \in \mathcal{F}_M$ such that

$$J(f^*) = \inf_{f \in \mathcal{F}_M} J(f);$$

such a function f^* is called an optimal control. Here, $J(f)$ is a functional defined by

$$J(f) := \inf_{u \in \mathcal{S}(f)} \pi_f(u), \tag{1.4}$$

where $f \in \mathcal{F}_M$ is any control and $\mathcal{S}(f)$ is the set of all solutions to (P) associated with control f. In addition, $\pi_f(u)$ is the functional of $u \in \mathcal{S}(f)$ defined by:

$$\pi_f(u) := \frac{1}{2} \int_0^T |u(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2} \int_0^T |f(t)|_{V^*}^2 dt, \tag{1.5}$$

where $u_{ad} \in L^2(0,T;V)$ is a given target profile and $|\cdot|_V$ (resp. $|\cdot|_{V^*}$) is the norm of V (resp. V^*).

There is a vast amount of literature on optimal control problems for (parabolic or elliptic) variational inequalities. For instance, see [8, 14, 15, 17, 24, 25, 27, 28, 29, 36]. In particular, Lions [25], Neittaanmäki and Tiba [28], and Neittaanmäki et al. [29, Section 3.1.3.1] discussed singular control problems. Indeed, using the admissible pairs and the adapted penalization method, Neittaanmäki et al. [29] discussed a singular control problem for linear elliptic equations of second-order with the homogeneous Dirichlet boundary condition. However, as (P) is an abstract time-dependent doubly nonlinear evolution equation, it seems very hard to directly apply the penalization method established in [17, 25, 27, 28, 29] to our problem.

The theory of nonlinear evolution equations is useful in any systematic study of variational inequalities. For instance, many mathematicians have studied nonlinear evolution equations of the form:

$$u'(t) + \partial\varphi^t(u(t)) \ni f(t) \quad \text{in } H \quad \text{for a.a. } t \in (0,T), \tag{1.6}$$

where $\varphi^t : H \to \mathbb{R} \cup \{\infty\}$ is a time-dependent proper, l.s.c., and convex function for each $t \in [0,T]$. For fundamental results on (1.6), we refer to [15, 19, 30, 35]. In particular, Hu–Papageorgiou [15] treated some optimal control problems for (1.6). Furthermore, the optimal control of parameter-dependent evolution equations for (1.6) has previously been considered (cf. [15, 31]).

Doubly nonlinear evolution equations have been studied, for instance, by Kenmochi–Pawlow [21], in which nonlinear evolution equations of the following type were discussed:

$$\frac{d}{dt}\partial\psi(u(t)) + \partial\varphi^t(u(t)) \ni f(t) \quad \text{in } H \quad \text{for a.a. } t \in (0,T), \tag{1.7}$$

where $\psi : H \to \mathbb{R} \cup \{\infty\}$ is a proper, l.s.c., and convex function. The abstract results for (1.7) can be applied to elliptic-parabolic equations. From the viewpoint of (1.7), Hoffmann et al. [14] studied optimal control problems for quasi-linear elliptic-parabolic variational inequalities with time-dependent constraints. Additionally, Kadoya–Kenmochi [16] touched on the optimal shape design of elliptic-parabolic equations.

Akagi [2], Arai [3], Aso et al. [4, 5], Colli [11], Colli–Visintin [12], and Senba [32] investigated the following type of doubly nonlinear evolution equation:

$$\partial\psi^t(u'(t)) + \partial\varphi(u(t)) \ni f(t) \quad \text{in } H \quad \text{for a.a. } t \in (0,T). \tag{1.8}$$

Note that the second term $\partial\varphi$ in (1.8) is independent of time and, in the case of double time-dependent subdifferentials such as (1.2), no general theoretical results have yet been derived.

In [22], we evolved the abstract theory of (1.2). As mentioned above, one interesting feature is that (1.2) is not generally well-posed; namely, it lacks the uniqueness of solutions. In this respect, Farshbaf-Shaker and Yamazaki [13] studied the optimal control problem without the uniqueness of solutions in the state system (1.8) by employing the idea of Kadoya et al. [17, 27]; more precisely, they used cost functionals formulated by (1.4) and (1.5). In this paper, we show the existence of optimal control for (OP) under the abstract doubly time-dependent evolution equation (P) with a non-monotone perturbation $g(t,\cdot)$. Although there are some mathematical results dealing with optimal control without the uniqueness of solutions for state equations (cf. [13, 17, 25, 27, 28, 29]), it is still difficult to establish an approximation procedure for the singular control problem (OP). In this paper, we systematically investigate (OP). To this end, we recall the precise construction of solutions to (P), although this has already been discussed in [22], and then propose an approximation procedure from a numerical point of view.

In [22, Section 5], we also introduced the following doubly nonlinear quasi-variational evolution equation governed by double time-dependent subdifferentials:

$$\text{(QP)}\begin{cases} \partial_*\psi^t(u'(t)) + \partial_*\varphi^t(u;u(t)) + g(t,u(t)) \ni f(t) \quad \text{in } V^* \quad \text{for a.a. } t \in (0,T), \\ u(0) = u_0 \quad \text{in } V, \end{cases} \tag{1.9}$$

where $\varphi^t(v;z)$ is a time-dependent, non-negative, continuous convex function in $z \in V$, and $(t,v) \in [0,T] \times L^2(0,T;V)$ is a parameter that determines the convex function $\varphi^t(v;\cdot)$ on V. The dependence of function v upon $\varphi^t(v;\cdot)$ is, in general, allowed to be non-local (see Section 11). Moreover, the subdifferential $\partial_*\varphi^t(v;z)$ of $\varphi^t(v;z)$ is single-valued, linear, and bounded with respect to z from V into V^*. Under such a set-up, we showed the existence of a solution to (QP) in [22]; however, the uniqueness question was not discussed. For the systematic investigation of (QP) and the corresponding optimal control problem, we use the same approach to that of (P) and (OP), namely, we carefully recall the construction of solutions to (QP) described in [22, Section 5], and propose an approximation procedure for them. During this derivation, we shall show the non-uniqueness of solutions to (QP), giving an example, and present a sufficient condition to ensure the uniqueness of solutions to (QP). Finally, we consider the singular optimal control problem for the state system (QP) and its approximation from a numerical point of view.

The novelties of this work (although items (a) and (d) are somewhat reliant on [22]) are as follows:

(a) We show the existence of solutions to (P) for each $f \in L^2(0,T;V^*)$ and $u_0 \in V$.

(b) We show the existence of optimal control for (OP).

(c) We propose an approximation procedure for (P) and (OP), and clarify the relationship between the original problems and their approximations.

(d) We show the existence of solutions to (QP) for each $f \in L^2(0,T;V^*)$ and $u_0 \in V$.

(e) We show that (QP) is generally not a well-posed state system by giving an example of the non-uniqueness of solutions to (QP). Moreover, we discuss the uniqueness of solutions to (QP) under some additional condition.

(f) We formulate a singular optimal control problem for (QP).

(g) We establish an approximate procedure to investigate the singular optimal control problem for (QP) from a numerical point of view.

The remainder of this paper is organized as follows. In Section 2, we state the abstract result of the existence–uniqueness of solutions to (P) for each $f \in L^2(0,T;V^*)$ and $u_0 \in V$. In Section 3, we give a proof of the existence of solutions to (P), which is the main result corresponding to item (a). In Section 4, concerning the singular optimal control problem (OP), we give a proof of the main result corresponding to item (b). In Section 5, we propose a general approximate procedure for (P) and (OP) and construct a solution as the limit of the approximate optimal controls using the results of item (c). In Section 6, we state the solvability result of (QP) for each $f \in L^2(0,T;V^*)$ and $u_0 \in V$ and give a proof of the main result corresponding to item (d). In Section 7, we discuss the uniqueness question of solutions to (QP), which gives an answer to item (e). In Section 8, we consider the singular optimal control problem for (QP) and give a proof of the main result, corresponding to item (f). In Section 9, we establish an approximation procedure to construct an optimal control for (QP) and give a proof of the main result, corresponding to item (g). In Section 10, we consider another type of singular optimal control problem for doubly nonlinear parameter-dependent evolution state equations. Additionally, we give another approximation procedure for the optimal control of (QP). In the final section, we apply our general results to some model problems: parabolic variational and quasi-variational inequalities with time-dependent constraints.

Notation

Throughout this paper, let H be a (real) Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|_H$. Let V be a (real) uniformly convex Banach space with the uniformly convex dual space V^*; denote by $|\cdot|_V$ and $|\cdot|_{V^*}$ the norms of V and V^*, respectively. Assume that $V \subset H$, V is dense in H, and $V \hookrightarrow H \hookrightarrow V^*$, where $\hookrightarrow$ denotes the compact embedding. Therefore, (V, H, V^*) is the standard triplet and

$$\langle u, v\rangle = (u, v) \quad \text{for } u \in H \text{ and } v \in V,$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between V^* and V.

Let $F : V \to V^*$ be the duality mapping.

We now list some notation and definitions of subdifferentials of convex functions. Let $\phi : V \to \mathbb{R} \cup \{\infty\}$ be a proper (i.e., not identically equal to infinity), l.s.c., and convex function. Then, the effective domain $D(\phi)$ is defined by

$$D(\phi) := \{z \in V;\ \phi(z) < \infty\}.$$

The subdifferential $\partial_*\phi : V \to V^*$ of ϕ is a possibly multi-valued operator from V into V^*, and is defined by $z^* \in \partial_*\phi(z)$ if and only if

$$z \in D(\phi) \quad \text{and} \quad \langle z^*, y - z\rangle \le \phi(y) - \phi(z) \quad \text{for all } \ y \in V.$$

Its graph is the set $\{[z, z^*] \in V \times V^* \mid z^* \in \partial_*\phi(z)\}$, which is often identified with $\partial_*\phi$, namely, $z^* \in \partial_*\phi(z)$ is denoted by $[z, z^*] \in \partial_*\phi$.

For various properties and related notions of a proper, l.s.c., convex function ϕ and its subdifferential $\partial_*\phi$, we refer to the monographs by Barbu [7, 9]. In particular, for those in Hilbert spaces, we refer to the monographs by Brézis [10].

We also recall a notion of convergence for convex functions, developed by Mosco [26].

Definition 1.1 (cf. [26])**.** *Let ϕ, ϕ_n ($n \in \mathbb{N}$) be proper, l.s.c., and convex functions on V. Then, we say that ϕ_n converges to ϕ on V in the sense of Mosco [26] as $n \to \infty$ if the following two conditions are satisfied:*

(i) *for any subsequence $\{\phi_{n_k}\}_{k\in\mathbb{N}} \subset \{\phi_n\}_{n\in\mathbb{N}}$, if $z_k \to z$ weakly in V as $k \to \infty$, then*

$$\liminf_{k\to\infty} \phi_{n_k}(z_k) \geq \phi(z);$$

(ii) *for any $z \in D(\phi)$, there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ in V such that*

$$z_n \to z \text{ in } V \text{ as } n \to \infty \quad \textit{and} \quad \lim_{n\to\infty} \phi_n(z_n) = \phi(z).$$

For some important characterizations of the Mosco convergence of convex functions, we refer to the monographs by Attouch [6] and Kenmochi [20].

2 Solvability of (P)

We begin with the notion of a solution to (P).

Definition 2.1. *Given $f \in L^2(0,T;V^*)$ and $u_0 \in V$, the function $u : [0,T] \to V$ is called a solution to* (P)*, or* (P; f, u_0) *when the data f and u_0 are indicated, on $[0,T]$, if and only if the following conditions are satisfied:*

(i) *$u \in W^{1,2}(0,T;V)$.*

(ii) *There exists a function $\xi \in L^2(0,T;V^*)$ such that*

$$\xi(t) \in \partial_*\psi^t(u'(t)) \quad \textit{in } V^* \textit{ for a.a. } t \in (0,T),$$

$$\xi(t) + \partial_*\varphi^t(u(t)) + g(t,u(t)) = f(t) \quad \textit{in } V^* \textit{ for a.a. } t \in (0,T).$$

(iii) *$u(0) = u_0$ in V.*

Now, we list some assumptions on ψ^t, φ^t, $g(t,\cdot)$, and F.

We suppose that the duality mapping $F : V \to V^*$ is strongly monotone; more precisely, there is a positive constant C_F such that

$$\langle Fz_1 - Fz_2, z_1 - z_2\rangle \geq C_F|z_1 - z_2|_V^2, \quad \forall z_1, z_2 \in V. \tag{2.1}$$

(*Assumption (A)*)

Let $\psi^t(\cdot) : V \to \mathbb{R} \cup \{\infty\}$ be a proper, l.s.c., and convex function with $D(\psi^t) \subset V$ for all $t \in [0,T]$, and assume:

(A1) If $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ and $t \in [0,T]$ with $t_n \to t$ as $n \to \infty$, then

$$\psi^{t_n}(\cdot) \to \psi^t(\cdot) \text{ in the sense of Mosco [26] as } n \to \infty.$$

(A2) There exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\psi^t(z) \geq C_1|z|_V^2 - C_2, \quad \forall t \in [0,T],\ \forall z \in D(\psi^t).$$

(A3) $\partial_*\psi^t(0) \ni 0$ for all $t \in [0,T]$ and $\psi^{(\cdot)}(0) \in L^1(0,T)$.

(*Assumption (B)*)

Let $\varphi^t(\cdot) : V \to \mathbb{R} \cup \{\infty\}$ be a non-negative, finite, continuous, and convex function with $D(\varphi^t) = V$ for all $t \in [0,T]$, and assume:

(B1) For each $t \in [0,T]$, the subdifferential $\partial_*\varphi^t : D(\partial_*\varphi^t) = V \to V^*$ is linear and uniformly bounded, i.e., there exists a constant $C_3 > 0$ such that

$$|\partial_*\varphi^t(z)|_{V^*} \leq C_3|z|_V, \quad \forall t \in [0,T],\ \forall z \in V.$$

(B2) $\varphi^0(0) = 0$ and there exists a constant $C_4 > 0$ such that

$$\varphi^0(z) \geq C_4|z|_V^2, \quad \forall z \in V.$$

(B3) There is a function $\alpha \in W^{1,1}(0,T)$ such that

$$|\varphi^t(z) - \varphi^s(z)| \leq |\alpha(t) - \alpha(s)|\varphi^s(z), \quad \forall s,t \in [0,T],\ \forall z \in V.$$

(*Assumption (C)*)

Let g be a single-valued operator from $[0,T] \times V$ into V^* such that $g(t,z)$ is strongly measurable in $t \in [0,T]$ for each $z \in V$, and assume:

(C1) For each $t \in [0,T]$, the operator $z \to g(t,z)$ is continuous from V_w into V^*, i.e., if $z_n \to z$ weakly in V as $n \to \infty$, then $g(t,z_n) \to g(t,z)$ in V^* as $n \to \infty$, where V_w is the linear space V with the weak topology.

(C2) $g(t,\cdot)$ is uniformly Lipschitz from V into V^*, i.e., there is a constant $L_g > 0$ such that

$$|g(t,z_1) - g(t,z_2)|_{V^*} \leq L_g|z_1 - z_2|_V, \quad \forall t \in [0,T],\ \forall z_i \in V\ (i = 1,2).$$

Remark 2.1. *The assumption* (B3) *is one of the standard time-dependence conditions of convex functions in the theory of evolution equations generated by time-dependent subdifferentials (cf. [19, 30, 35]).*

Condition (B2) is slightly weaker than that required in [22, Section 2]. However, the following lemma shows that it is sufficient to assume (B2), as long as (B1) and (B3) are required together. In fact, we have:

Lemma 2.1. *Suppose that Assumption (B) holds. Then, the following inequalities hold:*

(i)

$$\frac{C_4}{|\alpha'|_{L^1(0,T)}+1}|z|_V^2 \le \varphi^t(z) \le \left(|\alpha'|_{L^1(0,T)}+1\right) C_3|z|_V^2, \quad \forall t\in[0,T],\ \forall z\in V. \tag{2.2}$$

(ii)

$$\langle \partial_*\varphi^t(z), z\rangle \ge \frac{C_4}{|\alpha'|_{L^1(0,T)}+1}|z|_V^2, \quad \forall t\in[0,T],\ \forall z\in V.$$

Proof. We show (i). We observe from (B1) with $t=0$, (B2), and the definition of $\partial_*\varphi^0$ that

$$\varphi^0(z)=\varphi^0(z)-\varphi^0(0)\le \langle \partial_*\varphi^0(z), z\rangle \le |\partial_*\varphi^0(z)|_{V^*}|z|_V \le C_3|z|_V^2,$$

and also, by (B2),

$$C_4|z|_V^2 \le \varphi^0(z) \le C_3|z|_V^2, \quad \forall z\in V. \tag{2.3}$$

Note from (B3) with $s=0$ that

$$|\varphi^t(z)-\varphi^0(z)| \le |\alpha(t)-\alpha(0)|\varphi^0(z) \le \int_0^t |\alpha'(\tau)|d\tau\cdot\varphi^0(z), \quad \forall t\in[0,T],\ \forall z\in V,$$

which implies that

$$\varphi^t(z) \le \left(|\alpha'|_{L^1(0,T)}+1\right)\varphi^0(z), \quad \forall t\in[0,T],\ \forall z\in V.$$

Therefore, it follows from (2.3) that

$$\varphi^t(z) \le \left(|\alpha'|_{L^1(0,T)}+1\right) C_3|z|_V^2, \quad \forall t\in[0,T],\ \forall z\in V. \tag{2.4}$$

Similarly, note from (B3) that

$$|\varphi^0(z)-\varphi^t(z)| \le |\alpha(0)-\alpha(t)|\varphi^t(z) \le \int_0^t |\alpha'(\tau)|d\tau\cdot\varphi^t(z), \quad \forall t\in[0,T],\ \forall z\in V,$$

which implies that

$$\varphi^0(z) \le \left(|\alpha'|_{L^1(0,T)}+1\right)\varphi^t(z), \quad \forall t\in[0,T],\ \forall z\in V.$$

Hence, we infer from (B2) (cf. (2.3)) that

$$\varphi^t(z) \ge \frac{1}{|\alpha'|_{L^1(0,T)}+1}\varphi^0(z) \ge \frac{C_4}{|\alpha'|_{L^1(0,T)}+1}|z|_V^2, \quad \forall t\in[0,T],\ \forall z\in V. \tag{2.5}$$

Thus, we conclude from (2.4) and (2.5) that (2.2) holds.

Now, we show (ii). To this end, we note from (2.2) that

$$\varphi^t(0)=0, \quad \forall t\in[0,T].$$

Therefore, we observe from the definition of $\partial_*\varphi^t$, and (2.2) (cf. (2.5)) that

$$\langle \partial_*\varphi^t(z), z\rangle \ge \varphi^t(z)-\varphi^t(0)=\varphi^t(z) \ge \frac{C_4}{|\alpha'|_{L^1(0,T)}+1}|z|_V^2, \quad \forall t\in[0,T],\ \forall z\in V.$$

Thus, we conclude (ii), and the proof of Lemma 2.1 is complete. □

Remark 2.2 (cf. [22, Remark 1]). *We derive from* (B1) *and* (i) *of Lemma 2.1 that the subdifferential* $\partial_*\varphi^t$ *satisfies*

$$C_3|z|_V^2 \geq \langle \partial_*\varphi^t(z), z\rangle \geq \varphi^t(z) \geq \frac{C_4}{|\alpha'|_{L^1(0,T)}+1}|z|_V^2, \quad \forall z \in V, \ \forall t \in [0,T]. \tag{2.6}$$

Additionally, it follows from (B3) *that the function* $t \to \partial_*\varphi^t(z)$ *is weakly continuous from* $[0,T]$ *into* V^* *for all* $z \in V$. *Indeed, let* z *be any element in* V. *Furthermore, let* $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ *and* $t \in [0,T]$ *with* $t_n \to t$ *as* $n \to \infty$. *Then, note from* (B1) *that* $t \to \partial_*\varphi^t(z)$ *is bounded in* V^*. *Hence, taking a subsequence if necessary (still denoted by* $\{t_n\}_{n\in\mathbb{N}}$*), we observe that*

$$\partial_*\varphi^{t_n}(z) \to \xi^* \quad \textit{weakly in } V^* \textit{ for some } \xi^* \in V^* \textit{ as } n \to \infty. \tag{2.7}$$

From the definition of $\partial_*\varphi^{t_n}$*, we infer that*

$$\langle \partial_*\varphi^{t_n}(z), v - z\rangle \leq \varphi^{t_n}(v) - \varphi^{t_n}(z), \quad \forall v \in V.$$

Letting $n \to \infty$*, we observe from* (B3) *that*

$$\langle \xi^*, v - z\rangle \leq \varphi^t(v) - \varphi^t(z), \quad \forall v \in V,$$

which implies that $\xi^* \in \partial_*\varphi^t(z)$. *As* $\partial_*\varphi^t$ *is single-valued (cf.* (B1)*), we conclude that* $\xi^* = \partial_*\varphi^t(z)$ *and* (2.7) *holds without extracting any subsequence from* $\{t_n\}_{n\in\mathbb{N}}$. *Thus, the function* $t \to \partial_*\varphi^t(z)$ *is weakly continuous from* $[0,T]$ *into* V^* *for all* $z \in V$.

We now state the first main result of this paper, which is concerned with the existence of a solution to problem (P) on $[0,T]$.

Theorem 2.1 (cf. [22, Theorem 1]). *Suppose that Assumptions (A), (B), and (C) hold. Then, for each* $f \in L^2(0,T;V^*)$ *and* $u_0 \in V$*, there exists at least one solution* u *to* (P;f, u_0) *on* $[0,T]$. *Moreover, there exists a constant* $N_0 > 0$*, independent of* f *and* u_0*, such that*

$$\int_0^T \psi^t(u'(t))dt + \sup_{t\in[0,T]} \varphi^t(u(t)) \leq N_0 \left(\varphi^0(u_0) + |f|^2_{L^2(0,T;V^*)} + 1\right) \tag{2.8}$$

for any solution u *to* (P;f, u_0) *on* $[0,T]$.

The above theorem was proved in [22, Theorem 1]. However, we shall carefully repeat the proof in Section 3 to make use of a similar idea in the singular optimal control problem (OP) and clarify the similarity between Theorem 2.1 and Theorem 5.1, as well as Theorem 10.2, which is treated later.

It is known that solutions to (P;f, u_0) are not, in general, unique (cf. [22, Example 4.1]). We can show the uniqueness of solutions under the additional assumption on $\partial_*\psi^t$ stated below.

Theorem 2.2 (cf. [22, Theorem 2]). *Suppose that Assumptions (A), (B), and (C) are satisfied. In addition, assume that* $\partial_*\psi^t$ *is strongly monotone in* V^**; more precisely,* (A4) *There exists a constant* $C_5 > 0$ *such that*

$$\langle z_1^* - z_2^*, z_1 - z_2 \rangle \geq C_5 |z_1 - z_2|_V^2, \quad \forall [z_i, z_i^*] \in \partial_* \psi^t \ (i = 1, 2), \ \forall t \in [0, T].$$

Then, (P;f, u_0) *has at most one solution.*

This theorem was proved in [22, Theorem 2], and it will be generalized to the case of doubly nonlinear quasi-variational problems of the form (QP) (cf. Theorem 7.1). The proof is included in that of Theorem 7.1 as a special case. Therefore, we omit here the detailed proof of Theorem 2.2.

Remark 2.3. *Colli [11, Theorem 5] and Colli–Visintin [12, Remark 2.5] gave several criteria for the uniqueness of the following type of doubly nonlinear time-independent evolution equations:*

$$\partial \psi(u'(t)) + \partial \varphi(u(t)) \ni f(t) \text{ in } H \text{ for a.a. } t \in (0, T). \tag{2.9}$$

For instance, if $\partial \varphi$ *is linear and positive in* H *and* $\partial \psi$ *is strictly monotone in* H*, then the solution to the Cauchy problem for* (2.9) *is unique.*

Remark 2.4. *When* $g(t, \cdot)$ *is Lipschitz from* V *into* H *in the sense that*

$$|g(t, z_1) - g(t, z_2)|_H \leq L'_g |z_1 - z_2|_H, \quad \forall z_1, z_2 \in V$$

for a positive constant L'_g*, condition* (A4) *in Theorem 2.2 can be replaced by the following: There exists a constant* $C'_5 > 0$ *such that*

$$\langle z_1^* - z_2^*, z_1 - z_2 \rangle \geq C_5 |z_1 - z_2|_H^2, \quad \forall [z_i, z_i^*] \in \partial_* \psi^t \ (i = 1, 2), \ \forall t \in [0, T]. \tag{2.10}$$

This is easily checked by a slight modification of the proof given in [22, Theorem 2].

3 Existence of solutions to (P; f, u_0)

In this section, we discuss the solvability of (P; f, u_0) for each $f \in L^2(0, T; V^*)$ and $u_0 \in V$.

One of the main objectives of this paper is to establish a systematic approach to singular optimal control problems. To this end, it is very important to carefully review the construction of solutions to the state system (P; f, u_0) and its approximate state systems, as these could be used in proving the convergence of their solutions with respect to the data f and u_0 (cf. Proposition 4.1 and Proposition 5.2). Therefore, in this section, we repeat the detailed construction of solutions to (P; f, u_0), although this was covered in [22, Theorem 1].

Throughout this section, we suppose that all the assumptions of Theorem 2.1 are made. We construct a solution to (P; f, u_0) by considering the convergence of approximate solutions of (P; f, u_0). Indeed, for each $\varepsilon \in (0,1]$, we consider the following problem, denoted by $(\mathrm{P})_\varepsilon$, or $(\mathrm{P}; f, u_0)_\varepsilon$ when the data are specified:

$$(\mathrm{P})_\varepsilon \begin{cases} \varepsilon F u_\varepsilon'(t) + \partial_* \psi^t(u_\varepsilon'(t)) + \partial_* \varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) \ni f(t) \ \text{ in } V^* \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t \in (0,T), \\ u_\varepsilon(0) = u_0 \ \text{ in } V, \end{cases} \tag{3.1}$$

where $F : V \to V^*$ is the duality mapping. Based on this, the solutions to (P) are to be obtained through the limiting process for $(\mathrm{P})_\varepsilon$ as $\varepsilon \to 0$.

We begin with the following key lemma, which gives an a priori estimate of solutions to $(\mathrm{P})_\varepsilon$.

Lemma 3.1 (cf. [19, Lemma 2.1.1], [22, Lemma 1])**.** *Suppose that Assumption (B) holds. Let $v \in W^{1,1}(0,T;V)$. Then, we have:*

$$\frac{d}{dt}\varphi^t(v(t)) - \langle \partial_* \varphi^t(v(t)), v'(t) \rangle \le |\alpha'(t)| \varphi^t(v(t)), \quad a.a.\ t \in (0,T). \tag{3.2}$$

Proof. Lemma 3.1 will be proved using a similar approach as for the proof of [22, Lemma 1]. Indeed, we observe from (2.2) that $\varphi^t(v(t))$ is bounded on $[0,T]$. Therefore, we infer from (B3) that $\varphi^t(v(t))$ is absolutely continuous on $[0,T]$. Taking account of this fact and the definition of subdifferential $\partial_*\varphi^t$, we have:

$$\begin{aligned} &\varphi^t(v(t)) - \varphi^s(v(s)) - \langle \partial_* \varphi^t(v(t)), v(t) - v(s) \rangle \\ \le &\varphi^t(v(s)) - \varphi^s(v(s)) \\ \le &|\alpha(t) - \alpha(s)| \varphi^s(v(s)) \quad \text{ for all } s, t \in [0,T]. \end{aligned}$$

Thus, dividing the above inequality by $t - s$ and letting $s \uparrow t$, we get (3.2). □

Taking Lemma 3.1 into account, we can prove the existence–uniqueness of solutions to $(\mathrm{P})_\varepsilon$ for each $\varepsilon \in (0,1]$ as follows.

Proposition 3.1 (cf. [22, Proposition 1])**.** *Suppose that Assumptions (A), (B), and (C) are satisfied. Then, for each $\varepsilon \in (0,1]$, $f \in L^2(0,T;V^*)$, and $u_0 \in V$, there exists a unique solution $u_\varepsilon \in W^{1,2}(0,T;V)$ to $(\mathrm{P}; f, u_0)_\varepsilon$ on $[0,T]$ satisfying $u_\varepsilon(0) = u_0$ in V, and the following statements hold:*

(•) *There exists a function $\xi_\varepsilon \in L^2(0,T;V^*)$ such that*

$$\xi_\varepsilon(t) \in \partial_* \psi^t(u_\varepsilon') \quad \text{in } V^* \text{ for a.a. } t \in (0,T),$$

$$\varepsilon F u_\varepsilon'(t) + \xi_\varepsilon(t) + \partial_* \varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) = f(t) \quad \text{in } V^* \text{ for a.a. } t \in (0,T).$$

Moreover, there exists a constant $N_1 > 0$, independent of ε, f, and u_0, such that

$$\int_0^T \psi^t(u_\varepsilon'(t))dt + \sup_{t \in [0,T]} \varphi^t(u_\varepsilon(t)) \le N_1 \left(\varphi^0(u_0) + |f|^2_{L^2(0,T;V^*)} + 1 \right). \tag{3.3}$$

Proof. We easily observe that the approximate problem $(\mathrm{P}; f, u_0)_\varepsilon$ can be reformulated as:

$$\begin{cases} u_\varepsilon'(t) = (\varepsilon F + \partial_* \psi^t)^{-1} \left(f(t) - \partial_* \varphi^t(u_\varepsilon(t)) - g(t, u_\varepsilon(t)) \right) \quad \text{in } V \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t \in (0,T), \\ u_\varepsilon(0) = u_0 \quad \text{in } V. \end{cases} \tag{3.4}$$

Putting

$$\mathcal{B}(t) z^* := (\varepsilon F + \partial_* \psi^t)^{-1} z^* \quad \text{for all } z^* \in V^*$$

and

$$\mathcal{F}(t, z) := f(t) - \partial_* \varphi^t(z) - g(t, z) \quad \text{for all } z \in V,$$

we show the existence–uniqueness of solutions to (3.4) by applying the Cauchy–Lipschitz–Picard existence theorem.

To this end, we first show that the operator $\mathcal{B}(t) z^* : [0,T] \times V^* \to V$ is Lipschitz in $z^* \in V^*$, and is bounded and continuous in $t \in [0,T]$. We fix any $t \in [0,T]$ to show that $z^* \in V^* \mapsto \mathcal{B}(t) z^* \in V$ is Lipschitz continuous. Setting $z_i = \mathcal{B}(t) z_i^*$ $(i = 1, 2)$, we observe that

$$z_i^* = \varepsilon F z_i + z_{i,*} \quad \text{for some } z_{i,*} \in \partial_* \psi^t(z_i).$$

By (2.1) and the monotonicity of $\partial_* \psi^t(\cdot)$, we have:

$$\begin{aligned} \langle z_1^* - z_2^*, z_1 - z_2 \rangle = & \langle \varepsilon F z_1 + z_{1,*} - \varepsilon F z_2 - z_{2,*}, z_1 - z_2 \rangle \\ \geq & \varepsilon \langle F z_1 - F z_2, z_1 - z_2 \rangle \\ \geq & \varepsilon C_F |z_1 - z_2|_V^2. \end{aligned}$$

Hence, we conclude that

$$|\mathcal{B}(t) z_1^* - \mathcal{B}(t) z_2^*|_V = |z_1 - z_2|_V \leq \frac{1}{\varepsilon C_F} |z_1^* - z_2^*|_{V^*}.$$

Thus, the operator $\mathcal{B}(t) z^*$ is Lipschitz in $z^* \in V^*$ for all $t \in [0,T]$.

Next, we fix any $z^* \in V^*$ to show that $t \in [0,T] \mapsto \mathcal{B}(t) z^* \in V$ is bounded. Setting $z^t := \mathcal{B}(t) z^*$, we observe from the definition of $\mathcal{B}(t)$ that

$$z^* = \varepsilon F z^t + z_*^t \quad \text{for some } z_*^t \in \partial_* \psi^t(z^t). \tag{3.5}$$

Hence, we infer from (3.5) and the monotonicity of $\partial_* \psi^t(\cdot)$ with $\partial_* \psi^t(0) \ni 0$ (cf. (A3)) that

$$\begin{aligned} |z^t|_V^2 = \langle F z^t, z^t \rangle = & \left\langle \frac{1}{\varepsilon} z^* - \frac{1}{\varepsilon} z_*^t, z^t \right\rangle \\ = & \frac{1}{\varepsilon} \langle z^*, z^t \rangle - \frac{1}{\varepsilon} \langle z_*^t, z^t \rangle \\ \leq & \frac{1}{\varepsilon} |z^*|_{V^*} |z^t|_V, \end{aligned}$$

which implies that

$$|\mathcal{B}(t) z^*|_V = |z^t|_V \leq \frac{1}{\varepsilon} |z^*|_{V^*} \quad \text{for all } t \in [0,T]. \tag{3.6}$$

Thus, the operator $\mathcal{B}(t)z^*$ is bounded in $t \in [0,T]$ for all $z^* \in V^*$.

In addition, we fix any $z^* \in V^*$ to show that $t \in [0,T] \mapsto \mathcal{B}(t)z^* \in V$ is continuous. Setting $z^t := \mathcal{B}(t)z^*$, we observe from the definition of $\mathcal{B}(t)$ that

$$\varepsilon F z^t + \partial_* \psi^t(z^t) \ni z^*.$$

Let $\{s_n\}_{n\in\mathbb{N}} \subset [0,T]$ with $s_n \to t$ as $n \to \infty$. Note that $z^{s_n} \in D(\partial_*\psi^{s_n})$ and

$$z^* = \varepsilon F z^{s_n} + z_*^{s_n} \quad \text{for some } z_*^{s_n} \in \partial_*\psi^{s_n}(z^{s_n}). \tag{3.7}$$

Additionally, note from (A1) that $\partial_*\psi^{s_n}$ converges to $\partial_*\psi^t$ in the sense of its graph as $n \to \infty$ (cf. [6, 20]); namely, for $[z^t, z^* - \varepsilon F z^t] \in \partial_*\psi^t$, there exists a sequence $\{[z_n, z_n^*]\}_{n\in\mathbb{N}} \subset V \times V^*$ such that $[z_n, z_n^*] \in \partial_*\psi^{s_n}$ in $V \times V^*$,

$$z_n \to z^t \ \text{ in } V \ \text{ and } \ z_n^* \to z^* - \varepsilon F z^t \ \text{ in } V^* \ \text{ as } n \to \infty. \tag{3.8}$$

As the dual space V^* is uniformly convex, the duality mapping F is uniformly continuous on every bounded subset of V. Therefore, we observe from (3.8) that

$$z_n^* + \varepsilon F z_n \to z^* - \varepsilon F z^t + \varepsilon F z^t = z^* \ \text{ in } V^* \ \text{ as } n \to \infty. \tag{3.9}$$

Note from (3.6) that $\{z^{s_n}\}_{n\in\mathbb{N}}$ is bounded in V. Hence, we infer from (2.1), (3.7), (3.9), and the monotonicity of $\partial_*\psi^{s_n}$ that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle z_n^* + \varepsilon F z_n - z^*, z_n - z^{s_n} \rangle \\ &= \lim_{n\to\infty} \langle z_n^* + \varepsilon F z_n - \varepsilon F z^{s_n} - z_*^{s_n}, z_n - z^{s_n} \rangle \\ &\geq \limsup_{n\to\infty} \varepsilon \langle F z_n - F z^{s_n}, z_n - z^{s_n} \rangle \\ &\geq \varepsilon C_F \limsup_{n\to\infty} |z_n - z^{s_n}|_V^2, \end{aligned}$$

which implies from (3.8) that

$$z^{s_n} = \mathcal{B}(s_n)z^* \to z^t = \mathcal{B}(t)z^* \ \text{ in } V \text{ as } s_n \to t.$$

Thus, the operator $\mathcal{B}(t)z^*$ is continuous in $t \in [0,T]$ for all $z^* \in V^*$.

Similarly, it follows from (B1), (B3), (C2), and $f \in L^2(0,T;V^*)$ that the operator $\mathcal{F}(t,z) : [0,T] \times V \to V^*$ is strongly measurable in $t \in [0,T]$ and Lipschitz in $z \in V$.

We now show the existence–uniqueness of solutions to (3.4), i.e., $(\mathrm{P}; f, u_0)_\varepsilon$ on $[0,T]$. To this end, we define the operator $S : C([0,T];V) \to C([0,T];V)$ by:

$$S(u)(t) := u_0 + \int_0^t \mathcal{B}(s)[\mathcal{F}(s,u(s))]ds, \ \forall t \in [0,T], \ \forall u \in C([0,T];V).$$

Note that the operator $\mathcal{B}(\cdot)[\mathcal{F}(\cdot,\cdot)] : [0,T] \times V \to V$ satisfies the Carathéodory condition, $\mathcal{B}(\cdot)[\mathcal{F}(\cdot,z)]$ is Lipschitz in $z \in V$, and $\mathcal{B}(\cdot)[\mathcal{F}(\cdot,u)] \in L^1(0,T;V)$ for all $u \in C([0,T];V)$. Therefore, by the Cauchy–Lipschitz–Picard existence theorem, we can show that S has a unique fixed point $u \in C([0,T_0];V)$ for some small $T_0 \in (0,T]$, which is a unique solution to $(\mathrm{P}; f, u_0)_\varepsilon$ on $[0, T_0]$. By repeating this local existence argument as above, a unique solution u_ε to $(\mathrm{P}; f, u_0)_\varepsilon$ is obtained on the time interval $[0, T]$.

Finally, we derive the a priori estimate (3.3). Multiplying (3.1) by u'_ε, we get:

$$\begin{aligned}\langle \varepsilon F u'_\varepsilon(t), u'_\varepsilon(t)\rangle + \langle \xi_\varepsilon(t), u'_\varepsilon(t)\rangle + \langle \partial_*\varphi^t(u_\varepsilon(t)), u'_\varepsilon(t)\rangle + \langle g(t, u_\varepsilon(t)), u'_\varepsilon(t)\rangle \\ = \langle f(t), u'_\varepsilon(t)\rangle \quad \text{for a.a. } t \in (0, T),\end{aligned} \tag{3.10}$$

with $\xi_\varepsilon \in L^2(0, T; V^*)$ satisfying $\xi_\varepsilon(t) \in \partial_*\psi^t(u'_\varepsilon(t))$ in V^* for a.a. $t \in (0, T)$.

By (A3), the definitions of F and $\partial_*\psi^t$, and Lemma 3.1, we have:

$$\langle \varepsilon F u'_\varepsilon(t), u'_\varepsilon(t)\rangle = \varepsilon |u'_\varepsilon(t)|_V^2, \tag{3.11}$$

$$\langle \xi_\varepsilon(t), u'_\varepsilon(t)\rangle \ge \psi^t(u'_\varepsilon(t)) - \psi^t(0), \tag{3.12}$$

$$\langle \partial_*\varphi^t(u_\varepsilon(t)), u'_\varepsilon(t)\rangle \ge \frac{d}{dt}\varphi^t(u_\varepsilon(t)) - |\alpha'(t)|\varphi^t(u_\varepsilon(t)) \tag{3.13}$$

for a.a. $t \in (0, T)$. Additionally, from (A2), (C2), Lemma 2.1(i), and Schwarz's inequality, it follows that

$$\begin{aligned}&|\langle g(t, u_\varepsilon(t)), u'_\varepsilon(t)\rangle| \le |g(t, u_\varepsilon(t))|_{V^*}|u'_\varepsilon(t)|_V \\ \le& \frac{C_1}{4}|u'_\varepsilon(t)|_V^2 + \frac{1}{C_1}|g(t, u_\varepsilon(t))|_{V^*}^2 \\ \le& \frac{1}{4}\psi^t(u'_\varepsilon(t)) + \frac{C_2}{4} + \frac{1}{C_1}\left(|g(t, 0)|_{V^*} + L_g|u_\varepsilon(t)|_V\right)^2 \\ \le& \frac{1}{4}\psi^t(u'_\varepsilon(t)) + \frac{C_2}{4} + \frac{2|g(t, 0)|_{V^*}^2}{C_1} + \frac{2L_g^2(|\alpha'|_{L^1(0,T)} + 1)}{C_1C_4}\varphi^t(u_\varepsilon(t))\end{aligned} \tag{3.14}$$

and

$$|\langle f(t), u'_\varepsilon(t)\rangle| \le \frac{C_1}{4}|u'_\varepsilon(t)|_V^2 + \frac{1}{C_1}|f(t)|_{V^*}^2 \le \frac{1}{4}\psi^t(u'_\varepsilon(t)) + \frac{C_2}{4} + \frac{1}{C_1}|f(t)|_{V^*}^2 \tag{3.15}$$

for a.a. $t \in (0, T)$.

Using (3.11)–(3.15), it follows from (3.10) that:

$$\begin{aligned}&\varepsilon|u'_\varepsilon(t)|_V^2 + \frac{1}{2}\psi^t(u'_\varepsilon(t)) + \frac{d}{dt}\varphi^t(u_\varepsilon(t)) \\ \le\ & M_1\left(|\alpha'(t)| + 1\right)\varphi^t(u_\varepsilon(t)) + M_2(|f(t)|_{V^*}^2 + \psi^t(0) + |g(t, 0)|_{V^*}^2 + 1) \\ &\qquad \text{for a.a. } t \in (0, T),\end{aligned} \tag{3.16}$$

where $M_1, M_2 > 0$ are constants independent of $\varepsilon \in (0, 1]$; for instance:

$$M_1 = \frac{2L_g^2(|\alpha'|_{L^1(0,T)} + 1)}{C_1C_4} + 1 \quad \text{and} \quad M_2 = \frac{2}{C_1} + \frac{C_2}{2} + 1.$$

Multiplying (3.16) by $e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}$ gives:

$$\varepsilon e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}|u'_\varepsilon(t)|_V^2 + \frac{1}{2}e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}(\psi^t(u'_\varepsilon(t)) + C_2),$$

$$+\frac{d}{dt}\left\{e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}\varphi^t(u_\varepsilon(t))\right\} \tag{3.17}$$

$$\leq \frac{C_2}{2}e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau} + M_2e^{-\int_0^t M_1(|\alpha'(\tau)|+1)d\tau}(|f(t)|_{V^*}^2 + \psi^t(0) + |g(t,0)|_{V^*}^2 + 1)$$
$$=: M_3(t).$$

Integrating (3.17) in time, we obtain:

$$\begin{aligned}&\int_0^T \psi^t(u_\varepsilon'(t))dt + \sup_{t\in[0,T]} \varphi^t(u_\varepsilon(t))\\ \leq\ & 3e^{\int_0^T M_1(|\alpha'(\tau)|+1)d\tau}\left\{\varphi^0(u_0) + \int_0^T M_3(\tau)d\tau\right\}.\end{aligned} \tag{3.18}$$

It is easy to observe from the above inequality that (3.3) holds for some positive constant N_1 independent of $\varepsilon \in (0,1]$, f, and u_0. Thus, the proof of Proposition 3.1 is complete. □

By taking the limit as $\varepsilon \to 0$, we have proved Theorem 2.1 concerning the existence of solutions to (P) on $[0,T]$.

Proof of Theorem 2.1. Let u_ε be a unique solution to $(\mathrm{P}; f, u_0)_\varepsilon$ on $[0,T]$, as obtained in Proposition 3.1. Then, there exists a function $\xi_\varepsilon \in L^2(0,T;V^*)$ such that

$$\xi_\varepsilon(t) \in \partial_*\psi^t(u_\varepsilon'(t)) \ \text{ in } V^* \ \text{ for a.a. } t \in (0,T) \tag{3.19}$$

and

$$\varepsilon F u_\varepsilon'(t) + \xi_\varepsilon(t) + \partial_*\varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) = f(t) \ \text{ in } V^* \text{ for a.a. } t \in (0,T). \tag{3.20}$$

From (A2), (2.2), (3.3), and the Ascoli–Arzelà theorem, we can derive a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ and a function $u \in W^{1,2}(0,T;V)$ such that

$$\left.\begin{aligned} u_{\varepsilon_n} \to u \quad & \text{weakly in } W^{1,2}(0,T;V),\\ & \text{in } C([0,T];H),\\ & \text{weakly-}* \text{ in } L^\infty(0,T;V)\end{aligned}\right\} \tag{3.21}$$

and

$$u_{\varepsilon_n}(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0,T] \tag{3.22}$$

as $n \to \infty$.

Here, for each $t \in [0,T]$, we define the function Ψ^t on $L^2(0,t;V)$ by:

$$\Psi^t(z) := \int_0^t \psi^s(z(s))ds, \quad \forall z \in L^2(0,t;V). \tag{3.23}$$

Then, Ψ^t is a proper, l.s.c., and convex function on $L^2(0,t;V)$ for each $t \in [0,T]$ (cf. [10, Proposition 2.16] and [19, Section 0.3]). Therefore, from (3.21) and the weak lower semicontinuity of Ψ^t, it follows that

$$\int_0^t \psi^\tau(u'(\tau))d\tau \leq \liminf_{n\to\infty}\int_0^t \psi^\tau(u_{\varepsilon_n}'(\tau))d\tau \leq \tilde{N}_1 \ \text{ for all } t \in [0,T],$$

where $\tilde{N}_1 =: N_1 \left(\varphi^0(u_0) + |f|^2_{L^2(0,T;V^*)} + 1\right)$ is the same constant as in (3.3).

Additionally, we infer from (C1), (C2), (2.2), (3.3), (3.21), and the Lebesgue dominated convergence theorem that

$$g(\cdot, u_{\varepsilon_n}(\cdot)) \to g(\cdot, u(\cdot)) \ \text{ in } L^2(0,T;V^*) \ \text{ as } n \to \infty. \tag{3.24}$$

Now, we show that $u_{\varepsilon_n} \to u$ in $C([0,T];V)$ as $n \to \infty$. To this end, we multiply (3.20) by $u'_{\varepsilon_n} - u'$ to obtain:

$$\begin{aligned} &\langle \varepsilon_n F u'_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle \xi_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &\quad + \langle \partial_* \varphi^t(u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle g(t, u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ = \ &\langle f(t), u'_{\varepsilon_n}(t) - u'(t)\rangle \quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{3.25}$$

Here, by the definition of $\partial_* \psi^t$ (cf. (3.19)), we have:

$$\langle \xi_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) - u'(t)\rangle \geq \psi^t(u'_{\varepsilon_n}(t)) - \psi^t(u'(t)) \quad \text{for a.a. } t \in (0,T). \tag{3.26}$$

As $\partial_* \varphi^t$ is linear from V into V^* (cf. (B1)), it follows from Lemma 3.1 that

$$\begin{aligned} &\langle \partial_* \varphi^t(u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ = \ &\langle \partial_* \varphi^t(u_{\varepsilon_n}(t) - u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle \partial_* \varphi^t(u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ \geq \ &\frac{d}{dt}\varphi^t(u_{\varepsilon_n}(t) - u(t)) - |\alpha'(t)|\varphi^t(u_{\varepsilon_n}(t) - u(t)) \\ &\quad + \langle \partial_* \varphi^t(u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{3.27}$$

In addition, we have:

$$\begin{aligned} &\langle g(t, u_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ = \ &\langle g(t, u_{\varepsilon_n}(t)) - g(t, u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle + \langle g(t, u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &\quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{3.28}$$

Therefore, from (3.25)–(3.28), we obtain that:

$$\begin{aligned} &\frac{d}{dt}\varphi^t(u_{\varepsilon_n}(t) - u(t)) + \langle g(t, u_{\varepsilon_n}(t)) - g(t, u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ \leq \ &|\alpha'(t)|\varphi^t(u_{\varepsilon_n}(t) - u(t)) + L(t) + \psi^t(u'(t)) - \psi^t(u'_{\varepsilon_n}(t)) \quad \text{for a.a. } t \in (0,T), \end{aligned} \tag{3.29}$$

where $L(\cdot)$ is the function defined by:

$$\begin{aligned} L(t) := &\langle f(t) - \partial_* \varphi^t(u(t)) - g(t, u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &+ \varepsilon_n |F u'_{\varepsilon_n}(t)|_{V^*} |u'_{\varepsilon_n}(t) - u'(t)|_V \quad \text{for a.a. } t \in (0,T). \end{aligned}$$

Multiplying (3.29) by $e^{-\int_0^t |\alpha'(\tau)| d\tau}$, we get:

$$\begin{aligned} &\frac{d}{dt}\left\{ e^{-\int_0^t |\alpha'(\tau)| d\tau} \varphi^t(u_{\varepsilon_n}(t) - u(t)) \right\} \\ &\quad + e^{-\int_0^t |\alpha'(\tau)| d\tau} \langle g(t, u_{\varepsilon_n}(t)) - g(t, u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ \leq \ &e^{-\int_0^t |\alpha'(\tau)| d\tau} L(t) + e^{-\int_0^t |\alpha'(\tau)| d\tau} \psi^t(u'(t)) - e^{-\int_0^t |\alpha'(\tau)| d\tau} \psi^t(u'_{\varepsilon_n}(t)). \end{aligned} \tag{3.30}$$

Integrating (3.30) in time and noting that $\varphi^0(0) = 0$ (cf. (2.2)), we obtain:

$$
\begin{aligned}
& e^{-\int_0^t |\alpha'(\tau)|d\tau} \varphi^t(u_{\varepsilon_n}(t) - u(t)) \\
& \qquad + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} \langle g(s, u_{\varepsilon_n}(s)) - g(s, u(s)), u'_{\varepsilon_n}(s) - u'(s) \rangle ds \\
\leq & \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} L(s) ds + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} (\psi^s(u'(s)) + C_2) ds \\
& \qquad - \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} (\psi^s(u'_{\varepsilon_n}(s)) + C_2) ds
\end{aligned}
\tag{3.31}
$$

for all $t \in [0, T]$.

Here, we define the function $\tilde{\Psi}^t$ on $L^2(0, t; V)$ for each $t \in [0, T]$ by:

$$
\tilde{\Psi}^t(z) := \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} (\psi^s(z(s)) + C_2) ds, \quad \forall z \in L^2(0, t; V). \tag{3.32}
$$

Then, $\tilde{\Psi}^t$ is a proper, l.s.c., and convex function on $L^2(0, t; V)$ for each $t \in [0, T]$ (cf. [10, Proposition 2.16] and [19, Section 0.3]). Therefore, from (3.21) and the weak lower semicontinuity of $\tilde{\Psi}^t$, we observe that

$$
\begin{aligned}
\limsup_{n \to 0} \Bigg\{ & \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} (\psi^s(u'(s)) + C_2) ds \\
& \qquad - \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} (\psi^s(u'_{\varepsilon_n}(s)) + C_2) ds \Bigg\} \leq 0.
\end{aligned}
\tag{3.33}
$$

Additionally, we infer from (3.3), (3.21), and (3.24) that

$$
\begin{aligned}
& \lim_{n \to 0} \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} \langle g(s, u_{\varepsilon_n}(s)) - g(s, u(s)), u'_{\varepsilon_n}(s) - u'(s) \rangle ds \\
= & \lim_{n \to 0} \int_0^t \langle e^{-\int_0^s |\alpha'(\tau)|d\tau} g(s, u_{\varepsilon_n}(s)) - e^{-\int_0^s |\alpha'(\tau)|d\tau} g(s, u(s)), u'_{\varepsilon_n}(s) - u'(s) \rangle ds \\
= & \ 0
\end{aligned}
\tag{3.34}
$$

and

$$
\begin{aligned}
& \lim_{n \to 0} \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} L(s) ds \\
= & \lim_{n \to 0} \int_0^t \langle e^{-\int_0^s |\alpha'(\tau)|d\tau} f(s), u'_{\varepsilon_n}(s) - u'(s) \rangle ds \\
& \qquad - \lim_{n \to 0} \int_0^t \langle e^{-\int_0^s |\alpha'(\tau)|d\tau} \partial_* \varphi^s(u(s)), u'_{\varepsilon_n}(s) - u'(s) \rangle ds \\
& \qquad - \lim_{n \to 0} \int_0^t \langle e^{-\int_0^s |\alpha'(\tau)|d\tau} g(s, u(s)), u'_{\varepsilon_n}(s) - u'(s) \rangle ds \\
= & \ 0.
\end{aligned}
\tag{3.35}
$$

Therefore, we see from (3.31), (3.33), (3.34), and (3.35) that

$$
\limsup_{n \to 0} e^{-\int_0^t |\alpha'(\tau)|d\tau} \varphi^t(u_{\varepsilon_n}(t) - u(t)) \leq 0 \ \text{ uniformly in } t \in [0, T].
$$

Hence,

$$\limsup_{n\to 0} \varphi^t(u_{\varepsilon_n}(t) - u(t)) \le 0 \;\; \text{uniformly in } t \in [0,T],$$

which implies by (2.2) that

$$u_{\varepsilon_n} \to u \;\; \text{in } C([0,T];V) \;\; \text{as } n \to \infty. \tag{3.36}$$

Now, we show that u is a solution to $(\mathrm{P}; f, u_0)$ on $[0,T]$. Note that, from (B1), (2.2), (3.3), (3.36), and the Lebesgue dominated convergence theorem,

$$\partial_* \varphi^{(\cdot)}(u_{\varepsilon_n}(\cdot)) \to \partial_* \varphi^{(\cdot)}(u(\cdot)) \;\; \text{in } L^2(0,T;V^*) \text{ as } n \to \infty. \tag{3.37}$$

Additionally, by (A2), (3.3), and $\varepsilon_n \downarrow 0$, we have

$$\varepsilon_n F u'_{\varepsilon_n} \to 0 \;\; \text{in } L^2(0,T;V^*) \text{ as } n \to \infty. \tag{3.38}$$

As a consequence, (3.20), (3.24), (3.37), and (3.38) imply that

$$\{\xi_{\varepsilon_n}\}_{n\in\mathbb{N}} \text{ is bounded in } L^2(0,T;V^*).$$

Therefore, taking a subsequence if necessary (still denoted by $\{\varepsilon_n\}_{n\in\mathbb{N}}$), we observe that:

$$\xi_{\varepsilon_n} \to \xi \;\; \text{weakly in } L^2(0,T;V^*) \;\; \text{for some } \xi \in L^2(0,T;V^*) \text{ as } n \to \infty. \tag{3.39}$$

Hence, we infer from (3.20), (3.24), and (3.37)–(3.39) that:

$$\begin{gathered} \xi_{\varepsilon_n} = f - \partial_* \varphi^{(\cdot)}(u_{\varepsilon_n}) - g(\cdot, u_{\varepsilon_n}) - \varepsilon_n F u'_{\varepsilon_n} \to f - \partial_* \varphi^{(\cdot)}(u) - g(\cdot, u) = \xi \\ \text{in } L^2(0,T;V^*) \text{ as } n \to \infty. \end{gathered} \tag{3.40}$$

Thus, from (3.19), (3.20), (3.40), and the demi-closedness of the maximal monotone operator $\partial_* \psi^{(\cdot)}$ in $L^2(0,T;V^*)$, we conclude that

$$\xi \in \partial_* \psi^{(\cdot)}(u') \;\; \text{in } L^2(0,T;V^*), \tag{3.41}$$

or, equivalently ([18, Proposition 1.1 and Lemma 3.3]), that

$$\xi(t) \in \partial_* \psi^t(u'(t)) \text{ in } V^* \text{ for a.a. } t \in (0,T).$$

Additionally, from (3.36), we have $u(0) = u_0$ in V. Therefore, u is a solution to $(\mathrm{P}; f, u_0)$ on $[0,T]$.

Note that, from a priori estimate (3.3) and the convergence results (3.21), (3.36) and the lower semi-continuity of Ψ^t in (3.23), the bounded estimate (2.8) holds by setting $N_0 := N_1$.

Finally, we show that estimate (2.8) is valid for any solution to $(\mathrm{P}; f, u_0)$. Let u be any solution to $(\mathrm{P}; f, u_0)$ on $[0,T]$. Then, u is also a solution to the following equation for every $\varepsilon > 0$:

$$\begin{gathered} \varepsilon F u'(t) + \xi(t) + \partial_* \varphi^t(u(t)) + g(t, u(t)) = f(t) + \varepsilon F u'(t) \;\; \text{in } V^* \text{ for a.a. } t \in (0,T), \\ \xi(t) \in \partial_* \psi^t(u'(t)) \;\; \text{in } V^* \;\; \text{for a.a. } t \in (0,T). \end{gathered}$$

Therefore, by Proposition 3.1 (cf. (3.3)), we have

$$\int_0^T \psi^t(u'(t))dt + \sup_{t\in[0,T]} \varphi^t(u(t)) \le N_1 \left(\varphi^0(u_0) + |f + \varepsilon F u'|^2_{L^2(0,T;V^*)} + 1 \right).$$

Letting $\varepsilon \to 0$, we conclude that estimate (2.8) is valid for any solution u to $(\mathrm{P}; f, u_0)$.

Thus, the proof of Theorem 2.1 is complete. □

4 Singular optimal control problem (OP)

In this section, we consider the singular optimal control problem (OP). In Theorem 2.2, we showed the uniqueness of solutions to state system (P) under the additional assumption (A4). However, it seems that Assumption (A4) is too strong when we consider a class of interesting variational inequalities (cf. [22, Section 6] and Proposition 11.1 in Section 11).

Now, we state the main result of this paper, which is directed toward the existence of an optimal control for problem (OP) without the uniqueness of solutions to state system (P).

Theorem 4.1. *Suppose that Assumptions (A), (B), and (C) hold. Let u_0 be any initial datum in V, and let u_{ad} be a given function in $L^2(0,T;V)$. Then,* (OP) *has at least one optimal control $f^* \in \mathcal{F}_M$, namely,*

$$J(f^*) = \inf_{f \in \mathcal{F}_M} J(f),$$

where $J(\cdot)$ is the cost functional of (OP) *defined by* (1.4) *and* (1.5)*.*

We begin with the following result on the convergence of solutions to (P), which is a key component in the proof of Theorem 4.1.

Proposition 4.1. *Suppose that Assumptions (A), (B), and (C) are satisfied. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V^*)$, $\{u_{0,n}\}_{n\in\mathbb{N}} \subset V$, $f \in L^2(0,T;V^*)$, and $u_0 \in V$. Assume that*

$$f_n \to f \text{ in } L^2(0,T;V^*), \tag{4.1}$$

$$u_{0,n} \to u_0 \text{ in } V \tag{4.2}$$

as $n \to \infty$. Let u_n be a solution to (P;$f_n, u_{0,n}$) *on $[0,T]$. Then, there exist a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ and a function $u \in W^{1,2}(0,T;V)$ such that u is a solution to* (P;f, u_0) *on $[0,T]$ and*

$$u_{n_k} \to u \quad \text{in } C([0,T];V) \quad \text{as } k \to \infty. \tag{4.3}$$

Proof. As $\{u_{0,n}\}_{n\in\mathbb{N}}$ is bounded in V by (4.2), we observe from (2.3) that

$$\varphi^0(u_{0,n}) \text{ is bounded in } n \geq 1 \tag{4.4}$$

and

$$\lim_{n\to\infty} \varphi^0(u_{0,n} - u_0) = 0. \tag{4.5}$$

From (A2), (2.2), the bounded estimate (2.8), (4.1), (4.4), and the Ascoli–Arzelà theorem, we derive the existence of a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ and a function $u \in W^{1,2}(0,T;V)$ satisfying $n_k \to \infty$,

$$\left.\begin{array}{ll} u_{n_k} \to u & \text{weakly in } W^{1,2}(0,T;V), \\ & \text{in } C([0,T];H), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;V), \end{array}\right\} \tag{4.6}$$

and

$$u_{n_k}(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0,T]$$

as $k \to \infty$. From (4.6) and the weak lower semicontinuity of Ψ^t given in (3.23), we observe that

$$\int_0^t \psi^\tau(u'(\tau))d\tau \le \liminf_{k\to\infty} \int_0^t \psi^\tau(u'_{n_k}(\tau))d\tau < +\infty \quad \text{for all } t \in [0,T].$$

Additionally, by (C1), (C2), (2.2), (2.8), (4.6), and the Lebesgue dominated convergence theorem,

$$g(\cdot, u_{n_k}(\cdot)) \to g(\cdot, u(\cdot)) \quad \text{in } L^2(0,T;V^*) \quad \text{as } k \to \infty. \tag{4.7}$$

Next, we show that $u_{n_k} \to u$ in $C([0,T];V)$ as $k \to \infty$. To this end, we multiply (P;f_{n_k}, u_{0,n_k}) by $u'_{n_k} - u'$. Then, just as for the derivation of (3.29), we have:

$$\begin{aligned} &\frac{d}{dt}\varphi^t(u_{n_k}(t) - u(t)) + \langle g(t, u_{n_k}(t)) - g(t,u(t)), u'_{\varepsilon_n}(t) - u'(t)\rangle \\ &\quad -\langle f_{n_k}(t) - f(t), u'_{n_k}(t) - u'(t)\rangle \\ \le\ & |\alpha'(t)|\varphi^t(u_{n_k}(t) - u(t)) + \tilde{L}(t) + \psi^t(u'(t)) - \psi^t(u'_{n_k}(t)) \quad \text{for a.a. } t \in (0,T), \end{aligned} \tag{4.8}$$

where $\tilde{L}(\cdot)$ is the function defined by:

$$\tilde{L}(t) := \langle f(t) - \partial_*\varphi^t(u(t)) - g(t,u(t)), u'_{n_k}(t) - u'(t)\rangle \quad \text{for a.a. } t \in (0,T).$$

Multiplying (4.8) by $e^{-\int_0^t |\alpha'(\tau)|d\tau}$ and integrating in time, we get:

$$\begin{aligned} &e^{-\int_0^t |\alpha'(\tau)|d\tau}\varphi^t(u_{n_k}(t) - u(t)) \\ &\quad + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}\langle g(s, u_{n_k}(s)) - g(s,u(s)), u'_{n_k}(s) - u'(s)\rangle ds \\ &\quad - \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}\langle f_{n_k}(s) - f(s), u'_{n_k}(s) - u'(s)\rangle ds \\ \le\ & \varphi^0(u_{0,n_k} - u_0) + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}\tilde{L}(s)ds + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}(\psi^s(u'(s)) + C_2)ds \\ &\quad - \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}(\psi^s(u'_{n_k}(s)) + C_2)ds \end{aligned} \tag{4.9}$$

for all $t \in [0,T]$.

Using similar arguments to (3.33)–(3.35), we infer from (4.9) with (4.1), (4.2), (4.5)–(4.7), and the weak lower semicontinuity of $\tilde{\Psi}^t$ given by (3.32) that

$$\limsup_{k\to\infty} e^{-\int_0^t |\alpha'(\tau)|d\tau}\varphi^t(u_{n_k}(t) - u(t)) \le 0 \quad \text{uniformly in } t \in [0,T],$$

or, equivalently,

$$\limsup_{k\to\infty} \varphi^t(u_{n_k}(t) - u(t)) \le 0 \quad \text{uniformly in } t \in [0,T],$$

which implies by (2.2) that

$$u_{n_k} \to u \quad \text{in } C([0,T];V) \text{ as } k \to \infty. \tag{4.10}$$

Finally, we show that u is a solution to (P; f, u_0) on $[0,T]$. From (B1), (2.2), (2.8), (4.10), and the Lebesgue dominated convergence theorem, it follows that

$$\partial_*\varphi^{(\cdot)}(u_{n_k}(\cdot)) \to \partial_*\varphi^{(\cdot)}(u(\cdot)) \text{ in } L^2(0,T;V^*) \text{ as } k\to\infty. \tag{4.11}$$

As u_{n_k} is a solution to (P;f_{n_k}, u_{0,n_k}) on $[0,T]$, there exists a function $\xi_{n_k} \in L^2(0,T;V^*)$ such that

$$\xi_{n_k}(t) \in \partial_*\psi^t(u'_{n_k}(t)) \text{ in } V^* \text{ for a.a. } t\in(0,T)$$

and

$$\xi_{n_k}(t) + \partial_*\varphi^t(u_{n_k}(t)) + g(t,u_{n_k}(t)) = f_{n_k}(t) \text{ in } V^* \text{ for a.a. } t\in(0,T). \tag{4.12}$$

By (4.12) with (4.1), (4.7), and (4.11), we see that

$$\{\xi_{n_k}\}_{k\in\mathbb{N}} \text{ is bounded in } L^2(0,T;V^*).$$

Therefore, taking a subsequence if necessary (still denoted by $\{n_k\}_{k\in\mathbb{N}}$), we observe that:

$$\xi_{n_k} \to \xi \text{ weakly in } L^2(0,T;V^*) \text{ for some } \xi\in L^2(0,T;V^*) \text{ as } k\to\infty. \tag{4.13}$$

In addition, we infer from (4.1), (4.7), (4.11), (4.12), and (4.13) that:

$$\partial_*\psi^{(\cdot)}(u'_{n_k}) \ni \xi_{n_k} \to \xi = f - \partial_*\varphi^{(\cdot)}(u) - g(\cdot,u) \text{ in } L^2(0,T;V^*) \text{ as } k\to\infty. \tag{4.14}$$

Thus, from (4.6), (4.14), and the demi-closedness of the maximal monotone operator $\partial_*\psi^{(\cdot)}$ in $L^2(0,T;V^*)$, we infer that

$$\xi \in \partial_*\psi^{(\cdot)}(u') \text{ in } L^2(0,T;V^*),$$

or, equivalently,

$$\xi(t) \in \partial_*\psi^t(u'(t)) \text{ in } V^* \text{ for a.a. } t\in(0,T).$$

From (4.10), we have $u(0) = u_0$ in V. Hence, we conclude that u is a solution to (P; f, u_0) on $[0,T]$. Thus, the proof of Proposition 4.1 is complete. □

Using the above convergence result of solutions to state problem (P), we prove the main theorem in our paper (Theorem 4.1), which is concerned with the existence of an optimal control for problem (OP).

Proof of Theorem 4.1. We are going to prove the existence of an optimal control for (OP) without the uniqueness of solutions to state problem (P).

Note that, from (1.4) and (1.5), $J(f) \ge 0$ for all $f\in\mathcal{F}_M$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_M$ be a minimizing sequence of the functional J on $\mathcal{F}_M$, namely,

$$d^* := \inf_{f\in\mathcal{F}_M} J(f) = \lim_{n\to\infty} J(f_n).$$

By the definition in (1.4) of $J(f_n)$, for each n, there is a solution $u_n \in \mathcal{S}(f_n)$ such that

$$\pi_{f_n}(u_n) < J(f_n) + \frac{1}{n}. \tag{4.15}$$

Here, we observe from $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_M$ and (1.3) that

$$\{f_n\}_{n\in\mathbb{N}} \text{ is bounded in } W^{1,2}(0,T;V^*) \cap L^2(0,T;H).$$

Thus, by the Aubin compactness theorem (cf. [23, Chapter1, Section 5]), there is a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ and a function $f^* \in \mathcal{F}_M$ such that

$$\left.\begin{array}{ll} f_{n_k} \to f^* & \text{weakly in } W^{1,2}(0,T;V^*), \\ & \text{weakly in } L^2(0,T;H), \\ & \text{in } L^2(0,T;V^*) \end{array}\right\} \tag{4.16}$$

as $k \to \infty$.

Now, taking a subsequence if necessary, we infer from Proposition 4.1 that there is a solution u^* to (P;f^*, u_0) on $[0,T]$ satisfying

$$u_{n_k} \to u^* \text{ in } C([0,T];V) \ \text{ as } k \to \infty. \tag{4.17}$$

Therefore, it follows from (4.15)–(4.17) and $u^* \in \mathcal{S}(f^*)$ that

$$\begin{aligned} d^* &= \inf_{f\in\mathcal{F}_M} J(f) \le J(f^*) = \inf_{u\in\mathcal{S}(f^*)} \pi_{f^*}(u) \\ &\le \pi_{f^*}(u^*) = \frac{1}{2}\int_0^T |u^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f^*(t)|_{V^*}^2 dt \\ &= \lim_{k\to\infty} \pi_{f_{n_k}}(u_{n_k}) \\ &\le \lim_{k\to\infty} \left\{ J(f_{n_k}) + \frac{1}{n_k} \right\} \\ &= \lim_{k\to\infty} J(f_{n_k}) = d^*. \end{aligned}$$

Hence, we have $d^* = \inf_{f\in\mathcal{F}_M} J(f) = J(f^*)$, which implies that $f^* \in \mathcal{F}_M$ is an optimal control for (OP). Thus, the proof of Theorem 4.1 is complete. □

5 Approximation for (P) and (OP)

The non-uniqueness situation of state problem (P), as in Section 4, makes the numerical approach to (OP) quite difficult. In this section, we establish an approximation procedure to (P) and (OP) from the viewpoint of numerical analysis.

Throughout this section, we fix the initial datum $u_0 \in V$. We begin by setting up approximate problems for (P). For each $\varepsilon \in (0,1]$ and each $h \in L^2(0,T;V)$, we consider (P; $f + \varepsilon F h, u_0)_\varepsilon$ as the approximate problem to (P; f, u_0):

$$\begin{cases} \varepsilon F u_\varepsilon'(t) + \partial_*\psi^t(u_\varepsilon'(t)) + \partial_*\varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) \ni f(t) + \varepsilon F h(t) \ \text{ in } V^* \\ \hfill \text{for a.a. } t \in (0,T), \\ u_\varepsilon(0) = u_0 \ \text{ in } V, \end{cases} \tag{5.1}$$

where $F : V \to V^*$ is the duality mapping (cf. (2.1)).

We immediately obtain the following from Proposition 3.1.

Proposition 5.1 (cf. Proposition 3.1)**.** *Suppose that Assumptions (A), (B), and (C) hold. Then, for each* $\varepsilon \in (0,1]$*,* $f \in L^2(0,T;V^*)$*,* $h \in L^2(0,T;V)$*, and* $u_0 \in V$*, there exists a unique solution* u_ε *to* $(\mathrm{P}; f + \varepsilon Fh, u_0)_\varepsilon$ *on* $[0,T]$*, namely,* u_ε *satisfies* $u_\varepsilon \in W^{1,2}(0,T;V)$*,* $u_\varepsilon(0) = u_0$ *in* V *and the following holds:*

(•) *There exists a function* $\xi_\varepsilon \in L^2(0,T;V^*)$ *such that*

$$\xi_\varepsilon(t) \in \partial_* \psi^t(u'_\varepsilon(t)) \quad \text{in } V^* \quad \text{for a.a. } t \in (0,T),$$

$$\varepsilon F u'_\varepsilon(t) + \xi_\varepsilon(t) + \partial_* \varphi^t(u_\varepsilon(t)) + g(t, u_\varepsilon(t)) = f(t) + \varepsilon F h(t) \quad \text{in } V^* \text{ for a.a. } t \in (0,T).$$

Moreover, there exists a constant $N_2 > 0$*, independent of* ε*,* f*,* h*, and* u_0*, such that*

$$\begin{aligned} & \int_0^T \psi^t(u'_\varepsilon(t))dt + \sup_{t \in [0,T]} \varphi^t(u_\varepsilon(t)) \\ \leq \ & N_2 \left(\varphi^0(u_0) + |f|^2_{L^2(0,T;V^*)} + |h|^2_{L^2(0,T;V)} + 1 \right). \end{aligned} \tag{5.2}$$

It is easy to see that Proposition 5.1 is a direct consequence of Proposition 3.1 with estimate (5.2) as well.

We now state the first main result of this section, which is concerned with the relationship between $(\mathrm{P}; f, u_0)$ and $(\mathrm{P}; f + \varepsilon Fh, u_0)_\varepsilon$.

Theorem 5.1. *Suppose that Assumptions (A), (B), and (C) hold. Let* $f \in L^2(0,T;V^*)$ *and* $u_0 \in V$*. Then, we have:*

(i) *Let* $\varepsilon \in (0,1]$ *and let* $\{h_\varepsilon\}_{\varepsilon \in (0,1]}$ *be a bounded set in* $L^2(0,T;V)$*. Additionally, let* u_ε *be a unique solution to* $(\mathrm{P}; f + \varepsilon F h_\varepsilon, u_0)_\varepsilon$ *on* $[0,T]$*. Then, there exist a sequence* $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset \{\varepsilon\}_{\varepsilon \in (0,1]}$ *with* $\varepsilon_n \to 0$ *(as* $n \to \infty$*) and a function* $u \in W^{1,2}(0,T;V)$ *such that* u *is a solution to* $(\mathrm{P}; f, u_0)$ *on* $[0,T]$ *and*

$$u_{\varepsilon_n} \to u \quad \text{in } C([0,T];V) \quad \text{as } n \to \infty.$$

(ii) *Let* u *be any solution to* $(\mathrm{P}; f, u_0)$ *on* $[0,T]$*. Then, there exist sequences* $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset (0,1]$ *with* $\varepsilon_n \to 0$ *(as* $n \to \infty$*),* $\{f_n\}_{n \in \mathbb{N}} \subset L^2(0,T;V^*)$*,* $\{h_n\}_{n \in \mathbb{N}} \subset L^2(0,T;V)$*, and* $\{u_{\varepsilon_n}\}_{n \in \mathbb{N}} \subset W^{1,2}(0,T;V)$ *such that* u_{ε_n} *is a unique solution to* $(\mathrm{P}; f_n + \varepsilon_n F h_n, u_0)_{\varepsilon_n}$ *on* $[0,T]$*,* $\{h_n\}_{n \in \mathbb{N}}$ *is bounded in* $L^2(0,T;V)$*, and*

$$u_{\varepsilon_n} \to u \quad \text{in } C([0,T];V), \qquad f_n \to f \quad \text{in } L^2(0,T;V^*) \quad \text{as } n \to \infty.$$

Proof. We first show (i). As $\{h_\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in $L^2(0,T;V)$ and $F : V \to V^*$ is the duality mapping, we observe that

$$\varepsilon F h_\varepsilon \to 0 \quad \text{in } L^2(0,T;V^*) \quad \text{as } \varepsilon \to 0.$$

Therefore, assertion (i) can be shown in a similar manner to the proof of Theorem 2.1 (cf. Section 3).

Next, we show (ii). To this end, let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ be a sequence with $\varepsilon_n \to 0$ as $n \to \infty$. Additionally, let u be any solution to $(\mathrm{P}; f, u_0)$ on $[0,T]$. Note that $u \in W^{1,2}(0,T;V)$ and the following equation holds:

$$\partial_* \psi^t(u'(t)) + \partial_* \varphi^t(u(t)) + g(t, u(t)) \ni f(t) \quad \text{in } V^* \text{ for a.a. } t \in (0,T). \tag{5.3}$$

Adding $\varepsilon_n F u'(t)$ to both sides in (5.3), we observe that the function u is also a solution to $(\mathrm{P}; f + \varepsilon_n F u', u_0)_{\varepsilon_n}$ on $[0,T]$. Hence, we conclude that assertion (ii) holds for $u_{\varepsilon_n} := u$, $f_n := f$ and $h_n := u'$.

Thus, Theorem 5.1 has been proved. □

Next, we consider an approximate problem for (OP), fixing an initial datum $u_0 \in V$. Note from (A2) and (2.8) that any solution u to $(\mathrm{P}; f, u_0)$ on $[0,T]$ satisfies the following estimate:

$$\int_0^T |u'(t)|_V^2 dt \le \frac{N_0 \left(\varphi^0(u_0) + |f|^2_{L^2(0,T;V^*)} + 1 \right) + C_2 T}{C_1}. \tag{5.4}$$

Therefore, we take and fix a positive number $N > 0$ so that

$$N^2 \ge \frac{N_0 \left(\varphi^0(u_0) + M^2 + 1 \right) + C_2 T}{C_1}, \tag{5.5}$$

where $M > 0$ is the same positive constant as in the control space $\mathcal{F}_M$ (cf. (1.3)).

For each $\varepsilon \in (0,1]$, we consider a perturbation of the control space $\mathcal{H}_N^\varepsilon$ defined by

$$\mathcal{H}_N^\varepsilon := \left\{ h \in W^{1,2}(0,T;V) \cap L^2(0,T;X) \ ; \ \begin{array}{r} |h|_{L^2(0,T;V)} \le N, \\ |h'|_{L^2(0,T;V)} \le \varepsilon^{-1} N, \\ |h|_{L^2(0,T;X)} \le \varepsilon^{-1} N \end{array} \right\}, \tag{5.6}$$

where X is a reflexive Banach space such that X is densely and compactly embedded into V.

Now, for each $\varepsilon \in (0,1]$, we study the following control problem for the state system $(\mathrm{P}; f + \varepsilon F h, u_0)_\varepsilon$, denoted by $(\mathrm{OP})_\varepsilon$:

Problem $(\mathrm{OP})_\varepsilon$: Find a control $(f_\varepsilon^*, h_\varepsilon^*) \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$, called an optimal control, such that

$$J_\varepsilon(f_\varepsilon^*, h_\varepsilon^*) = \inf_{(f,h) \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon} J_\varepsilon(f, h).$$

Here, $J_\varepsilon(f,h)$ is the cost functional defined by

$$J_\varepsilon(f,h) := \frac{1}{2} \int_0^T |u_\varepsilon(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2} \int_0^T |f(t)|_{V^*}^2 dt + \frac{\varepsilon}{2} \int_0^T |h(t)|_V^2 dt, \tag{5.7}$$

where (f,h) is any control in $\mathcal{F}_M \times \mathcal{H}_N^\varepsilon$, u_ε is a unique solution to the state system $(\mathrm{P}; f + \varepsilon F h, u_0)_\varepsilon$, and $u_{ad} \in L^2(0,T;V)$ is the target profile.

Note that $(\mathrm{OP})_\varepsilon$ is the standard optimal control problem, because the state system $(\mathrm{P}; f + \varepsilon F h, u_0)_\varepsilon$ has a unique solution on $[0,T]$.

We now state the second result of this section, which is concerned with the relationship between (OP) and $(\mathrm{OP})_\varepsilon$.

Theorem 5.2. *Suppose that Assumptions (A), (B), and (C) hold. Let* $u_0 \in V$ *and* $u_{ad} \in L^2(0,T;V)$*. Then, we have:*

(i) *For each* $\varepsilon \in (0,1]$, $(\mathrm{OP})_\varepsilon$ *has at least one optimal control* $(f_\varepsilon^*, h_\varepsilon^*) \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$*, namely,*

$$J_\varepsilon(f_\varepsilon^*, h_\varepsilon^*) = \inf_{(f,h)\in\mathcal{F}_M\times\mathcal{H}_N^\varepsilon} J_\varepsilon(f,h).$$

(ii) *Let* $\varepsilon \in (0,1]$*, and let* $(f_\varepsilon^*, h_\varepsilon^*) \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ *be an optimal control of the approximate problem* $(\mathrm{OP})_\varepsilon$*. Additionally, assume that*

(D) *For any function* $h \in L^2(0,T;V)$ *with* $|h|_{L^2(0,T;V)} \le N$*, there exists a sequence* $\{h_\varepsilon\}_{\varepsilon\in(0,1]}$ *of functions* $h_\varepsilon \in \mathcal{H}_N^\varepsilon$ *such that*

$$h_\varepsilon \to h \quad \text{in } L^2(0,T;V) \text{ as } \varepsilon \to 0.$$

Then, there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$ *with* $\varepsilon_n \to 0$ $(n \to \infty)$ *such that any weak limit function* f^* *of* $\{f_{\varepsilon_n}^*\}_{n\in\mathbb{N}}$ *in* $L^2(0,T;V^*)$ *is an optimal control for* (OP).

Remark 5.1. *The main point of Assumption* (D) *is to guarantee the compactness of* $\mathcal{H}_N^\varepsilon$ *in* $L^2(0,T;V)$*. In any application treated in Section 11, Assumption* (D) *is automatically checked by the usual smoothness argument (e.g., the regularization method using the mollifier and the convolution [1, Sections 2.28 and 3.16]). For instance, in the case of* $V = W^{1,p}(\Omega)$, $2 \le p < \infty$*, Assumption* (D) *is easily verified by choosing* $W^{2,p}(\Omega)$ *as the space* X*.*

The following convergence result for solutions is a key component in the proof of Theorem 5.2.

Proposition 5.2. *Suppose that Assumptions (A), (B), and (C) hold. Let* $\{f_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V^*)$, $\{u_{0,n}\}_{n\in\mathbb{N}} \subset V$, $f \in L^2(0,T;V^*)$*, and* $u_0 \in V$*. Assume that*

$$f_n \to f \text{ in } L^2(0,T;V^*), \quad u_{0,n} \to u_0 \text{ in } V \quad \text{as } n \to \infty.$$

Then, the following statements hold:

(i) *Assume* $\{h_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V)$, $h \in L^2(0,T;V)$ *and*

$$h_n \to h \quad \text{in } L^2(0,T;V) \quad \text{as } n \to \infty. \tag{5.8}$$

For a fixed parameter $\varepsilon \in (0,1]$*, let* u_n *be a unique solution to* $(\mathrm{P}; f_n + \varepsilon F h_n, u_{0,n})_\varepsilon$ *on* $[0,T]$*. Then, there is a function* $u \in W^{1,2}(0,T;V)$ *such that* u *is a unique solution to* $(\mathrm{P}; f + \varepsilon F h, u_0)_\varepsilon$ *on* $[0,T]$ *and*

$$u_n \to u \quad \text{in } C([0,T];V) \quad \text{as } n \to \infty.$$

(ii) *Assume that* $\{h_n\}_{n\in\mathbb{N}}$ *is a bounded set in* $L^2(0,T;V)$. *Let* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ *with* $\varepsilon_n \to 0$ *(as* $n \to \infty$*). Let* u_{ε_n} *be a unique solution to* $(\mathrm{P};f_n+\varepsilon_n F h_n, u_{0,n})_{\varepsilon_n}$ *on* $[0,T]$. *Then, there exist a subsequence* $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ *and a function* $u \in W^{1,2}(0,T;V)$ *such that* u *is a solution to* $(\mathrm{P};f,u_0)$ *on* $[0,T]$ *and*

$$u_{\varepsilon_{n_k}} \to u \quad \text{in } C([0,T];V) \quad \text{as } k \to \infty.$$

Proof. We first prove (i). From the uniform convexity of V and V^* and the properties of the duality mapping $F : V \to V^*$, it follows that

$$Fh_n \to Fh \ \text{ in } L^2(0,T;V^*) \ \text{ as } n \to \infty.$$

Therefore, similar to the proof of Proposition 4.1, we can prove assertion (i).

Next, we show (ii). As $\{h_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;V)$, we observe that

$$\varepsilon_n F h_n \to 0 \ \text{ in } L^2(0,T;V^*) \ \text{ as } n \to \infty.$$

Therefore, assertion (ii) is proved in a similar way to Theorem 2.1 (cf. Theorem 5.1(i)), and the detailed proof is omitted. □

Additionally, the following convergence result for solutions is a key component in the proof of Theorem 5.2(ii).

Proposition 5.3. *Suppose that Assumptions (A), (B), (C), and* (D) *hold. Let* $f \in \mathcal{F}_M$ *and* $u_0 \in V$. *Additionally, let* u *be any solution to* $(\mathrm{P};f,u_0)$ *on* $[0,T]$. *Then, there are sequences* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ *with* $\varepsilon_n \to 0$ *(as* $n \to \infty$*),* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_M$, $\{h_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V)$ *with* $h_n \in \mathcal{H}_N^{\varepsilon_n}$, *and* $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset W^{1,2}(0,T;V)$ *such that* u_{ε_n} *is a unique solution to* $(\mathrm{P};f_n+\varepsilon_n F h_n, u_0)_{\varepsilon_n}$ *on* $[0,T]$, *and*

$$u_{\varepsilon_n} \to u \quad \text{in } C([0,T];V), \qquad f_n \to f \quad \text{in } L^2(0,T;V^*) \quad \text{as } n \to \infty.$$

Proof. Note from (5.4) and (5.5) that the solution u to $(\mathrm{P};f,u_0)$ on $[0,T]$ satisfies the following:

$$|u'|_{L^2(0,T;V)} \le N, \tag{5.9}$$

$$\partial_*\psi^t(u'(t)) + \partial_*\varphi^t(u(t)) + g(t,u(t)) \ni f(t) \ \text{ in } V^* \text{ for a.a. } t \in (0,T). \tag{5.10}$$

Let $\delta \in (0,1]$ be any constant. Then, adding $\delta F u'(t)$ to both sides of (5.10), we observe that the function u is also a solution to $(\mathrm{P}; f+\delta F u', u_0)_\delta$ on $[0,T]$ (cf. Theorem 5.1(ii)).

By (5.9) and assumption (D), there exist a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,1]$ with $\varepsilon_k \to 0$ and a sequence $\{h_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of functions $h_{\varepsilon_k} \in \mathcal{H}_N^{\varepsilon_k}$ such that

$$h_{\varepsilon_k} \to u' \ \text{ in } L^2(0,T;V) \ \text{ as } k \to \infty. \tag{5.11}$$

Let $\{\delta_\ell\}_{\ell\in\mathbb{N}}$ be a sequence in $(0,1]$ so that $\delta_\ell \to 0$ as $\ell \to \infty$. Now, for a fixed number δ_ℓ, we consider the approximate system $(\mathrm{P}; f+\delta_\ell F h_{\varepsilon_k}, u_0)_{\delta_\ell}$ on $[0,T]$. Then, taking a subsequence if necessary (still denoted by $\{\varepsilon_k\}_{k\in\mathbb{N}}$), we observe from Proposition 5.2(i) with (5.11) that a unique solution $u^{\ell}_{\varepsilon_k} \in W^{1,2}(0,T;V)$ to $(\mathrm{P}; f + \delta_\ell F h_{\varepsilon_k}, u_0)_{\delta_\ell}$ on $[0,T]$ converges to the one $\tilde{u}^\ell$ to $(\mathrm{P}; f + \delta_\ell F u', u_0)_{\delta_\ell}$ on $[0,T]$ in the following sense:

$$u^{\ell}_{\varepsilon_k} \to \tilde{u}^\ell \ \text{ in } C([0,T];V) \ \text{ as } k \to \infty.$$

As u is also a solution to $(\mathrm{P}; f + \delta_\ell F u', u_0)_{\delta_\ell}$ on $[0,T]$, we infer from the uniqueness of solutions to $(\mathrm{P}; f + \delta_\ell F u', u_0)_{\delta_\ell}$ that $u = \tilde{u}^\ell$, and hence

$$u^{\ell}_{\varepsilon_k} \to u \ \text{ in } C([0,T];V) \ \text{ as } k \to \infty.$$

Note from $h_{\varepsilon_k} \in \mathcal{H}^{\varepsilon_k}_N$ that $\{h_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;V)$; more precisely,

$$|h_{\varepsilon_k}|_{L^2(0,T;V)} \le N \text{ for all } k \ge 1.$$

Therefore, from the diagonal argument with respect to the parameters k and ℓ, we verify the validity of Proposition 5.3. Indeed, taking $\delta_n := \varepsilon_n$, we derive the convergence by setting $u_{\varepsilon_n} := u^n_{\varepsilon_n}$, $f_n := f$, and $h_n := h_{\varepsilon_n}$. Thus, the proof of Proposition 5.3 is complete. □

Now, let us prove Theorem 5.2, which is concerned with the relationship between (OP) and $(\mathrm{OP})_\varepsilon$.

Proof of Theorem 5.2. We first prove Theorem 5.2(i). Using the standard argument with Proposition 5.2(i), we can show Theorem 5.2(i) concerning the existence of an optimal control for $(\mathrm{OP})_\varepsilon$. Indeed, let $\varepsilon \in (0,1]$ be fixed. Then, we observe from (5.7) that $J_\varepsilon(f,h) \ge 0$ for all $(f,h) \in \mathcal{F}_M \times \mathcal{H}^\varepsilon_N$. Let $\{(f_n,h_n)\}_{n\in\mathbb{N}} \subset \mathcal{F}_M \times \mathcal{H}^\varepsilon_N$ be a minimizing sequence such that

$$d^*_\varepsilon := \inf_{(f,h)\in\mathcal{F}_M\times\mathcal{H}^\varepsilon_N} J_\varepsilon(f,h) = \lim_{n\to\infty} J_\varepsilon(f_n,h_n).$$

Here, we observe from $\{(f_n,h_n)\}_{n\in\mathbb{N}} \subset \mathcal{F}_M \times \mathcal{H}^\varepsilon_N$, (1.3), and (5.6) that

$$\{f_n\}_{n\in\mathbb{N}} \text{ is bounded in } W^{1,2}(0,T;V^*) \cap L^2(0,T;H),$$

$$\{h_n\}_{n\in\mathbb{N}} \text{ is bounded in } W^{1,2}(0,T;V) \cap L^2(0,T;X).$$

Thus, with the help of the Aubin compactness theorem (cf. [23, Chapter 1, Section 5]), there exist a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ and a function $(f^*,h^*) \in \mathcal{F}_M \times \mathcal{H}^\varepsilon_N$ such that

$$\left.\begin{aligned} f_{n_k} \to f^* \quad & \text{weakly in } W^{1,2}(0,T;V^*), \\ & \text{weakly in } L^2(0,T;H), \\ & \text{in } L^2(0,T;V^*) \end{aligned}\right\} \tag{5.12}$$

and

$$\left.\begin{aligned} h_{n_k} \to h^* \quad & \text{weakly in } W^{1,2}(0,T;V), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{in } L^2(0,T;V) \end{aligned}\right\} \tag{5.13}$$

as $k \to \infty$.

Let u_{n_k} be a unique solution to (P;$f_{n_k} + \varepsilon F h_{n_k}, u_0)_\varepsilon$ on $[0,T]$. Then, we infer from Proposition 5.2(i) with (5.12) and (5.13) that there is a unique solution u^* to (P;$f^* + \varepsilon F h^*, u_0)_\varepsilon$ on $[0,T]$ satisfying

$$u_{n_k} \to u^* \text{ in } C([0,T];V) \text{ as } k \to \infty. \tag{5.14}$$

Therefore, it follows from (5.12)–(5.14) that

$$J_\varepsilon(f^*, h^*) = \lim_{k\to\infty} J_\varepsilon(f_{n_k}, h_{n_k}) = \inf_{(f,h)\in\mathcal{F}_M\times\mathcal{H}_N^\varepsilon} J_\varepsilon(f,h) = d_\varepsilon^*,$$

which implies that $(f^*, h^*) \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ is an optimal control for (OP)$_\varepsilon$. Thus, the proof of Theorem 5.2(i) is complete.

Next, we prove Theorem 5.2(ii) by approximating the admissible optimal pair for (OP).

Define $d^* := \inf_{f\in\mathcal{F}_M} J(f)$ and let $\tilde{f}^*$ be any optimal control for (OP) with its optimal state $\tilde{u}^*$, namely, $\tilde{u}^* \in S(\tilde{f}^*)$ and

$$d^* = J(\tilde{f}^*) = \frac{1}{2}\int_0^T |\tilde{u}^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |\tilde{f}^*(t)|_{V^*}^2 dt.$$

Now, we approximate the admissible optimal pair $(\tilde{u}^*, \tilde{f}^*)$ for (OP) by applying Proposition 5.3. Indeed, we observe from Proposition 5.3 that there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ with $\varepsilon_n \to 0$ (as $n \to \infty$), $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_M$, $\{h_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V)$ with $h_n \in \mathcal{H}_N^{\varepsilon_n}$, and $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset W^{1,2}(0,T;V)$ such that u_{ε_n} is a unique solution to (P; $f_n + \varepsilon_n F h_n, u_0)_{\varepsilon_n}$ on $[0,T]$,

$$u_{\varepsilon_n} \to \tilde{u}^* \text{ in } C([0,T];V) \text{ as } n \to \infty, \tag{5.15}$$

and

$$f_n \to \tilde{f}^* \text{ in } L^2(0,T;V^*) \text{ as } n \to \infty. \tag{5.16}$$

Note from $h_n \in \mathcal{H}_N^{\varepsilon_n}$ that $\{h_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;V)$; more precisely,

$$|h_n|_{L^2(0,T;V)} \le N \text{ for all } n \ge 1.$$

Therefore, from (1.4), (1.5), (5.7), (5.15), and (5.16), it follows that

$$\begin{aligned} d^* = J(\tilde{f}^*) &= \frac{1}{2}\int_0^T |\tilde{u}^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |\tilde{f}^*(t)|_{V^*}^2 dt \\ &= \lim_{n\to\infty} \left\{ \frac{1}{2}\int_0^T |u_{\varepsilon_n}(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f_n(t)|_{V^*}^2 dt \right\} \\ &= \lim_{n\to\infty} J_{\varepsilon_n}(f_n, h_n) \\ &\ge \limsup_{n\to\infty} d_{\varepsilon_n}^*, \end{aligned} \tag{5.17}$$

where $d_{\varepsilon_n}^* := \inf_{(f,h)\in\mathcal{F}_M\times\mathcal{H}_N^{\varepsilon_n}} J_{\varepsilon_n}(f,h)$.

Now, let $\{(f_{\varepsilon_n}^*, h_{\varepsilon_n}^*)\}_{n\in\mathbb{N}}$ be any sequence of optimal controls $(f_{\varepsilon_n}^*, h_{\varepsilon_n}^*)$ for (OP)$_{\varepsilon_n}$. In addition, let $u_{\varepsilon_n}^*$ be a unique solution to (P; $f_{\varepsilon_n}^* + \varepsilon_n F h_{\varepsilon_n}^*, u_0)_{\varepsilon_n}$ on $[0,T]$. Then, it follows from $(f_{\varepsilon_n}^*, h_{\varepsilon_n}^*) \in \mathcal{F}_M \times \mathcal{H}_N^{\varepsilon_n}$, (1.3) and (5.6) that

$$\{f_{\varepsilon_n}^*\}_{n\in\mathbb{N}} \text{ is bounded in } W^{1,2}(0,T;V^*) \cap L^2(0,T;H),$$

$$\{h^*_{\varepsilon_n}\}_{n\in\mathbb{N}} \text{ is bounded in } L^2(0,T;V). \tag{5.18}$$

Therefore, by the Aubin compactness theorem (cf. [23, Chapter 1, Section 5]), there exist a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ and a function $f^* \in \mathcal{F}_M$ such that

$$\left.\begin{aligned} f^*_{\varepsilon_{n_k}} \to f^* \quad & \text{weakly in } W^{1,2}(0,T;V^*), \\ & \text{weakly in } L^2(0,T;H), \\ & \text{in } L^2(0,T;V^*) \end{aligned}\right\} \tag{5.19}$$

as $k \to \infty$. Then, taking a subsequence if necessary, we infer from Proposition 5.2(ii) with (5.18) and (5.19) that there is a solution u^* to (P;f^*, u_0) on $[0,T]$ satisfying

$$u^*_{\varepsilon_{n_k}} \to u^* \ \text{ in } C([0,T];V) \ \text{ as } k \to \infty. \tag{5.20}$$

Next, taking a subsequence if necessary, we choose a subsequence of $\{n_k\}_{k\in\mathbb{N}}$ (still denoted by $\{n_k\}_{k\in\mathbb{N}}$) so that

$$\liminf_{n\to\infty} d^*_{\varepsilon_n} = \lim_{k\to\infty} d^*_{\varepsilon_{n_k}}.$$

Therefore, it follows from (1.4), (1.5), (5.18)–(5.20), and $u^* \in \mathcal{S}(f^*)$ that

$$\begin{aligned} &\liminf_{n\to\infty} d^*_{\varepsilon_n} \\ =& \lim_{k\to\infty} d^*_{\varepsilon_{n_k}} = \lim_{k\to\infty} J_{\varepsilon_{n_k}}(f^*_{\varepsilon_{n_k}}, h^*_{\varepsilon_{n_k}}) \\ =& \lim_{k\to\infty} \left\{ \frac{1}{2}\int_0^T |u^*_{\varepsilon_{n_k}}(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f^*_{\varepsilon_{n_k}}(t)|^2_{V^*} dt + \frac{\varepsilon_{n_k}}{2}\int_0^T |h^*_{\varepsilon_{n_k}}(t)|^2_V dt \right\} \\ =& \frac{1}{2}\int_0^T |u^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f^*(t)|^2_{V^*} dt \\ =& \pi_{f^*}(u^*) \\ \geq& J(f^*) \\ \geq& d^*. \end{aligned} \tag{5.21}$$

On account of (5.21) and inequality (5.17), we conclude that

$$d^* = \lim_{n\to\infty} d^*_{\varepsilon_n} = J(f^*).$$

Hence, $f^* \in \mathcal{F}_M$ is an optimal control for (OP) and u^* is its optimal state. Thus, the proof of Theorem 5.2 is complete. □

6 Solvability of (QP)

In this section, we consider a doubly nonlinear quasi-variational evolution equation, as introduced in [22, Section 5], of the form:

$$\text{(QP)} \begin{cases} \partial_*\psi^t(u'(t)) + \partial_*\varphi^t(u;u(t)) + g(t,u(t)) \ni f(t) \ \text{ in } V^* \text{ for a.a. } t \in (0,T), \\ u(0) = u_0 \ \text{ in } V, \end{cases}$$

where $\psi^t(z)$ and $g(t,z)$ are as in (P), and $\varphi^t(v;z)$ is precisely formulated below.

(Assumption (B'))

Setting

$$D_0 := \left\{ v \in W^{1,2}(0,T;V) \;\middle|\; \int_0^T \psi^t(v'(t))dt < \infty \right\},$$

we define a functional $\varphi^t : [0,T] \times D_0 \times V \to \mathbb{R}$ such that $\varphi^t(v;z)$ is non-negative, finite, continuous, and convex in $z \in V$ for any $t \in [0,T]$ and any $v \in D_0$, and

$$\varphi^t(v_1;z) = \varphi^t(v_2;z), \ \forall z \in V, \ \text{if } v_1 = v_2 \text{ on } [0,t],$$

for $v_i \in D_0, \ i = 1,2$. We assume the following:

(B1') The subdifferential $\partial_*\varphi^t(v;z)$ of $\varphi^t(v;z)$ with respect to $z \in V$ is linear and bounded from $D(\partial_*\varphi^t(v;\cdot)) = V$ into V^* for each $t \in [0,T]$ and $v \in D_0$, and there is a positive constant C_3' such that

$$|\partial_*\varphi^t(v;z)|_{V^*} \le C_3'|z|_V, \quad \forall z \in V, \ \forall v \in D_0, \ \forall t \in [0,T].$$

(B2') If $\{v_n\}_{n\in\mathbb{N}} \subset D_0$, $\sup_{n\in\mathbb{N}} \int_0^T \psi^t(v_n'(t))dt < \infty$ and $v_n \to v \in C([0,T];H)$ (as $n \to \infty$), then

$$\partial_*\varphi^t(v_n;z) \to \partial_*\varphi^t(v;z) \ \text{ in } V^*, \ \forall z \in V, \ \forall t \in [0,T] \ \text{ as } n \to \infty.$$

(B3') $\varphi^0(v;0) = 0$ for all $v \in D_0$. Moreover, there is a positive constant C_4' such that

$$\varphi^0(v;z) \ge C_4'|z|_V^2, \ \forall z \in V, \ \forall v \in D_0.$$

(B4') There is a function $\alpha \in W^{1,1}(0,T)$ such that

$$|\varphi^t(v;z) - \varphi^s(v;z)| \le |\alpha(t) - \alpha(s)|\varphi^s(v;z), \quad \forall z \in V, \ \forall v \in D_0, \ \forall s,t \in [0,T].$$

Similar to Lemma 2.1, we can state the following.

Lemma 6.1 (cf. Lemma 2.1). *Suppose that Assumption (B') is satisfied. Then, the following inequalities hold:*

(i)

$$\frac{C_4'}{|\alpha'|_{L^1(0,T)} + 1}|z|_V^2 \le \varphi^t(v;z) \le \left(|\alpha'|_{L^1(0,T)} + 1\right) C_3'|z|_V^2, \quad \forall t \in [0,T], \ \forall z \in V, \ \forall v \in D_0. \tag{6.1}$$

(ii)

$$\langle \partial_*\varphi^t(v;z), z\rangle \ge \frac{C_4'}{|\alpha'|_{L^1(0,T)} + 1}|z|_V^2, \quad \forall t \in [0,T], \ \forall z \in V, \ \forall v \in D_0.$$

Remark 6.1 (cf. Remark 2.2)**.** *From* (B1') *and Lemma 6.1(i), we can derive that the subdifferential* $\partial_* \varphi^t(v;\cdot)$ *satisfies*

$$C_3'|z|_V^2 \geq \langle \partial_* \varphi^t(v;z), z\rangle \geq \varphi^t(v;z) \geq \frac{C_4'}{|\alpha'|_{L^1(0,T)}+1}|z|_V^2, \quad \forall t \in [0,T],\ \forall z \in V,\ \forall v \in D_0, \tag{6.2}$$

and from (B4')*, we have that the function* $t \to \partial_* \varphi^t(v;z)$ *is weakly continuous from* $[0,T]$ *into* V^* *for all* $(v,z) \in D_0 \times V$*.*

Note that Assumption (B4') is also a typical time-dependence condition of convex functions (cf. [19, 30, 35], Remark 2.1). In a similar manner to the proof of Lemma 3.1 (cf. [22, Lemma 1]), we have the following:

Lemma 6.2 (cf. Lemma 3.1, [19, Lemma 2.1.1], [22, Lemma 1])**.** *Suppose that Assumption (B') holds. Let* $v \in D_0$ *and* $w \in W^{1,1}(0,T;V)$*. Then, we have:*

$$\frac{d}{dt}\varphi^t(v;w(t)) - \langle \partial_* \varphi^t(v;w(t)), w'(t)\rangle \leq |\alpha'(t)|\varphi^t(v;w(t)), \quad a.a.\ t \in (0,T). \tag{6.3}$$

For each $v \in D_0$, we consider the doubly nonlinear evolution equation, denoted by $(\mathrm{QP})^v$, or $(\mathrm{QP};f,u_0)^v$ when the data are indicated, on $[0,T]$:

$$(\mathrm{QP})^v \begin{cases} \partial_*\psi^t(u'(t)) + \partial_*\varphi^t(v;u(t)) + g(t,u(t)) \ni f(t) \ \text{ in } V^* \ \text{ for a.a. } t \in (0,T), \\ u(0) = u_0 \ \text{ in } V, \end{cases}$$

which is used in the proof of Theorem 6.1 and Section 10.

Definition 6.1. (I) *Given* $v \in D_0$*,* $f \in L^2(0,T;V^*)$*, and* $u_0 \in V$*, the function* $u : [0,T] \to V$ *is called a solution to* $(\mathrm{QP};f,u_0)^v$ *on* $[0,T]$ *if the following conditions are satisfied:*

(i) $u \in W^{1,2}(0,T;V)$.

(ii) *There exists a function* $\xi \in L^2(0,T;V^*)$ *such that*

$$\xi(t) \in \partial_*\psi^t(u'(t)) \quad \text{in } V^* \text{ for a.a. } t \in (0,T),$$

$$\xi(t) + \partial_*\varphi^t(v;u(t)) + g(t,u(t)) = f(t) \quad \text{in } V^* \text{ for a.a. } t \in (0,T).$$

(iii) $u(0) = u_0$ *in* V*.*

(II) *Given* $f \in L^2(0,T;V^*)$ *and* $u_0 \in V$*, the function* $u : [0,T] \to V$ *is called a solution to* (QP)*, or* $(\mathrm{QP};f,u_0)$ *when the data are indicated, on* $[0,T]$ *if* u *is a solution to* $(\mathrm{QP};f,u_0)^v$ *on* $[0,T]$ *with* $v = u$*.*

We now state the existence result for problem (QP) on $[0,T]$.

Theorem 6.1 (cf. [22, Theorem 3]). *Suppose that Assumptions (A), (B'), and (C) are satisfied. Let f be any function in $L^2(0,T;V^*)$ and u_0 be any element in V. Then,* $(\mathrm{QP}; f, u_0)$ *admits at least one solution u on $[0,T]$. Moreover, there exists a constant $N_3 > 0$, independent of f and u_0, such that*

$$\int_0^T \psi^t(u'(t))dt + \sup_{t\in[0,T]} \varphi^t(u;u(t)) \le N_3 \left(|u_0|_V^2 + |f|^2_{L^2(0,T;V^*)} + 1 \right) \tag{6.4}$$

for any solution u to $(\mathrm{QP}; f, u_0)$ *on* $[0,T]$.

Proof. Theorem 6.1 can be proved using a very similar approach to that in [22, Theorem 3]. We repeat it here to clarify the connection between the above theorem and the approach to its optimal control problem (see Theorem 8.1).

By considering approximate problems for (QP), we are going to prove Theorem 6.1. Indeed, for each $\varepsilon \in (0,1]$, we consider the following approximate Cauchy problem for any given $v \in D_0$:

$$(\mathrm{QP})^v_\varepsilon \begin{cases} \varepsilon F u'(t) + \partial_*\psi^t(u'(t)) + \partial_*\varphi^t(v;u(t)) + g(t,u(t)) \ni f(t) \ \text{ in } V^* \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t \in (0,T), \\ u(0) = u_0 \ \text{ in } V. \end{cases} \tag{6.5}$$

Then, $(\mathrm{QP})^v_\varepsilon$ can be considered as $(\mathrm{P}; f, u_0)_\varepsilon$ with $\partial_*\varphi^t(\cdot)$ replaced by $\partial_*\varphi^t(v;\cdot)$. By virtue of Proposition 3.1, problem $(\mathrm{QP})^v_\varepsilon$ possesses one and only one solution u in the same sense as **(I)** of Definition 6.1, and has the estimate

$$\begin{aligned} &\int_0^T \{\varepsilon|u'(t)|_V^2 + \psi^t(u'(t))\}dt + \sup_{t\in[0,T]} \varphi^t(v;u(t)) \\ \le\ & N_0 \left(\varphi^0(v;u_0) + |f|^2_{L^2(0,T;V^*)} + 1 \right). \end{aligned} \tag{6.6}$$

From (6.1) of Lemma 6.1, it follows that

$$\varphi^0(v;u_0) \le \left(|\alpha'|_{L^1(0,T)} + 1 \right) C_3' |u_0|_V^2. \tag{6.7}$$

Now, setting

$$\tilde{N}_3 := N_0 \left(\left(|\alpha'|_{L^1(0,T)} + 1 \right) C_3' |u_0|_V^2 + |f|^2_{L^2(0,T;V^*)} + 1 \right)$$

and

$$X(u_0) := \left\{ v \in W^{1,2}(0,T;V) \ \middle|\ v(0) = u_0, \quad \int_0^T \psi^t(v'(t))dt \le \tilde{N}_3 \right\},$$

we define a mapping $\mathcal{S} : X(u_0) \to X(u_0)$ that maps $v \in X(u_0) \subset D_0$ to a unique solution u of (6.5), namely, $\mathcal{S}v = u$; from (6.6), note that $u \in X(u_0)$. Clearly, $X(u_0)$ is non-empty, convex, and compact in $C([0,T];H)$.

Next, we show that $\mathcal{S}$ is continuous in $X(u_0)$ with respect to the topology of $C([0,T];H)$. Let $v \in C([0,T];H)$, and let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in $X(u_0)$ such that

$$v_n \to v \text{ in } C([0,T];H) \text{ as } n \to \infty, \tag{6.8}$$

and set $u_n = \mathcal{S}v_n$. Then, we see that $v \in X(u_0)$, $v_n \to v$ weakly in $W^{1,2}(0,T;V)$, and $\sup_{n\in\mathbb{N}} \int_0^T \psi^t(v_n'(t))dt \le \tilde{N}_3$. From (6.6) and (6.7), it follows that there exist a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, denoted by $\{u_n\}_{n\in\mathbb{N}}$ again, and a function $u \in W^{1,2}(0,T;V)$ such that

$$u_n \to u \text{ in } C([0,T];H), \text{ weakly in } W^{1,2}(0,T;V) \quad \text{as } n \to \infty \tag{6.9}$$

and

$$u_n(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0,T] \text{ as } n \to \infty.$$

We now show that $u_n \to u$ in $C([0,T];V)$ as $n \to \infty$. To this end, note that $u_n (= \mathcal{S}v_n)$ is a unique solution to the following equation with $u_n(0) = u_0$ in V:

$$\varepsilon F u_n'(t) + \partial_* \psi^t(u_n'(t)) + \partial_* \varphi^t(v_n; u_n(t)) + g(t, u_n(t)) \ni f(t) \text{ in } V^* \quad \text{for a.a. } t \in (0,T), \tag{6.10}$$

As for (3.29) in the proof of Theorem 2.1, we obtain the following by multiplying (6.10) by $u_n'(s) - u'(s)$ for $t = s$ and using (2.1) and Lemma 6.2:

$$\begin{aligned} &\varepsilon C_F |u_n'(s) - u'(s)|_V^2 + \frac{d}{ds}\varphi^s(v_n; u_n(s) - u(s)) \\ \le\ & |\alpha'(s)|\varphi^s(v_n; u_n(s) - u(s)) + \bar{L}_n(s) + \psi^s(u'(s)) - \psi^s(u_n'(s)) \\ & \qquad \text{for a.a. } s \in (0,T), \end{aligned} \tag{6.11}$$

where

$$\begin{aligned} \bar{L}_n(s) = {} & \langle f(s) - \partial_* \varphi^s(v_n; u(s)) - g(s, u_n(s)), u_n'(s) - u'(s) \rangle \\ & - \varepsilon \langle F u'(s), u_n'(s) - u'(s) \rangle \qquad \text{for a.a. } s \in (0,T). \end{aligned}$$

Multiplying (6.11) by $e^{-\int_0^s |\alpha'(\tau)|d\tau}$ and integrating in time, we use $\varphi^0(v_n; 0) = 0$ (cf. (6.1)) to obtain:

$$\begin{aligned} & \varepsilon C_F \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} |u_n'(s) - u'(s)|_V^2 ds + e^{-\int_0^t |\alpha'(\tau)|d\tau} \varphi^t(v_n; u_n(t) - u(t)) \\ \le\ & \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} \bar{L}_n(s) ds + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} (\psi^s(u'(s)) + C_2) ds \\ & - \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} (\psi^s(u_n'(s)) + C_2) ds \end{aligned} \tag{6.12}$$

for all $t \in [0,T]$.

Here, note from (B1'), (B2'), (6.8), and the Lebesgue dominated convergence theorem that

$$\partial_* \varphi^{(\cdot)}(v_n; u) \to \partial_* \varphi^{(\cdot)}(v; u) \text{ in } L^2(0,T;V^*) \text{ as } n \to \infty. \tag{6.13}$$

As in the case of (3.33)–(3.35), we infer from (6.12) with (C1), (6.9), (6.13), and the weak lower semicontinuity of $\tilde{\Psi}^t$ given by (3.32) that

$$\limsup_{n\to\infty} e^{-\int_0^t |\alpha'(\tau)|d\tau} \varphi^t(v_n; u_n(t) - u(t)) \le 0 \quad \text{uniformly in } t \in [0,T] \tag{6.14}$$

and

$$\limsup_{n\to\infty} \varepsilon C_F \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau} |u_n'(s) - u'(s)|_V^2 ds \le 0, \quad \forall t \in [0,T]. \tag{6.15}$$

Hence, we observe from (6.14) that

$$\limsup_{n\to\infty} \varphi^t(v_n; u_n(t) - u(t)) \le 0 \quad \text{uniformly in } t \in [0,T],$$

which implies by (6.1) that

$$u_n \to u \ \text{ in } C([0,T];V) \text{ as } n \to \infty. \tag{6.16}$$

Additionally, we infer from (6.9) and (6.15) that

$$u_n' \to u' \text{ in } L^2(0,T;V) \text{ as } n \to \infty. \tag{6.17}$$

We now show that u is a solution to $(\mathrm{QP})_\varepsilon^v$ on $[0,T]$, namely, $u = \mathcal{S}v$.

From (B1'), (B2'), and (6.16), we see that

$$\partial_* \varphi^t(v_n; u_n(t)) \to \partial_* \varphi^t(v; u(t)) \ \text{ in } V^* \text{ for all } t \in [0,T] \text{ as } n \to \infty.$$

Therefore, from the Lebesgue dominated convergence theorem, it follows that

$$\partial_* \varphi^{(\cdot)}(v_n; u_n(\cdot)) \to \partial_* \varphi^{(\cdot)}(v; u(\cdot)) \ \text{ in } L^2(0,T;V^*) \text{ as } n \to \infty. \tag{6.18}$$

Additionally, we observe from (6.17) that

$$\varepsilon F u_n' \to \varepsilon F u' \ \text{ in } L^2(0,T;V^*) \text{ as } n \to \infty. \tag{6.19}$$

Hence, it follows from (C2), (6.16), and (6.18) that

$$\begin{aligned} \varepsilon F u_n' + \partial_* \psi^{(\cdot)}(u_n') \ni \xi_n &:= f - \partial_* \varphi^{(\cdot)}(v_n; u_n) - g(\cdot, u_n) \\ &\to f - \partial_* \varphi^{(\cdot)}(v; u) - g(\cdot, u) =: \xi \text{ in } L^2(0,T;V^*) \end{aligned}$$

as $n \to \infty$. Thus, from (6.17), (6.19), and the demi-closedness of the maximal monotone operator $\partial_* \psi^{(\cdot)}$ in $L^2(0,T;V^*)$, we infer that

$$\xi \in \varepsilon F u' + \partial_* \psi^{(\cdot)}(u') \ \text{ in } L^2(0,T;V^*),$$

or, equivalently,

$$\xi(t) \in \varepsilon F u'(t) + \partial_* \psi^t(u'(t)) \text{ in } V^* \text{ for a.a. } t \in (0,T).$$

Additionally, we observe from (6.16) that $u(0) = u_0$ in V. Therefore, we conclude that u is a solution to $(\mathrm{QP})_\varepsilon^v$ on $[0,T]$, namely, $u = \mathcal{S}v$. From the uniqueness of solutions to $(\mathrm{QP})_\varepsilon^v$, we conclude that $\mathcal{S}v_n = u_n \to u = \mathcal{S}v$ in $C([0,T];V)$, and hence in $C([0,T];H)$, without extracting any subsequence from $\{u_n\}_{n\in\mathbb{N}}$. Thus, $\mathcal{S}$ is continuous in $X(u_0)$ with respect to the topology of $C([0,T];H)$. Therefore, by the Schauder fixed point theorem, $\mathcal{S}$ has at least one fixed point u in $X(u_0)$. This is a solution to $(\mathrm{QP})_\varepsilon^v$ with $v = u$.

We showed above that, for every small $\varepsilon > 0$, the Cauchy problem

$$(\mathrm{QP}; f, u_0)_\varepsilon \begin{cases} \varepsilon F u_\varepsilon'(t) + \partial_* \psi^t(u_\varepsilon'(t)) + \partial_* \varphi^t(u_\varepsilon; u_\varepsilon(t)) + g(t, u_\varepsilon(t)) \ni f(t) \text{ in } V^* \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t \in (0,T), \\ u_\varepsilon(0) = u_0 \ \text{ in } V \end{cases}$$

admits at least one solution $u_\varepsilon \in W^{1,2}(0,T;V)$ in the same sense as **(II)** in Definition 6.1, and has the estimate

$$\varepsilon \int_0^T |u_\varepsilon'(t)|_V^2 dt + \int_0^T \psi^t(u_\varepsilon'(t))dt + \sup_{t\in[0,T]} \varphi^t(u_\varepsilon; u_\varepsilon(t)) \le \tilde{N}_3, \quad \forall \varepsilon \in (0,1]. \tag{6.20}$$

Therefore, we can choose a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \downarrow 0$ (as $n \to \infty$) and a function $u \in D_0$ so that

$$\begin{gathered} u_n := u_{\varepsilon_n} \to u \text{ in } C([0,T];H), \text{ weakly in } W^{1,2}(0,T;V) \text{ as } n \to \infty, \\ u_n(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0,T] \text{ as } n \to \infty, \\ \varepsilon_n u_n' \to 0 \text{ in } L^2(0,T;V) \text{ as } n \to \infty, \\ \sup_{n\in\mathbb{N}} \int_0^T \psi^t(u_n'(t))dt \le \tilde{N}_3. \end{gathered}$$

In a similar manner to the case of (6.16), we have:

$$u_n := u_{\varepsilon_n} \to u \text{ in } C([0,T];V) \text{ as } n \to \infty.$$

Therefore, in the same way as for the proof of convergence for Theorem 2.1, we can infer from (B1'), (B2'), and (C1) that the limit u satisfies

$$\begin{cases} \partial_* \psi^t(u'(t)) + \partial_* \varphi^t(u; u(t)) + g(t, u(t)) \ni f(t) \ \text{ in } V^* \text{ for a.a. } t \in (0,T), \\ u(0) = u_0 \ \text{ in } V. \end{cases}$$

Hence, u is a required solution to (QP;f, u_0) on $[0,T]$. In addition, in the same way as in the proof of the a priori estimate (3.18) (cf. (2.8)), we conclude that (6.4) holds by setting $N_3 := \tilde{N}_3$. Moreover, estimate (6.4) holds for any solution to (QP;f, u_0) on $[0,T]$; this is easily seen, as in the proof of Theorem 2.1. Thus, the proof of Theorem 6.1 is complete. □

7 Uniqueness of solutions to (QP)

In this section, we show that a solution to (QP;f, u_0) on $[0,T]$ is not, in general, unique. Indeed, we give a counterexample for the uniqueness of solutions to (QP;f, u_0) as follows.

Example 7.1 (cf. [11, Section 2], [22, Example 4.1])**.** Let $\Omega = (0,1)$, and set $V = H^1(\Omega)$ and $H = L^2(\Omega)$. Additionally, let $Q := (0,T) \times \Omega$, and let ρ be a prescribed obstacle function in $C(\overline{Q})$ such that

$$1 \le \rho(t,x) \le \rho^*, \quad \forall (t,x) \in \overline{Q}, \tag{7.1}$$

where ρ^* is a positive constant.

Now, for each $t \in [0,T]$, define a closed convex subset $K(t)$ of V by

$$K(t) := \{z \in V \ ; \ |z(x)| \le \rho(t,x), \ |z_x(x)| \le \rho(t,x) \ \text{ for a.a. } x \in \Omega\} .$$

Then, we consider the following quasi-variational problem with constraint:

$$\begin{cases} u_t(t) \in K(t) \text{ for a.a. } t \in (0,T), \\ \displaystyle\int_\Omega a(t,x,u(t,x))u_x(t,x)(u_{xt}(t,x) - w_x(x))dx \le 0 \\ \quad \text{for all } w \in K(t) \text{ and a.a. } t \in (0,T), \\ u(0,x) = 0, \quad x \in \Omega, \end{cases} \tag{7.2}$$

where $0 < T < +\infty$, and $a(t,x,r)$ is a prescribed function on $\overline{Q} \times \mathbb{R}$ satisfying the following conditions:

$$\begin{cases} a_* \le a(t,x,r) \le a^*, \quad \forall (t,x) \in \overline{Q}, \ \forall r \in \mathbb{R}, \\ |a(t_1,x,r_1) - a(t_2,x,r_2)| \le L_a(|t_1 - t_2| + |r_1 - r_2|), \\ \qquad \forall t_i \in [0,T], \ r_i \in \mathbb{R}, \ i = 1,2, \ \forall x \in \overline{\Omega}, \end{cases} \tag{7.3}$$

where a_*, a^* and L_a are positive constants.

Here, for each $t \in [0,T]$, the time-dependent convex functional ψ^t is defined by

$$\psi^t(z) := I_{K(t)}(z) = \begin{cases} 0, & \text{if } z \in K(t), \\ +\infty, & \text{otherwise}, \end{cases} \qquad \forall z \in V. \tag{7.4}$$

Furthermore, the (t,v)-dependent functional $\varphi^t(v;z)$ is given by

$$\varphi^t(v;z) := \frac{1}{2}\int_\Omega a(t,x,v(t,x))|z_x(x)|^2 dx + \frac{1}{2}\int_\Omega |z(x)|^2 dx \\ \text{for all } t \in [0,T], \ \forall v \in D_0, \ \forall z \in V, \tag{7.5}$$

where

$$D_0 = \left\{ v \in W^{1,2}(0,T;V) \mid v'(t) \in K(t) \text{ for a.a. } t \in [0,T] \right\} .$$

Then, we have (cf. [22, Section 6]):

(i) $z^* \in \partial_* \psi^t(z)$ if and only if

$$z^* \in V^*, \ z \in K(t) \text{ and } \langle z^*, w - z \rangle \le 0 \text{ for all } w \in K(t),$$

(ii) $\langle \partial_* \varphi^t(v;z), w \rangle = \displaystyle\int_\Omega a(t,x,v(t,x))z_x(x)w_x(x)dx + \int_\Omega z(x)w(x)dx$ for all $z, w \in V$ and $v \in D_0$

for all $t \in [0,T]$.

Additionally, we observe that problem (7.2) can be reformulated as (QP;0,0) with $g(t,z) = -z$. Using similar arguments to those in [22, Section 6], it is easy to check

Assumptions (A), *(B')*, and *(C)*. Therefore, applying Theorem 6.1, problem (7.2) has at least one solution u on $[0,T]$.

Moreover, note that, for each constant $c \in (0,1)$, the function u_c defined by

$$u_c(t,x) := c(1-\exp(-t)) \quad \text{for all } (t,x) \in (0,T)\times\Omega$$

is a solution to (7.2) on $[0,T]$. Indeed, we observe that

$$(u_c)_t(t,x) = c\exp(-t), \quad (u_c)_x(t,x) = 0, \quad (u_c)_{xt}(t,x) = 0$$

for all $(t,x) \in (0,T)\times\Omega$. Therefore,

$$(u_c)_t(t) \in K(t) \text{ for a.a. } t \in (0,T).$$

Hence, we easily observe that, for each $c \in (0,1)$, the function u_c satisfies (7.2). Thus, $\{u_c\}_{c\in(0,1)}$ provides an infinite family of solutions to (7.2) on $[0,T]$.

From the counterexample above, note that the uniqueness of solutions to (QP) is not generally expected. However, if $\partial_*\psi^t$ is strictly monotone from V into V^* and $\partial_*\varphi^t(v;\cdot)$ is Lipschitz in $v \in D_0$, we have the following uniqueness result for (QP).

Theorem 7.1 (cf. Theorem 2.2, [22, Theorem 2]). *Suppose that Assumptions (A), (B'), and (C) are satisfied. Let f be any function in $L^2(0,T;V^*)$ and u_0 be any element in V. In addition, assume the strict monotonicity condition* (A4) *of $\partial_*\psi^t$ in Theorem 2.2. Furthermore, assume that $\partial_*\varphi^t(v;\cdot)$ is Lipschitz in $v \in D_0$, i.e.,*

(B5') *There exists a positive constant $C_6 > 0$ such that*

$$|\partial_*\varphi^t(v_1;z) - \partial_*\varphi^t(v_2;z))|_{V^*} \le C_6|v_1(t)-v_2(t)|_V\,(1+|z|_V)\,,$$
$$\forall v_i \in D_0\ (i=1,2),\ \forall z \in D_0,\ \forall t \in [0,T].$$

Then, the solution to $(\mathrm{QP}; f, u_0)$ *on* $[0,T]$ *is unique.*

Proof. Using a quite standard argument (cf. [22, Theorem 2]), we prove Theorem 7.1. To this end, let u_i, $i=1,2$, be two solutions to $(\mathrm{QP}; f, u_0)$ on $[0,T]$. Then, by Theorem 6.1 (cf. (6.4)), we have $u_i \in W^{1,2}(0,T;V)$ and $u_i \in D_0$ for $i=1,2$.

Subtract $(\mathrm{QP}; f, u_0)$ for $i=2$ from that for $i=1$, and multiply the result by $u_1'-u_2'$. Then:

$$\begin{aligned}\langle \xi_1(t)-\xi_2(t), u_1'(t)-u_2'(t)\rangle + \langle \partial_*\varphi^t(u_1;u_1(t)) - \partial_*\varphi^t(u_2;u_2(t)), u_1'(t)-u_2'(t)\rangle \\ +\langle g(t,u_1(t)) - g(t,u_2(t)), u_1'(t)-u_2'(t)\rangle = 0 \quad \text{for a.a. } t\in(0,T),\end{aligned} \tag{7.6}$$

where $\xi_i(t) \in \partial_*\psi^t(u_i'(t))$ for a.a. $t \in (0,T)$ $(i=1,2)$. From (A4), we observe that

$$\langle \xi_1(t)-\xi_2(t), u_1'(t)-u_2'(t)\rangle \ge C_5|u_1'(t)-u_2'(t)|_V^2 \text{ for a.a. } t\in(0,T) \tag{7.7}$$

and, by Lemma 6.2 and (B5'), that

$$\begin{aligned}
& \langle \partial_* \varphi^t(u_1; u_1(t)) - \partial_* \varphi^t(u_2; u_2(t)), u_1'(t) - u_2'(t) \rangle \\
= \; & \langle \partial_* \varphi^t(u_1; u_1(t)) - \partial_* \varphi^t(u_2; u_1(t)), u_1'(t) - u_2'(t) \rangle \\
& \quad + \langle \partial_* \varphi^t(u_2; u_1(t) - u_2(t)), u_1'(t) - u_2'(t) \rangle \\
\geq \; & -|\partial_* \varphi^t(u_1; u_1(t)) - \partial_* \varphi^t(u_2; u_1(t))|_{V^*} |u_1'(t) - u_2'(t)|_V \\
& \quad + \frac{d}{dt} \varphi^t(u_2; u_1(t) - u_2(t)) - |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) \\
\geq \; & -C_6 |u_1(t) - u_2(t)|_V \left(1 + |u_1(t)|_V\right) |u_1'(t) - u_2'(t)|_V \\
& \quad + \frac{d}{dt} \varphi^t(u_2; u_1(t) - u_2(t)) - |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) \quad \text{for a.a.} t \in (0, T).
\end{aligned} \tag{7.8}$$

Therefore, we observe from (7.6)–(7.8) and (C2) with the help of the Schwarz inequality that

$$\begin{aligned}
& C_5 |u_1'(t) - u_2'(t)|_V^2 + \frac{d}{dt} \varphi^t(u_2; u_1(t) - u_2(t)) \\
\leq & |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) + C_6 |u_1(t) - u_2(t)|_V \left(1 + |u_1(t)|_V\right) |u_1'(t) - u_2'(t)|_V \\
& \quad + |g(t, u_1(t)) - g(t, u_2(t))|_{V^*} |u_1'(t) - u_2'(t)|_V \\
\leq & |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) + \frac{C_6^2}{C_5} |u_1(t) - u_2(t)|_V^2 \left(1 + |u_1(t)|_V\right)^2 + \frac{C_5}{4} |u_1'(t) - u_2'(t)|_V^2 \\
& \quad + \frac{1}{C_5} |g(t, u_1(t)) - g(t, u_2(t))|_{V^*}^2 + \frac{C_5}{4} |u_1'(t) - u_2'(t)|_V^2 \\
\leq & |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) + \frac{C_6^2}{C_5} |u_1(t) - u_2(t)|_V^2 \left(1 + |u_1(t)|_V\right)^2 \\
& \quad + \frac{L_g^2}{C_5} |u_1(t) - u_2(t)|_V^2 + \frac{C_5}{2} |u_1'(t) - u_2'(t)|_V^2
\end{aligned}$$

for a.a. $t \in (0, T)$. From the above inequality with (6.1), we infer that

$$\begin{aligned}
& \frac{C_5}{2} |u_1'(t) - u_2'(t)|_V^2 + \frac{d}{dt} \varphi^t(u_2; u_1(t) - u_2(t)) \\
\leq \; & K_1 (|\alpha'(t)| + |u_1(t)|_V^2 + 1) \varphi^t(u_2; u_1(t) - u_2(t)) \quad \text{for a.a. } t \in (0, T)
\end{aligned} \tag{7.9}$$

for some constant $K_1 > 0$, which is independent of u_i ($i = 1, 2$). Hence, applying the Gronwall inequality to (7.9), we conclude that

$$u_1(t) - u_2(t) = 0 \text{ in } V \text{ for all } t \in [0, T].$$

Thus, the proof of Theorem 7.1 is complete. □

Remark 7.1. *In [22, Example 4.1], the authors showed a counterexample for the uniqueness of solutions to the following type of doubly nonlinear evolution equations:*

$$\partial_* \psi^t(u'(t)) + \partial_* \varphi^t(u(t)) + g(t, u(t)) \ni f(t) \text{ in } V^* \text{ for a.a. } t \in (0, T). \tag{7.10}$$

Additionally, the uniqueness of solutions to (7.10) *was proved there under the additional condition* (A4) *regarding the strict monotonicity of* $\partial_* \psi^t$ *(cf. Theorem 2.2, [22, Theorem 2]).*

Remark 7.2. *As Example 7.1 demonstrates, for doubly nonlinear variational inequalities, the uniqueness of solutions is quite independent of its regularity in time. It seems to depend on the structure of the problem, and we need some strong conditions such as* (A4) *and* (B5') *to ensure unique solvability.*

8 Singular optimal control problem for (QP)

In Theorem 7.1, we derived the uniqueness of solutions to state system (QP) under additional assumptions (A4) and (B5'). However, Assumption (A4) cannot be expected in many of the applications we shall treat in Section 11. Therefore, in this section, without this assumption, we consider the singular optimal control problem that is a class of control problems formulated for the non-well-posed state system (QP).

Using arguments similar to those in Sections 4 and 5, we study the singular optimal control problem for quasi-variational evolution equations (QP). Indeed, for a fixed number $M > 0$, we consider the control space $\mathcal{F}_M$ defined by (1.3) to give the following singular optimal control problem for (QP), denoted by $(\mathrm{OP})_{\mathrm{QV}}$:

Problem $(\mathbf{OP})_{\mathrm{QV}}$: Find a control $f^* \in \mathcal{F}_M$, called an optimal control, such that

$$J_{\mathrm{QV}}(f^*) = \inf_{f \in \mathcal{F}_M} J_{\mathrm{QV}}(f).$$

Here, $J_{\mathrm{QV}}(f)$ is the cost functional defined by

$$J_{\mathrm{QV}}(f) := \inf_{u \in \mathcal{S}_{\mathrm{QV}}(f)} \pi_{\mathrm{QV},f}(u), \tag{8.1}$$

where $f \in \mathcal{F}_M$ is any control, and $\mathcal{S}_{\mathrm{QV}}(f)$ is the set of all solutions to $(\mathrm{QP}; f, u_0)$ associated with the control function f. In addition, for any solution u to the state system $(\mathrm{QP}; f, u_0)$, its functional $\pi_{\mathrm{QV},f}(u)$ is defined by

$$\pi_{\mathrm{QV},f}(u) := \frac{1}{2}\int_0^T |u(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f(t)|_{V^*}^2 dt, \tag{8.2}$$

where $u_{ad} \in L^2(0,T;V)$ is a given target profile.

Using a quite standard argument (cf. Proof of Theorem 4.1), we can show the existence of optimal control $f^* \in \mathcal{F}_M$ to Problem $(\mathrm{OP})_{\mathrm{QV}}$. Indeed, the following result on the convergence of solutions to $(\mathrm{QP}; f, u_0)$ is the key to the proof.

Proposition 8.1. *Suppose that Assumptions (A), (B'), and (C) are satisfied. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V^*)$ and $\{u_{0,n}\}_{n\in\mathbb{N}} \subset V$. Additionally, let $f \in L^2(0,T;V^*)$ and $u_0 \in V$. Assume that*

$$f_n \to f \ \text{in}\ L^2(0,T;V^*), \tag{8.3}$$

$$u_{0,n} \to u_0 \ \text{in}\ V \tag{8.4}$$

as $n \to \infty$. Let u_n be a solution to $(\mathrm{QP}; f_n, u_{0,n})$ on $[0,T]$. Then, there exist a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ and a function $u \in W^{1,2}(0,T;V)$ such that u is a solution to $(\mathrm{QP}; f, u_0)$ on $[0,T]$ and

$$u_{n_k} \to u \ \text{in}\ C([0,T];V) \quad \text{as}\ k \to \infty. \tag{8.5}$$

Proof. As u_n is a solution to (QP;$f_n, u_{0,n}$) on $[0, T]$, by the definition of solutions, there is a function $\xi_n \in L^2(0, T; V^*)$ such that

$$\xi_n(t) \in \partial_* \psi^t(u_n'(t)) \text{ in } V^* \text{ for a.a. } t \in (0, T), \tag{8.6}$$

$$\xi_n(t) + \partial_* \varphi^t(u_n; u_n(t)) + g(t, u_n(t)) = f_n(t) \text{ in } V^* \text{ for a.a. } t \in (0, T). \tag{8.7}$$

Note from (8.3) that $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0, T; V^*)$. In addition, note from (8.4) that $\{u_{0,n}\}_{n\in\mathbb{N}}$ is bounded in V. Therefore, from (A2), (6.1), (6.4), and the Ascoli–Arzelà theorem, it follows that there exist a sequence $\{n_k\}_{k\in\mathbb{N}}$ with $n_k \to \infty$ (as $k \to \infty$) and a function $u \in W^{1,2}(0, T; V)$ such that

$$\left.\begin{array}{ll} u_{n_k} \to u & \text{weakly in } W^{1,2}(0, T; V), \text{ in } C([0, T]; H) \\ & \text{and weakly-}* \text{ in } L^\infty(0, T; V) \text{ as } n \to \infty, \end{array}\right\} \tag{8.8}$$

$$u_{n_k}(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0, T] \text{ as } n \to \infty, \tag{8.9}$$

$$\int_0^t \psi^\tau(u'(\tau))d\tau \le \liminf_{k\to\infty} \int_0^t \psi^\tau(u'_{n_k}(\tau))d\tau \le \check{N}_3 \text{ for all } t \in [0, T],$$

where $\check{N}_3 := N_3 \left(|u_0|_V^2 + |f|^2_{L^2(0,T;V^*)} + 1\right)$ is the same constant as in (6.4).

Next, we show that $u_{n_k} \to u$ in $C([0, T]; V)$ as $k \to \infty$. As with (6.11) in the proof of Theorem 6.1, we multiply (QP;f_{n_k}, u_{0,n_k}) (cf. (8.7)) for $t = s$ by $u'_{n_k}(s) - u'(s)$ and use Lemma 6.2 to obtain:

$$\begin{aligned} & \frac{d}{ds}\varphi^s(u_{n_k}; u_{n_k}(s) - u(s)) \\ \le \ & |\alpha'(s)|\varphi^s(u_{n_k}; u_{n_k}(s) - u(s)) + \tilde{L}_{n_k}(s) + \psi^s(u'(s)) - \psi^s(u'_{n_k}(s)) \\ & \qquad \text{for a.a. } s \in (0, T), \end{aligned} \tag{8.10}$$

where $\tilde{L}_{n_k}(\cdot)$ is a function defined by:

$$\tilde{L}_{n_k}(s) := \langle f_{n_k}(s) - \partial_*\varphi^s(u_{n_k}; u(s)) - g(t, u_{n_k}(s)), u'_{n_k}(s) - u'(s)\rangle \text{ for a.a. } s \in (0, T).$$

Multiplying (8.10) by $e^{-\int_0^s |\alpha'(\tau)|d\tau}$ and integrating in time, we obtain:

$$\begin{aligned} & e^{-\int_0^t |\alpha'(\tau)|d\tau}\varphi^t(u_{n_k}; u_{n_k}(t) - u(t)) \\ \le \ & \varphi^0(u_{n_k}; u_{0,n_k} - u_0) + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}\tilde{L}_{n_k}(s)ds \\ & + \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}(\psi^s(u'(s)) + C_2)ds \\ & - \int_0^t e^{-\int_0^s |\alpha'(\tau)|d\tau}(\psi^s(u'_{n_k}(s)) + C_2)ds \end{aligned} \tag{8.11}$$

for all $t \in [0, T]$.

Here, note from (B1'), (B2'), (8.8), and the Lebesgue dominated convergence theorem that

$$\partial_*\varphi^{(\cdot)}(u_{n_k}; u) \to \partial_*\varphi^{(\cdot)}(u; u) \text{ in } L^2(0, T; V^*) \text{ as } k \to \infty. \tag{8.12}$$

In a similar way to the case of (3.33)–(3.35), we infer from (8.11) with (C1), (6.1), (8.3), (8.4), (8.8), (8.12), and the weak lower semicontinuity of $\tilde{\Psi}^t$ given by (3.32) that

$$\limsup_{k\to\infty} e^{-\int_0^t |\alpha'(\tau)|d\tau} \varphi^t(u_{n_k}; u_{n_k}(t) - u(t)) \le 0 \quad \text{uniformly in } t \in [0,T],$$

whence

$$\limsup_{k\to\infty} \varphi^t(u_{n_k}; u_{n_k}(t) - u(t)) \le 0 \quad \text{uniformly in } t \in [0,T],$$

which implies by (6.1) that

$$u_{n_k} \to u \ \text{ in } C([0,T];V) \text{ as } k \to \infty. \tag{8.13}$$

We now show that u is a solution to (QP;f, u_0) on $[0,T]$. We observe from (B1'), (B2'), and (8.13) that

$$\partial_*\varphi^t(u_{n_k}; u_{n_k}(t)) \to \partial_*\varphi^t(u; u(t)) \ \text{ in } V^* \text{ for all } t \in [0,T] \text{ as } k \to \infty.$$

Therefore, from the Lebesgue dominated convergence theorem, it follows that

$$\partial_*\varphi^{(\cdot)}(u_{n_k}; u_{n_k}) \to \partial_*\varphi^{(\cdot)}(u; u) \text{ in } L^2(0,T;V^*) \text{ as } k \to \infty. \tag{8.14}$$

From (C2), (8.3), (8.7), (8.13), and (8.14), we observe that

$$\begin{aligned}\partial_*\psi^{(\cdot)}(u'_{n_k}) \ni \xi_{n_k} &= f_{n_k} - \partial_*\varphi^{(\cdot)}(u_{n_k}; u_{n_k}) - g(\cdot, u_{n_k}(\cdot)) \\ &\to f - \partial_*\varphi^{(\cdot)}(u; u) - g(\cdot, u) =: \xi \text{ in } L^2(0,T;V^*)\end{aligned}$$

as $k \to \infty$. Thus, from (8.8) and the demi-closedness of the maximal monotone operator $\partial_*\psi^{(\cdot)}$ in $L^2(0,T;V^*)$, we infer that

$$\xi \in \partial_*\psi^{(\cdot)}(u') \ \text{ in } L^2(0,T;V^*),$$

or, equivalently,

$$\xi(t) \in \partial_*\psi^t(u'(t)) \text{ in } V^* \text{ for a.a. } t \in (0,T)$$

and

$$\xi(t) + \partial_*\varphi^t(u; u(t)) + g(t, u(t)) = f(t) \text{ in } V^* \text{ for a.a. } t \in (0,T).$$

Additionally, by (8.4) and (8.13), we see $u(0) = u_0$ in V. Therefore, we conclude that u is a solution to (QP; f, u_0) on $[0,T]$. Thus, the proof of Proposition 8.1 is complete. □

We now state the main result of this section, which is directed to the existence of an optimal control for $(\mathrm{OP})_{\mathrm{QV}}$ without the uniqueness of solutions to (QP; f, u_0).

Theorem 8.1. *Suppose that Assumptions (A), (B'), and (C) are satisfied. Let u_{ad} be a given function in $L^2(0,T;V)$ and u_0 be any element in V. Then, $(\mathrm{OP})_{\mathrm{QV}}$ has at least one optimal control $f^* \in \mathcal{F}_M$, namely,*

$$J_{\mathrm{QV}}(f^*) = \inf_{f\in\mathcal{F}_M} J_{\mathrm{QV}}(f),$$

where $J_{\mathrm{QV}}(\cdot)$ is the cost functional of $(\mathrm{OP})_{\mathrm{QV}}$, which is defined by (8.1) and (8.2).

Taking account of Proposition 8.1 concerning the convergence result of solutions to (QP;f, u_0), we can prove Theorem 8.1. Indeed, the proof of Theorem 8.1 is the same as that of Theorem 4.1. Thus, we omit the detailed proof.

9 Approximation for (QP) and $(\mathrm{OP})_{\mathrm{QV}}$

In Section 5, we established an approximate procedure to investigate the singular optimal control problem (OP). In this section, we study similar approximate problems for (QP) and $(\mathrm{OP})_{\mathrm{QV}}$ from the viewpoint of numerical analysis. To this end, we assume *(A)*, *(B')*, *(C)*, and (B5') in Theorem 7.1. However, (A4) is not assumed. In this case, $(\mathrm{QP}; f, u_0)$ is generally a non-well-posed state system, even if *(A)*, *(B')*, *(C)*, and (B5') are satisfied.

We begin with the setting of an approximation to $(\mathrm{QP}; f, u_0)$. Indeed, for each $\varepsilon \in (0,1]$, $f \in L^2(0,T;V^*)$, $h \in L^2(0,T;V)$, and $u_0 \in V$, we consider the following problem, denoted by $(\mathrm{QP}; f+\varepsilon Fh, u_0)_\varepsilon$:

$$(\mathrm{QP}; f+\varepsilon Fh, u_0)_\varepsilon \left\{ \begin{array}{l} \varepsilon F u_\varepsilon'(t) + \partial_* \psi^t(u_\varepsilon'(t)) + \partial_* \varphi^t(u_\varepsilon; u_\varepsilon(t)) + g(t, u_\varepsilon(t)) \\ \qquad\qquad \ni f(t) + \varepsilon F h(t) \text{ in } V^* \text{ for a.a. } t \in (0,T), \\ u_\varepsilon(0) = u_0 \text{ in } V, \end{array} \right. \tag{9.1}$$

where $F : V \to V^*$ is the duality mapping.

The existence–uniqueness of solutions to $(\mathrm{QP}; f+\varepsilon Fh, u_0)_\varepsilon$ is proved using a slight modification of the proof of Theorem 6.1 for each $\varepsilon \in (0,1]$, $f \in L^2(0,T;V^*)$, $h \in L^2(0,T;V)$, and $u_0 \in V$. In fact, we have:

Proposition 9.1 (cf. Theorem 6.1, [22, Theorem 3])**.** *Suppose that Assumptions (A), (B'), (C), and* (B5') *hold. Then, for each* $\varepsilon \in (0,1]$, $f \in L^2(0,T;V^*)$, $h \in L^2(0,T;V)$, *and* $u_0 \in V$, *there exists a unique solution* u_ε *to* $(\mathrm{QP}; f+\varepsilon Fh, u_0)_\varepsilon$ *on* $[0,T]$ *satisfying* $u_\varepsilon(0) = u_0$ *in* V *and:*

(•) *There exists a function* $\xi_\varepsilon \in L^2(0,T;V^*)$ *such that*

$$\xi_\varepsilon(t) \in \partial_* \psi^t(u_\varepsilon'(t)) \text{ in } V^* \text{ for a.a. } t \in (0,T),$$

$$\varepsilon F u_\varepsilon'(t) + \xi_\varepsilon(t) + \partial_* \varphi^t(u_\varepsilon; u_\varepsilon(t)) + g(t, u_\varepsilon(t)) = f(t) + \varepsilon F h(t) \text{ in } V^* \text{ for a.a. } t \in (0,T).$$

Moreover, there exists a positive constant $N_4 > 0$, *independent of* ε, f, h, *and* u_0, *such that*

$$\begin{aligned} & \int_0^T \psi^t(u_\varepsilon'(t))dt + \sup_{t\in[0,T]} \varphi^t(u_\varepsilon; u_\varepsilon(t)) \\ \leq\ & N_4 \left(|u_0|_V^2 + |f|^2_{L^2(0,T;V^*)} + |h|^2_{L^2(0,T;V)} + 1 \right). \end{aligned} \tag{9.2}$$

Proof. As the proof is quite similar to those of Theorem 6.1 and Theorem 7.1, we omit the detailed proof of Proposition 9.1. □

We now state the first result of this section, which is concerned with the relationship between $(\mathrm{QP}; f, u_0)$ and $(\mathrm{QP}; f+\varepsilon Fh, u_0)_\varepsilon$.

Theorem 9.1 (cf. Theorem 5.1)**.** *Suppose that Assumptions (A), (B'), (C), and* (B5') *hold. Let* $f \in L^2(0,T;V^*)$ *and* $u_0 \in V$. *Then, we have:*

(i) *Let* $\varepsilon \in (0,1]$ *and let* $\{h_\varepsilon\}_{\varepsilon\in(0,1]}$ *be a bounded set in* $L^2(0,T;V)$. *Additionally, let* u_ε *be a unique solution to* $(\mathrm{QP}; f+\varepsilon F h_\varepsilon, u_0)_\varepsilon$ *on* $[0,T]$. *Then, there exist a sequence* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$ *with* $\varepsilon_n \to 0$ *(as* $n\to\infty$*) and a function* $u \in W^{1,2}(0,T;V)$ *such that* u *is a solution to* $(\mathrm{QP}; f, u_0)$ *on* $[0,T]$ *and*

$$u_{\varepsilon_n} \to u \quad in\ C([0,T];V) \quad as\ n\to\infty.$$

(ii) *Let* u *be any solution to* $(\mathrm{QP}; f, u_0)$ *on* $[0,T]$. *Then, there are sequences* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ *with* $\varepsilon_n \to 0$ *(as* $n\to\infty$*),* $\{f_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V^*)$, $\{h_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V)$, *and* $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset W^{1,2}(0,T;V)$ *such that* u_{ε_n} *is a unique solution to* $(\mathrm{QP}; f_n + \varepsilon_n F h_n, u_0)_{\varepsilon_n}$, $\{h_n\}_{n\in\mathbb{N}}$ *is bounded in* $L^2(0,T;V)$, *and*

$$u_{\varepsilon_n} \to u \quad in\ C([0,T];V), \qquad f_n \to f \quad in\ L^2(0,T;V^*) \quad as\ n\to\infty.$$

Proof. The proof is quite similar to that of Theorem 5.1. Indeed, assertion (i) can be shown in a way similar to the proof of Theorem 6.1 (cf. [22, Theorem 3]). In addition, assertion (ii) can be shown by adding $\varepsilon_n F u'(t)$ to both sides of the equation for $(\mathrm{QP}; f, u_0)$ and setting $u_{\varepsilon_n} := u$, $f_n := f$, and $h_n := u'$ (cf. (ii) of Theorem 5.1). □

Next, by the same approach as in Section 5, let us consider an approximation to optimal control problem $(\mathrm{OP})_{\mathrm{QV}}$. To this end, we fix an initial datum $u_0 \in V$, and note from (A2) and (6.4) that any solution u to $(\mathrm{QP}; f, u_0)$ on $[0,T]$ satisfies the following estimate:

$$\int_0^T |u'(t)|_V^2 dt \le \frac{N_3\left(|u_0|_V^2 + |f|^2_{L^2(0,T;V^*)} + 1\right) + C_2 T}{C_1}. \tag{9.3}$$

Therefore, we take and fix a positive number $N > 0$ so that

$$N^2 \ge \frac{N_3\left(|u_0|_V^2 + M^2 + 1\right) + C_2 T}{C_1}, \tag{9.4}$$

where $M > 0$ is the same positive constant as in the control space $\mathcal{F}_M$ (cd. (1.3)).

For each $\varepsilon \in (0,1]$, let $\mathcal{H}_N^\varepsilon$ be the perturbation of the control space defined by (5.6). Then, we study the following control problem for approximate state system $(\mathrm{QP}; f + \varepsilon F h_\varepsilon, u_0)_\varepsilon$, denoted by $(\mathrm{OP})_{\mathrm{QV},\varepsilon}$:

Problem $(\mathbf{OP})_{\mathrm{QV},\varepsilon}$**:** Find a control $(f_\varepsilon^*, h_\varepsilon^*) \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$, called an optimal control, such that

$$J_{\mathrm{QV},\varepsilon}(f_\varepsilon^*, h_\varepsilon^*) = \inf_{(f,h)\in\mathcal{F}_M\times\mathcal{H}_N^\varepsilon} J_{\mathrm{QV},\varepsilon}(f,h).$$

Here, $J_{\mathrm{QV},\varepsilon}(f,h)$ is the cost functional defined by

$$J_{\mathrm{QV},\varepsilon}(f,h) := \frac{1}{2}\int_0^T |u_\varepsilon(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f(t)|_{V^*}^2 dt + \frac{\varepsilon}{2}\int_0^T |h(t)|_V^2 dt, \tag{9.5}$$

where $(f,h) \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ is a control, u_ε is a unique solution to the approximate state system $(\mathrm{QP}; f + \varepsilon F h_\varepsilon, u_0)_\varepsilon$, and $u_{ad} \in L^2(0,T;V)$ is the target profile.

Let us state the second result of this section, which is concerned with the relationship between $(\mathrm{OP})_{\mathrm{QV}}$ and $(\mathrm{OP})_{\mathrm{QV},\varepsilon}$.

Theorem 9.2. *Suppose that Assumptions (A), (B'), (C), and* (B5') *hold. Let* $u_0 \in V$ *and* $u_{ad} \in L^2(0,T;V)$. *Then, we have:*

(i) *For each* $\varepsilon \in (0,1]$, $(\mathrm{OP})_{\mathrm{QV},\varepsilon}$ *has at least one optimal control* $(f_\varepsilon^*, h_\varepsilon^*) \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$, *namely,*

$$J_{\mathrm{QV},\varepsilon}(f_\varepsilon^*, h_\varepsilon^*) = \inf_{(f,h)\in\mathcal{F}_M\times\mathcal{H}_N^\varepsilon} J_{\mathrm{QV},\varepsilon}(f,h).$$

(ii) *Let* $\varepsilon \in (0,1]$, *and let* $(f_\varepsilon^*, h_\varepsilon^*) \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ *be any optimal control of the approximate problem* $(\mathrm{OP})_{\mathrm{QV},\varepsilon}$. *Additionally, suppose that Assumption* (D) *(cf. (ii) of Theorem 5.2) is satisfied. Then, any weak limit* f^* *of* $\{f_\varepsilon^*\}$ *in* $L^2(0,T;V^*)$ *(as* $\varepsilon \to 0$*) is an optimal control of* $(\mathrm{OP})_{\mathrm{QV}}$; *more precisely, if* $\varepsilon_n \to 0$ *and* $f_{\varepsilon_n}^* \to f^*$ *weakly in* $L^2(0,T;V^*)$ *(as* $n \to \infty$*), then the weak limit* f^* *is an optimal control for* $(\mathrm{OP})_{\mathrm{QV}}$.

We can prove Theorem 9.2 in the same way as Theorem 5.2, by making use of the following two propositions. The first is concerned with the convergence of solutions to $(\mathrm{QP}; f + \varepsilon F h_\varepsilon, u_0)_\varepsilon$ on $[0,T]$.

Proposition 9.2 (cf. Proposition 5.2)**.** *Suppose that Assumptions (A), (B'), (C), and* (B5') *hold. Let* $\{f_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V^*)$ *and let* $\{u_{0,n}\}_{n\in\mathbb{N}} \subset V$ *such that*

$$f_n \to f \text{ in } L^2(0,T;V^*), \quad u_{0,n} \to u_0 \text{ in } V \quad \text{as } n \to \infty.$$

Then, the following statements hold:

(i) *Assume that* $\{h_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V)$, $h \in L^2(0,T;V)$ *and*

$$h_n \to h \quad \text{in } L^2(0,T;V) \quad \text{as } n \to \infty.$$

For any fixed parameter $\varepsilon \in (0,1]$, *let* u_n *be a unique solution to* $(\mathrm{QP}; f_n + \varepsilon F h_n, u_{0,n})_\varepsilon$ *on* $[0,T]$. *Then, there is a function* $u \in W^{1,2}(0,T;V)$ *such that* u *is a unique solution to* $(\mathrm{QP}; f + \varepsilon F h, u_0)_\varepsilon$ *on* $[0,T]$ *and*

$$u_n \to u \text{ in } C([0,T];V) \text{ as } n \to \infty.$$

(ii) *Assume that* $\{h_n\}_{n\in\mathbb{N}}$ *is a bounded set in* $L^2(0,T;V)$. *Let* $\{\varepsilon_n\}_{n\in\mathbb{N}}$ *be a sequence in* $(0,1]$ *with* $\varepsilon_n \to 0$ *(as* $n \to \infty$*). Let* u_{ε_n} *be a unique solution to* $(\mathrm{QP}; f_n + \varepsilon_n F h_n, u_{0,n})_{\varepsilon_n}$ *on* $[0,T]$. *Then, there exist a subsequence* $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ *and a function* $u \in W^{1,2}(0,T;V)$ *such that* u *is a solution to* $(\mathrm{QP}; f, u_0)$ *on* $[0,T]$ *and*

$$u_{\varepsilon_{n_k}} \to u \text{ in } C([0,T];V) \text{ as } k \to \infty.$$

The proof of Proposition 9.2 is a slight modification of the proof of Proposition 5.2. Therefore, we omit the detailed proof.

The second proposition is the key to the proof of Theorem 9.2(ii).

Proposition 9.3 (cf. Proposition 5.3). *Suppose that Assumptions (A), (B'), (C),* (D), *and* (B5') *hold. Let* $f \in \mathcal{F}_M$ *and* $u_0 \in V$. *Additionally, let* u *be any solution to* (QP; f, u_0) *on* $[0,T]$. *Then, there exist sequences* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ *with* $\varepsilon_n \to 0$ *(as* $n \to \infty$*),* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_M$, $\{h_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V)$ *with* $h_n \in \mathcal{H}_N^{\varepsilon_n}$, *and* $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset W^{1,2}(0,T;V)$ *such that* u_{ε_n} *is a unique solution to* (QP;$f_n + \varepsilon_n F h_n, u_0)_{\varepsilon_n}$ *on* $[0,T]$, *and*

$$u_{\varepsilon_n} \to u \quad \text{in } C([0,T];V), \qquad f_n \to f \quad \text{in } L^2(0,T;V^*) \quad \text{as } n \to \infty.$$

Proposition 9.3 can be proved using a slight modification of the proof of Proposition 5.3. Therefore, we omit the detailed proof.

10 Control of parameter-dependent evolution equations

In this section, we discuss another type of singular optimal control problem associated with the following doubly nonlinear parameter-dependent evolution equation:

$$(\text{DP};w,f,u_0)\begin{cases} \partial_*\psi^t(u'(t)) + \partial_*\varphi^t(w;u(t)) + g(t,u(t)) \ni f(t) \text{ in } V^* \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } t \in (0,T), \\ u(0) = u_0 \quad \text{in } V. \end{cases} \tag{10.1}$$

Here, ψ^t, φ^t, $g(t,\cdot)$, and f are the same as in Section 6, and w is any function in D_0, where D_0 is the set introduced in *Assumption (B')*.

A function $u : [0,T] \to V$ is called a solution to (DP;w,f,u_0) on $[0,T]$ if u is a solution to (P;f,u_0) with $\partial_*\varphi^t(\cdot)$ replaced by $\partial_*\varphi^t(w;\cdot)$.

The following theorem is an immediate consequence of Theorems 2.1 and 2.2.

Theorem 10.1 (cf. Theorems 2.1 and 2.2). *Suppose that Assumptions (A), (B'), and (C) are satisfied. Then, for each* $w \in D_0$, $f \in L^2(0,T;V^*)$, *and* $u_0 \in V$, (DP; w,f,u_0) *admits at least one solution* u *on* $[0,T]$. *Additionally, there exists a constant* $N_5 > 0$, *independent of* $w \in D_0$, f, *and* u_0, *such that*

$$\int_0^T \psi^t(u'(t))dt + \sup_{t\in[0,T]} \varphi^t(w;u(t)) \le N_5 \left(|u_0|_V^2 + |f|^2_{L^2(0,T;V^*)} + 1\right) \tag{10.2}$$

for any solution u *to* (DP; w,f,u_0). *Additionally, assume* (A4). *Then, the solution* u *to* (DP; w,f,u_0) *is unique.*

Proof. From *Assumption (B')*, we observe that $\varphi^t(w;\cdot)$ satisfies *Assumption (B)* for any $w \in D_0$. Therefore, by applying Theorem 2.1, we obtain a solution to (DP;w,f,u_0) on $[0,T]$ for each $w \in D_0$, $f \in L^2(0,T;V^*)$, and $u_0 \in V$. The estimate (10.2) is a direct consequence of (2.8) in Theorem 2.1. Moreover, by applying Theorem 2.2, we observe from Assumption (A4) that the solution to (DP; w,f,u_0) on $[0,T]$ is unique. □

Using arguments similar to (P) and (QP), we consider the singular optimal control problem for (DP;w, f, u_0). To this end, for given positive numbers $M,\ M'$, we set up two control spaces: $\mathcal{F}_M$ given by (1.3) and $\mathcal{W}_{M'}$ given by

$$\mathcal{W}_{M'} := \left\{ w \in W^{1,2}(0,T;V)\ ;\ \int_0^T \psi^t(w'(t))dt \le M' \right\}. \tag{10.3}$$

Then, the singular optimal control problem with state (DP;w, f, u_0) is formulated as follows:

Problem ($\widetilde{\mathbf{OP}}$): Find an optimal control $(w^*, f^*) \in \mathcal{W}_{M'} \times \mathcal{F}_M$, namely,

$$\widetilde{J}(w^*, f^*) = \inf_{(w,f)\in \mathcal{W}_{M'} \times \mathcal{F}_M} \widetilde{J}(w, f).$$

Here, $\widetilde{J}(w, f)$ is the cost functional defined by

$$\widetilde{J}(w, f) := \inf_{u \in \mathcal{S}(w,f)} \widetilde{\pi}_{(w,f)}(u), \tag{10.4}$$

where $(w, f) \in \mathcal{W}_{M'} \times \mathcal{F}_M$ is any control, and $\mathcal{S}(w, f)$ is the set of all solutions to (DP;w, f, u_0) associated with control (w, f). Additionally, for any solution u to the state system (DP;w, f, u_0), its functional $\widetilde{\pi}_{(w,f)}(u)$ is defined by:

$$\widetilde{\pi}_{(w,f)}(u) := \frac{1}{2}\int_0^T |u(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |w(t)|_H^2 dt + \frac{1}{2}\int_0^T |f(t)|_{V^*}^2 dt$$

where $u_{ad} \in L^2(0, T; V)$ is a given target profile.

Using a proof similar to that of Proposition 8.1, we obtain the convergence result of solutions of (DP;w, f, u_0) with respect to $(w, f) \in \mathcal{W}_{M'} \times \mathcal{F}_M$. Therefore, we can easily prove the following theorem concerning the existence of an optimal control for ($\widetilde{\mathrm{OP}}$) without the uniqueness of solutions to (DP; w, f, u_0).

Theorem 10.2. *Suppose that Assumptions (A), (B'), and (C) are satisfied. Let u_{ad} be a given function in $L^2(0, T; V)$ and u_0 be any element in V. Then, ($\widetilde{\mathrm{OP}}$) has at least one optimal control $(w^*, f^*) \in \mathcal{W}_{M'} \times \mathcal{F}_M$, namely,*

$$\widetilde{J}(w^*, f^*) = \inf_{(w,f)\in \mathcal{W}_{M'} \times \mathcal{F}_M} \widetilde{J}(w, f).$$

The approximate problems for (DP;w, f, u_0) and ($\widetilde{\mathrm{OP}}$) are discussed as follows. For each $\varepsilon \in (0, 1]$ and $h \in L^2(0, T; V)$, the approximate state system is given by:

$$(\mathrm{DP}; w, f + \varepsilon F h, u_0)_\varepsilon \begin{cases} \varepsilon F u_\varepsilon'(t) + \partial_* \psi^t(u_\varepsilon'(t)) + \partial_* \varphi^t(w; u_\varepsilon(t)) + g(t, u_\varepsilon(t)) \\ \qquad\qquad\qquad \ni f(t) + \varepsilon F h(t) \text{ in } V^* \text{ for a.a. } t \in (0, T), \\ u_\varepsilon(0) = u_0 \text{ in } V, \end{cases}$$

where $F : V \to V^*$ is the duality mapping.

Then, using similar arguments to those in Sections 5 and 9, we see that:

(i) Problem $(\mathrm{DP}; w, f+\varepsilon Fh, u_0)_\varepsilon$ possesses one and only one solution u_ε in the same sense as $(\mathrm{P}; f+\varepsilon Fh, u_0)_\varepsilon$ with $\partial_*\varphi^t(\cdot)$ replaced by $\partial_*\varphi^t(w;\cdot)$ (cf. Proposition 3.1). Additionally, $(\mathrm{DP}; w, f+\varepsilon Fh, u_0)_\varepsilon$ approximates problem $(\mathrm{DP}; w, f, u_0)$ in the sense of Theorem 5.1.

(ii) Approximate optimal control problems, denoted by $(\widetilde{\mathrm{OP}})_\varepsilon$, are formulated by:

Problem $(\widetilde{\mathrm{OP}})_\varepsilon$**:** Find an optimal control $(w_\varepsilon^*, f_\varepsilon^*, h_\varepsilon^*) \in \mathcal{W}_{M'} \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$, namely,

$$\widetilde{J}_\varepsilon(w_\varepsilon^*, f_\varepsilon^*, h_\varepsilon^*) = \inf_{(w,f,h)\in\mathcal{W}_{M'}\times\mathcal{F}_M\times\mathcal{H}_N^\varepsilon} \widetilde{J}_\varepsilon(w,f,h).$$

Here, $\mathcal{H}_N^\varepsilon$ is a bounded perturbation of the control space in $L^2(0,T;V)$ given by (5.6) and

$$\widetilde{J}_\varepsilon(w,f,h) := \frac{1}{2}\int_0^T |u_\varepsilon(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |w(t)|_H^2 dt + \frac{1}{2}\int_0^T |f(t)|_{V^*}^2 dt + \frac{\varepsilon}{2}\int_0^T |h(t)|_V^2 dt,$$

where $(w,f,h) \in \mathcal{W}_{M'} \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ is a control, and u_ε is a unique solution to $(\mathrm{DP}; w, f+\varepsilon Fh, u_0)_\varepsilon$ on $[0,T]$.

(iii) Suppose that Assumption (D) holds. Then, problem $(\widetilde{\mathrm{OP}})_\varepsilon$ approximates $(\widetilde{\mathrm{OP}})$ as $\varepsilon \to 0$ in the sense of Theorem 5.2.

Remark 10.1. *There is a vast amount of literature on the optimal control of parameter-dependent problems. For instance, refer to [15, 31, 33, 34]. Note that* $(\mathrm{DP}; w, f, u_0)$ *is a new type of parameter-dependent evolution equation, and it is worthwhile evolving this system with related control problems.*

In the rest of this section, we attempt to find another approximation procedure for the singular optimal control problem $(\mathrm{OP})_{\mathrm{QV}}$ with state $(\mathrm{QP}; f, u_0)$ (see Section 6) as an application of Theorem 10.2.

For each $\delta \in (0,1]$, we consider the singular optimal control problem $(\widehat{\mathrm{DP}})_\delta$ with $(\mathrm{DP}; w, f, u_0)$ as the state problem for $w \in D_0$, $f \in L^2(0,T;V^*)$, and $u_0 \in V$ and with the cost functional $\widehat{E}_\delta$ defined by

$$\widehat{E}_\delta(w,f) := \inf_{u\in\mathcal{S}(w,f)} \widehat{\pi}_{(w,f)}^\delta(u), \tag{10.5}$$

where $\mathcal{S}(w,f)$ is the set of all solutions to $(\mathrm{DP}; w, f, u_0)$ and, for any $u \in \mathcal{S}(w,f)$,

$$\widehat{\pi}_{(w,f)}^\delta(u) := \frac{1}{2}\int_0^T |u(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f(t)|_{V^*}^2 dt + \frac{1}{2\delta}\int_0^T |u(t) - w(t)|_V^2 dt. \tag{10.6}$$

Additionally, we choose the set $\widehat{\mathcal{W}}_{M'}(u_0) \times \mathcal{F}_M$ as the control space, where

$$\widehat{\mathcal{W}}_{M'}(u_0) := \{w \in \mathcal{W}_{M'} \mid w(0) = u_0\}$$

with a positive constant M and $M' := N_5(|u_0|_V^2 + M^2 + 1)$. Then, for each $f \in \mathcal{F}_M$, (10.2) implies that

$$u \in \widehat{\mathcal{W}}_{M'}(u_0) \text{ for any solution } u \text{ to } (\mathrm{DP}; w, f, u_0) \text{ on } [0,T]. \tag{10.7}$$

Now, for each $\delta \in (0,1]$, the singular optimal control problem $(\widehat{\mathrm{DP}})_\delta$ is precisely formulated as follows:

Problem $(\widehat{\mathrm{DP}})_\delta$: Find an optimal control $(w_\delta^*, f_\delta^*) \in \widehat{\mathcal{W}}_{M'}(u_0) \times \mathcal{F}_M$, namely,

$$\widehat{E}_\delta(w_\delta^*, f_\delta^*) = \inf_{(w,f)\in\widehat{\mathcal{W}}_{M'}(u_0)\times\mathcal{F}_M} \widehat{E}_\delta(w, f).$$

Then, it follows from Proposition 8.1 and Theorem 8.1 (cf. Theorem 10.2) that, for each $\delta \in (0,1]$, problem $(\widehat{\mathrm{DP}})_\delta$ possesses an optimal control.

Moreover, we have:

Theorem 10.3. *Suppose that Assumptions (A), (B'), and (C) are satisfied. Let (w_δ^*, f_δ^*) be an optimal control of $(\widehat{\mathrm{DP}})_\delta$ for any $\delta \in (0,1]$, and let (w^*, f^*) be any weak limit of $\{(w_\delta^*, f_\delta^*)\}$ as $\delta \downarrow 0$, namely, there is a sequence $\{\delta_n\}$ with $\delta_n \downarrow 0$ (as $n \to \infty$) such that*

$$w_{\delta_n}^* \to w^* \text{ weakly in } L^2(0,T;V), \quad f_{\delta_n}^* \to f^* \text{ weakly in } L^2(0,T;V^*).$$

Then, f^* is an optimal control of $(\mathrm{OP})_{\mathrm{QV}}$, w^* is a solution of $(\mathrm{QP}; f^*, u_0)$, and

$$J_{\mathrm{QV}}(f^*) \ (= \inf_{f\in\mathcal{F}_M} J_{\mathrm{QV}}(f)) = \lim_{\delta\downarrow 0} \widehat{E}_\delta(w_\delta^*, f_\delta^*), \tag{10.8}$$

where J_{QV} is the functional on $\mathcal{F}_M$ given by (8.1) and (8.2).

Proof. We use the same notation as in the statement of the theorem. Let f be any element in $\mathcal{F}_M$ and $\pi_{\mathrm{QV},f}(\cdot)$ be the same functional defined by (8.2). Additionally, let $\mathcal{S}_{\mathrm{QV}}(f)$ be the set of all solutions to $(\mathrm{QP}; f, u_0)$. Then, $u \in \mathcal{S}_{\mathrm{QV}}(f)$ is also a solution to $(\mathrm{DP}; u, f, u_0)$ on $[0,T]$. Therefore, we see that

$$\widehat{E}_\delta(w_\delta^*, f_\delta^*) \le \widehat{\pi}_{(u,f)}^\delta(u) = \pi_{\mathrm{QV},f}(u), \ \forall u \in \mathcal{S}_{\mathrm{QV}}(f), \ \forall f \in \mathcal{F}_M,$$

whence

$$\widehat{E}_\delta(w_\delta^*, f_\delta^*) \le \inf_{f\in\mathcal{F}_M} J_{\mathrm{QV}}(f) =: d^*. \tag{10.9}$$

Now, let u_δ^* be any optimal state corresponding to $\widehat{E}_\delta(w_\delta^*, f_\delta^*)$, namely $u_\delta^* \in \mathcal{S}(w_\delta^*, f_\delta^*)$ and

$$\begin{aligned}\widehat{E}_\delta(w_\delta^*, f_\delta^*) &= \frac{1}{2}\int_0^T |u_\delta^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f_\delta^*(t)|_{V^*}^2 dt + \frac{1}{2\delta}\int_0^T |u_\delta^*(t) - w_\delta^*(t)|_V^2 dt \\ &\le d^*.\end{aligned} \tag{10.10}$$

As $w_\delta^* \in \widehat{\mathcal{W}}_{M'}(u_0)$, $u_\delta^* \in \widehat{\mathcal{W}}_{M'}(u_0)$ (cf. (10.7)), and $f_\delta^* \in \mathcal{F}_M$, according to the Aubin compactness theorem, there exist a subsequence $\{\delta_n\}_{n\in\mathbb{N}} \subset \{\delta\}_{\delta\in(0,1]}$, a function $(w^*, f^*) \in \widehat{\mathcal{W}}_{M'}(u_0) \times \mathcal{F}_M$, and a function $u^* \in \widehat{\mathcal{W}}_{M'}(u_0)$ such that $\delta_n \to 0$ as $n \to \infty$,

$$w_{\delta_n}^* \to w^* \text{ weakly in } W^{1,2}(0,T;V) \text{ and in } C([0,T];H),$$
$$w_{\delta_n}^*(t) \to w^*(t) \ \text{ weakly in } V,\ \forall t \in [0,T],$$
$$u_{\delta_n}^* \to u^* \text{ weakly in } W^{1,2}(0,T;V) \text{ and in } C([0,T];H),$$
$$u_{\delta_n}^*(t) \to u^*(t) \ \text{ weakly in } V,\ \forall t \in [0,T],$$

and

$$f_{\delta_n}^* \to f^* \ \text{ in } L^2(0,T;V^*)$$

as $n \to \infty$.

Therefore, by (10.10), $u_{\delta_n}^* - w_{\delta_n}^* \to 0$ in $L^2(0,T;V)$, which implies that

$$u^* = w^* \text{ in } L^2(0,T;V). \tag{10.11}$$

Moreover, as in the last part of the proof of Theorem 6.1, $u_{\delta_n}^*$ converges in $C([0,T];V)$ to a solution to (DP; w^*, f^*, u_0) that is equal to u^*. By (10.11), u^* is a solution to (QP;f^*, u_0) and (DP; w^*, f^*, u_0) on $[0,T]$.

Now, taking the limit of (10.10) as $\delta := \delta_n \downarrow 0$, we obtain

$$\begin{aligned}
d^* &\le J_{\mathrm{QV}}(f^*) = \inf_{u\in\mathcal{S}_{\mathrm{QV}}(f^*)} \pi_{\mathrm{QV},f^*}(u) \\
&\le \frac{1}{2}\int_0^T |u^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f^*(t)|_{V^*}^2 dt \\
&= \lim_{n\to\infty} \left\{ \frac{1}{2}\int_0^T |u_{\delta_n}^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f_{\delta_n}^*(t)|_{V^*}^2 dt \right\} \\
&\le \liminf_{n\to\infty} \left\{ \frac{1}{2}\int_0^T |u_{\delta_n}^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f_{\delta_n}^*(t)|_{V^*}^2 dt \right. \\
&\qquad\qquad \left. + \frac{1}{2\delta_n}\int_0^T |u_{\delta_n}^*(t) - w_{\delta_n}^*(t)|_V^2 dt \right\} \\
&= \liminf_{n\to\infty} \widehat{E}_{\delta_n}(w_{\delta_n}^*, f_{\delta_n}^*) \le \limsup_{n\to\infty} \widehat{E}_{\delta_n}(w_{\delta_n}^*, f_{\delta_n}^*) \le d^*,
\end{aligned}$$

and hence $d^* = J_{\mathrm{QV}}(f^*)$ with

$$\lim_{n\to\infty} \frac{1}{2\delta_n}\int_0^T |u_{\delta_n}^*(t) - w_{\delta_n}^*(t)|_V^2 dt = 0. \tag{10.12}$$

It is easy to see (10.8) from (10.12). □

Remark 10.2. *It is possible to prove Theorem 10.3 without using the results on* (OP)$_{\mathrm{QV}}$ *in Section 9; note that Assumption* (D) *is not required. In this sense, Theorem 10.3 provides us with another approximation procedure for* (OP)$_{\mathrm{QV}}$ *based on the theory of* (P;f, u_0) *and* (QP;f, u_0) *(see Sections 3, 4, and 6).*

11 Applications

In this final section, we consider five applications of the general results (Theorems 2.1–5.2, Theorems 6.1–9.2, and Theorems 10.1–10.3).

Let Ω be a bounded domain in $\mathbb{R}^N$ $(1 \le N < \infty)$ with smooth boundary $\Gamma := \partial\Omega$, and set

$$V := H_0^1(\Omega), \quad H := L^2(\Omega), \quad X := H^2(\Omega).$$

Additionally, let $T > 0$ be a given real number, $Q := (0,T) \times \Omega$, and $\Sigma := (0,T) \times \Gamma$.

11.1 Variational inequality with time-dependent gradient constraint

Let ρ be an obstacle function prescribed in $C(\overline{Q})$ such that

$$\rho_* \le \rho(t,x) \le \rho^*, \quad \forall (t,x) \in \overline{Q}, \tag{11.1}$$

where ρ_* and ρ^* are positive constants. Our constraint set $K(t)$ is defined for each $t \in [0,T]$ by

$$K(t) := \{z \in V \ ; \ |\nabla z(x)| \le \rho(t,x), \text{ a.a. } x \in \Omega\}.$$

Now, we consider the following variational inequality with time-dependent gradient constraint:

$$\left.\begin{aligned} & u_t(t) \in K(t) \quad \text{for a.a. } t \in (0,T), \\ & \tau \int_\Omega u_t(t,x)(u_t(t,x) - v(x))dx \\ & \quad + \int_\Omega a(t,x)\nabla u(t,x) \cdot \nabla(u_t(t,x) - v(x))dx \\ & \quad + \int_\Omega g(t,u(t,x))(u_t(t,x) - v(x))dx \\ & \le \int_\Omega f(t)(u_t(t,x) - v(x))dx \quad \text{for all } v \in K(t) \text{ and a.a. } t \in (0,T), \\ & u(0,x) = u_0(x), \quad x \in \Omega, \end{aligned}\right\} \tag{11.2}$$

where $\tau \ge 0$ is a constant, $g(\cdot,\cdot)$ is a Lipschitz continuous function on $[0,T] \times \mathbb{R}$, f is a function in $L^2(0,T;H)$, u_0 is an initial datum in V, and $a(\cdot,\cdot)$ is a function in $W^{1,1}(0,T;C(\overline{\Omega}))$ such that

$$a_* \le a(t,x) \le a^*, \quad \forall (t,x) \in \overline{Q},$$

where a_* and a^* are positive constants. All of them are prescribed as the data.

A function $u : [0,T] \to V$ is called a solution to (11.2) on $[0,T]$ if $u \in W^{1,2}(0,T;V)$ and all of the properties required in (11.2) are satisfied.

From Theorems 2.1 and 2.2, we obtain the existence–uniqueness result for problem (11.2).

Proposition 11.1 (cf. Theorems 2.1 and 2.2)**.** *Let $\tau \geq 0$ be a constant and $K(t)$ be as above. Then, for each $f \in L^2(0,T;H)$ and $u_0 \in V$, problem* (11.2) *admits at least one solution u on $[0,T]$. Moreover, if $\tau > 0$, then the solution u to* (11.2) *is unique.*

Proof. For each $t \in [0,T]$, define proper l.s.c., and convex functions ψ^t and φ^t on V by

$$\psi^t(z) := \frac{\tau}{2}\int_\Omega |z(x)|^2 dx + I_{K(t)}(z), \quad \forall z \in V \tag{11.3}$$

and

$$\varphi^t(z) := \frac{1}{2}\int_\Omega a(t,x)|\nabla z(x)|^2 dx, \quad \forall z \in V, \tag{11.4}$$

respectively, where $I_{K(t)}(\cdot)$ is the indicator function of $K(t)$, namely,

$$I_{K(t)}(z) := \begin{cases} 0, & \text{if } z \in K(t), \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easily observed that (11.2) can be reformulated in the abstract form (P; f, u_0). In addition, by an argument similar to that in [22, Application 1], we see that

(i) *Assumption (A)* holds. $z^* \in \partial_*\psi^t(z)$ if and only if $z^* \in V^*$, $z \in K(t)$ and

$$\tau \int_\Omega z(x)(z(x) - v(x))dx + \langle -z^*, z - v\rangle \leq 0, \quad \forall v \in K(t), \tag{11.5}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between V^* and $V(= H_0^1(\Omega))$.

(ii) *Assumption (B)* holds. In particular, (B3) holds for

$$\alpha(t) := \frac{1}{a_*}\int_0^t |a'(\tau)|_{C(\overline{\Omega})} d\tau, \quad \forall t \in [0,T].$$

Furthermore,

$$\langle \partial_*\varphi^t(z), v\rangle = \int_\Omega a(t,x)\nabla z(x)\cdot\nabla v(x)dx, \quad \forall z, v \in V, \ \forall t \in [0,T].$$

(iii) Condition (2.1) holds; more precisely, the duality mapping $F : V \to V^*$ is linear and

$$\langle Fz, v\rangle := \int_\Omega \nabla z(x)\cdot\nabla v(x)dx, \quad \forall z, v \in V;$$

hence, F is nothing but the Laplace operator $-\Delta$ under a homogeneous Dirichlet boundary condition on Γ.

Therefore, applying Theorem 2.1, we can show the existence of a solution to (11.2) on $[0,T]$. Moreover, if $\tau > 0$, it follows from (11.5) that $\partial_*\psi^t$ is strictly monotone from V into V^* in the sense of

$$\langle z_1^* - z_2^*, z_1 - z_2\rangle \geq \tau|z_1 - z_2|_H^2, \ \ \forall z_i \in D(\partial_*\psi^t), \ \forall z_i^* \in \partial_*\psi^t(z_i), \ i = 1, 2, \ \forall t \in [0,T],$$

namely, (2.10) holds. Hence, by Remark 2.4, we conclude that the solution to (11.2) is unique. Thus, Proposition 11.1 is obtained. □

We now discuss the singular optimal control problem associated with (11.2) in the case when $\tau = 0$. By Theorem 4.1 with $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$, we have the following result concerning the existence of an optimal control of (OP) for (11.2) with $\tau = 0$.

Proposition 11.2 (cf. Theorems 4.1)**.** *Let u_{ad} be a given function in $L^2(0,T;V)$, $u_0 \in V$, $K(t)$ be the same constraint set as in Proposition 11.1, and let $M > 0$ be a given constant. Assume $\tau = 0$. Then,* (OP) *has at least one optimal control $f^* \in \mathcal{F}_M$, namely,*

$$J(f^*) = \inf_{f \in \mathcal{F}_M} J(f),$$

where $\mathcal{F}_M$ is a control space defined by (1.3)*, and $J(\cdot)$ is the cost functional of* (OP) *defined by* (1.4) *and* (1.5)*.*

We employ the approximate system to (11.2), as proposed in Section 5, for the case $\tau = 0$. Indeed, for each $\varepsilon \in (0,1]$, we consider the following variational inequality with time-dependent gradient constraint $K(t)$:

$$\left.\begin{aligned} & u_t(t) \in K(t) \quad \text{for a.a. } t \in [0,T], \\ & \varepsilon \int_\Omega \nabla u_t(t,x) \cdot \nabla (u_t(t,x) - v(x)) dx \\ & \qquad + \int_\Omega a(t,x) \nabla u(t,x) \cdot \nabla (u_t(t,x) - v(x)) dx \\ & \qquad + \int_\Omega g(t,u(t,x))(u_t(t,x) - v(x)) dx \\ & \le \int_\Omega f(t)(u_t(t,x) - v(x)) dx + \varepsilon \int_\Omega \nabla h(t,x) \cdot \nabla (u_t(t,x) - v(x)) dx \\ & \qquad \text{for all } v \in K(t) \text{ and a.a. } t \in (0,T), \\ & \qquad u(0,x) = u_0(x), \quad x \in \Omega, \end{aligned}\right\} \tag{11.6}$$

where h is a given function in $L^2(0,T;V)$.

From the facts identified in the proof of Proposition 11.1, problem (11.6) can be reformulated in the abstract form $(\mathrm{P}; f + \varepsilon F h, u_0)_\varepsilon$. Therefore, Proposition 5.1 implies the existence–uniqueness of solutions to (11.6) on $[0,T]$. In addition, from the relationship between (11.2) and (11.6) shown by Theorem 5.1, we observe that (11.6) is an approximate problem for (11.2).

Moreover, using the standard smoothness arguments (e.g., regularization by the mollifier and the convolution [1, Sections 2.28 and 3.16]), we can check Assumption (D). Therefore, for each $\varepsilon \in (0,1]$, Theorem 5.2(i) implies that the approximate optimal control problem $(\mathrm{OP})_\varepsilon$ has an optimal control $(f_\varepsilon^*, h_\varepsilon^*)$. Additionally, by Theorem 5.2(ii) on the relationship between (OP) and $(\mathrm{OP})_\varepsilon$, we see that (OP) is approximated by $(\mathrm{OP})_\varepsilon$ as $\varepsilon \downarrow 0$; more precisely, there is a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ such that $\{f_{\varepsilon_n}^*\}$ is bounded in $L^2(0,T;V^*)$ and any weak limit function of $\{f_{\varepsilon_n}^*\}$ in $L^2(0,T;V^*)$ is an optimal control of (OP).

11.2 Quasi-variational inequality with time-dependent gradient constraint

In this subsection, we treat a quasi-variational inequality with gradient constraint for time derivatives, which is an application of the general results (Theorems 6.1, 7.1, 8.1, 9.1, and 9.2).

Let us consider problem (11.2) with the diffusion coefficient $a(t,x)$ replaced by $a(t,x,u)$:

$$\left.\begin{aligned} &u_t(t) \in K(t) \quad \text{for a.a. } t \in (0,T), \\ &\tau \int_\Omega u_t(t,x)(u_t(t,x)-v(x))dx \\ &\quad + \int_\Omega a(t,x,u(t,x))\nabla u(t,x)\cdot \nabla(u_t(t,x)-v(x))dx \\ &\quad + \int_\Omega g(t,u(t,x))(u_t(t,x)-v(x))dx \\ &\le \int_\Omega f(t)(u_t(t,x)-v(x))dx \quad \text{for all } v \in K(t) \text{ and a.a. } t\in(0,T), \\ &u(0,x) = u_0(x), \quad x \in \Omega, \end{aligned}\right\} \tag{11.7}$$

where $K(t)$, f, and u_0 are the same as in Section 11.1; the obstacle function ρ satisfies (11.1) as well. For the function $a(t,x,r)$, we suppose that

$$\left\{\begin{aligned} &a_* \le a(t,x,r) \le a^*, \quad \forall (t,x)\in \overline{Q},\ \forall r \in \mathbb{R}, \\ &|a(t_1,x,r_1)-a(t_2,x,r_2)| \le L_a(|t_1-t_2|+|r_1-r_2|), \\ &\qquad \forall t_i \in [0,T],\ r_i\in\mathbb{R},\ i=1,2,\ \forall x \in \overline{\Omega}, \end{aligned}\right. \tag{11.8}$$

where a_*, a^*, and L_a are positive constants.

A function $u : [0,T] \to V$ is called a solution to (11.7) on $[0,T]$ if $u \in W^{1,2}(0,T;V)$ and all of the properties required in (11.7) are satisfied.

From Theorem 6.1, the following existence result is obtained for problem (11.7).

Proposition 11.3 (cf. Theorem 6.1)**.** *Let $\tau \ge 0$ be a given constant. Then, for each $f \in L^2(0,T;H)$ and $u_0 \in V$, problem* (11.7) *admits at least one solution u on* $[0,T]$.

Proof. For each $t \in [0,T]$, let ψ^t be the proper l.s.c., and convex function on V defined by (11.3). Additionally, we define the (t,v)-dependent functional $\varphi^t(v;z)$ by

$$\varphi^t(v;z) := \frac{1}{2}\int_\Omega a(t,x,v(t,x))|\nabla z(x)|^2 dx, \quad \forall t \in [0,T],\ \forall v \in D_0,\ \forall z \in V, \tag{11.9}$$

where

$$D_0 = \left\{ v \in W^{1,2}(0,T;V) \mid v'(t) \in K(t) \text{ for a.a. } t \in [0,T] \right\}. \tag{11.10}$$

As mentioned in the proof of Proposition 11.1, *Assumptions (A), (C)*, and (2.1) are satisfied by ψ^t, $g(t,\cdot)$, and F, respectively. *Assumption (B')* is verified below:

(i) The subdifferential $\partial_*\varphi^t(v;\cdot)$ of $\varphi^t(v;\cdot)$ is given by

$$\langle \partial_*\varphi^t(v;z), w\rangle = \int_\Omega a(t,x,v(t,x))\nabla z(x)\cdot\nabla w(x)dx \tag{11.11}$$

for all $t \in [0,T]$, $v \in D_0$, and $z,\ w \in V$.

(ii) (B2') holds. Indeed, if $v_n \in D_0$, $\sup_{n\in\mathbb{N}}\int_0^T \psi^t(v_n'(t))dt < \infty$, and $v_n \to v$ in $C([0,T];H)$ as $n \to \infty$, then we have:

$$\begin{aligned} & |\langle \partial_*\varphi^t(v_n;z) - \partial_*\varphi^t(v;z), w\rangle| \\ \le\ & \int_\Omega |a(t,x,v_n(t,x)) - a(t,x,v(t,x))||\nabla z(x)||\nabla w(x)|dx \\ \le\ & \left(\int_\Omega |a(t,x,v_n(t,x)) - a(t,x,v(t,x))|^2|\nabla z(x)|^2dx\right)^{\frac{1}{2}} |w|_V, \\ & \qquad \forall z, w \in V \text{ and } \forall t \in [0,T], \end{aligned} \tag{11.12}$$

and the last integral converges to 0 as $n \to \infty$ by the Lebesgue dominated convergence theorem. Therefore, $\partial_*\varphi^t(v_n;z) \to \partial_*\varphi^t(v;z)$ in V^* as $n \to \infty$. Thus, (B2') holds.

(iii) (B4') holds. Indeed, by (11.1), we have

$$|\nabla v'(t,x)| \le \rho^* \ \text{ for a.a. } (t,x) \in Q,$$

which implies that

$$\begin{aligned} \sup_{t\in[0,T]} |v'(t,x)| + \sup_{t\in[0,T]} |\nabla v(t,x)| \le \bar\rho^*, \quad \text{a.a. } (t,x) \in Q,\ \forall v \in D_0 \\ \text{for some constant } \bar\rho^* > 0. \end{aligned} \tag{11.13}$$

Now, (B4') is verified using (11.1), (11.8), and (11.13) as follows:

$$\begin{aligned} & |\varphi^t(v;z) - \varphi^s(v;z)| \\ \le\ & \frac{1}{2}\int_\Omega |a(t,x,v(t,x)) - a(s,x,v(s,x))||\nabla z(x)|^2dx \\ \le\ & \frac{1}{2}\int_\Omega\int_s^t |a_\tau(\tau,x,v(\tau,x)) + a_v(\tau,x,v(\tau,x))v'(\tau,x)||\nabla z(x)|^2d\tau dx \\ \le\ & \frac{1}{a_*}(L_a + L_a\bar\rho^*)|t-s|\cdot\frac{1}{2}\int_\Omega a(s,x,v(s,x))|\nabla z(x)|^2dx \\ =\ & \frac{1}{a_*}L_a(1+\bar\rho^*)|t-s|\varphi^s(v;z), \qquad \forall v,z \in V,\ \forall s,t \in [0,T], \end{aligned}$$

where $a_\tau := \frac{\partial}{\partial\tau}a(\tau,x,v)$ and $a_v := \frac{\partial}{\partial v}a(\tau,x,v)$.

It is easy to see that (11.7) can be reformulated in the abstract form (QP; f, u_0). Therefore, by Theorem 6.1, problem (11.7) admits a solution. □

Moreover, we show that Assumption (B5') is satisfied. Let $v_i \in D_0$ $(i = 1, 2)$ and $z \in D_0$. Then, it follows from (11.8), (11.11), and (11.13) that:

$$\begin{aligned} & |\langle \partial_* \varphi^t(v_1; z) - \partial_* \varphi^t(v_2; z), w \rangle| \\ \le & \int_\Omega |a(t, x, v_1(t, x)) - a(t, x, v_2(t, x))| |\nabla z(x)| |\nabla w(x)| dx \\ \le & L_a \bar{\rho}^* \int_\Omega |v_1(t, x) - v_2(t, x)| |\nabla w(x)| dx \\ \le & L_a \bar{\rho}^* |v_1(t) - v_2(t)|_H |\nabla w|_H \\ \le & L_a \bar{\rho}^* |v_1(t) - v_2(t)|_H |w|_V, \quad \forall w \in V, \ \forall t \in [0, T]. \end{aligned} \tag{11.14}$$

We infer from (11.14) that (B5') holds.

We now discuss the singular optimal control problem for (11.7); for simplicity, we consider the case $\tau = 0$. By Theorem 8.1 with $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$, we have the following result concerning the existence of an optimal control of $(\mathrm{OP})_{\mathrm{QV}}$ for (11.7).

Proposition 11.4 (cf. Theorem 8.1)**.** *Let u_{ad} be a given function in $L^2(0, T; V)$, $u_0 \in V$, and let $M > 0$ be a given constant. Assume $\tau = 0$. Then,* $(\mathrm{OP})_{\mathrm{QV}}$ *has at least one optimal control $f^* \in \mathcal{F}_M$, namely,*

$$J_{\mathrm{QV}}(f^*) = \inf_{f \in \mathcal{F}_M} J_{\mathrm{QV}}(f),$$

where $J_{\mathrm{QV}}(\cdot)$ is the cost functional for $(\mathrm{OP})_{\mathrm{QV}}$*, which is defined by* (8.1) *and* (8.2)*.*

Next, we discuss the approximate system for (11.7) in the case $\tau = 0$. Note that (B5') is very important in studying approximate systems for (11.7) (cf. Theorem 9.1), as we need the uniqueness of solutions to the approximate state systems.

For each $\varepsilon \in (0, 1]$, we consider the following quasi-variational inequality with time-dependent gradient constraint $K(t)$:

$$\left. \begin{aligned} & u_t(t) \in K(t) \quad \text{for a.a. } t \in [0, T], \\ & \varepsilon \int_\Omega \nabla u_t(t, x) \cdot \nabla (u_t(t, x) - v(x)) dx \\ & \qquad + \int_\Omega a(t, x, u(t, x)) \nabla u(t, x) \cdot \nabla (u_t(t, x) - v(x)) dx \\ & \qquad + \int_\Omega g(t, u(t, x))(u_t(t, x) - v(x)) dx \\ & \le \int_\Omega f(x, t)(u_t(t, x) - v(x)) dx + \varepsilon \int_\Omega \nabla h(t, x) \cdot \nabla (u_t(t, x) - v(x)) dx \\ & \qquad \text{for all } v \in K(t) \text{ and a.a. } t \in (0, T), \\ & u(0, x) = u_0(x), \quad x \in \Omega, \end{aligned} \right\} \tag{11.15}$$

where h is a given function in $L^2(0, T; V)$.

As in the proof of Proposition 11.3, problem (11.15) can be reformulated in the abstract form $(\mathrm{QP}; f + \varepsilon F h, u_0)_\varepsilon$. Therefore, by Proposition 9.1, we have the existence–uniqueness of a solution to (11.15). In addition, it follows from Theorem 9.1 that there are sequences

$\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ and bounded sequences $\{h_n\}$ in $L^2(0,T;V)$ such that the solution u_n of (QP;$f+\varepsilon_n F h_n, u_0)_{\varepsilon_n}$ converges to that of (QP;f, u_0) in $C(0,T;V)$ (as $n \to \infty$). Namely, we observe that (11.15) is an approximate problem for (11.7) with $\tau = 0$.

Moreover, using the standard smoothness arguments (e.g., regularization by the mollifier and the convolution [1, Sections 2.28 and 3.16]), we can verify Assumption (D). Therefore, for each $\varepsilon \in (0,1]$, applying Theorem 9.2(i), we can show the existence of an optimal control $(f_\varepsilon^*, h_\varepsilon^*)$ of $(\mathrm{OP})_{\mathrm{QV},\varepsilon}$ for the approximate state system (11.15). Additionally, this converges to an optimal control of problem $(\mathrm{OP})_{\mathrm{QV}}$ in the sense of Theorem 9.2(ii); more precisely, there is a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ such that any weak limit of $\{f_{\varepsilon_n}^*\}$ (as $n \to \infty$) in $L^2(0,T;V^*)$ is an optimal control of $(\mathrm{OP})_{\mathrm{QV}}$.

11.3 Quasi-variational inequality with time-dependent non-local term

In this subsection, we consider a quasi-variational inequality with a time-dependent non-local term. Indeed, we define an operator $\mathcal{L} : L^2(0,T;H) \to L^2(0,T;H)$ by

$$(\mathcal{L}v)(t,x) := \int_\Omega \ell(t,x,\xi,v(t,\xi))d\xi + \ell_0(t,x) \quad \text{for } v \in L^2(0,T;H) \tag{11.16}$$

with given functions $\ell \in C(\overline{Q} \times \overline{\Omega} \times \mathbb{R})$ and $\ell_0 \in C(\overline{Q})$ satisfying the following conditions for a positive constant L_ℓ:

$$\begin{aligned} &|\ell(t_1,x,\xi,r_1) - \ell(t_2,x,\xi,r_2)| \le L_\ell\left(|t_1-t_2| + |r_1-r_2|\right), \\ &\qquad\qquad \forall t_i \in [0,T],\ \forall r_i \in \mathbb{R},\ i=1,2,\ \forall (x,\xi) \in \overline{\Omega} \times \overline{\Omega}, \end{aligned} \tag{11.17}$$

and

$$|\ell_0(t_1,x) - \ell_0(t_2,x)| \le L_\ell |t_1 - t_2|, \quad \forall t_i \in [0,T], i=1,2,\ \forall x \in \overline{\Omega}. \tag{11.18}$$

Now, consider the following quasi-variational inequality with a time-dependent non-local term:

$$\left.\begin{aligned} &u_t(t) \in K(t) \quad \text{for a.a. } t \in (0,T), \\ &\tau \int_\Omega u_t(t,x)(u_t(t,x) - v(x))dx \\ &\quad + \int_\Omega a(t,x,(\mathcal{L}u)(t,x))\nabla u(t,x) \cdot \nabla(u_t(t,x) - v(x))dx \\ &\quad + \int_\Omega g(t,u(t,x))(u_t(t,x) - v(x))dx \\ &\le \int_\Omega f(t)(u_t(t,x) - v(x))dx \quad \text{for all } v \in K(t) \text{ and a.a. } t \in (0,T), \\ &u(0,x) = u_0(x), \quad x \in \Omega, \end{aligned}\right\} \tag{11.19}$$

where $\tau \ge 0$, $K(\cdot)$, $g(\cdot,\cdot)$, f, and u_0 are the same as in Subsection 11.1, the obstacle function ρ satisfies (11.1), and the function $a(t,x,r)$ satisfies (11.8).

A function $u : [0, T] \to V$ is called a solution to (11.19) on $[0, T]$ if $u \in W^{1,2}(0, T; V)$ and all of the properties required in (11.19) are satisfied.

From Theorem 6.1, the following result is obtained for problem (11.19).

Proposition 11.5 (cf. Theorem 6.1). *Let $\tau \geq 0$ be a constant. Then, for each $f \in L^2(0, T; H)$ and $u_0 \in V$, problem* (11.19) *admits at least one solution u on* $[0, T]$.

Proof. For each $t \in [0, T]$, let ψ^t be the proper l.s.c., and convex function on V defined by (11.3) and $\varphi^t(v; z)$ be a (t, v)-dependent functional defined by

$$\varphi^t(v; z) := \frac{1}{2} \int_\Omega a(t, x, (\mathcal{L}v)(t, x)) |\nabla z(x)|^2 dx, \quad \forall t \in [0, T], \ \forall v \in D_0, \ \forall z \in V. \tag{11.20}$$

Then, it is easy to see that (11.19) can be reformulated in the abstract form (QP; f, u_0). Additionally, by arguments similar to [22, Application 2], we verify *Assumptions (A), (B')*, and (2.1) for the duality mapping $F : V \to V^*$. Indeed, as in the proof of Proposition 11.3, we check the following (i)–(iii):

(i) The subdifferential $\partial_* \varphi^t(v; \cdot)$ of $\varphi^t(v; \cdot)$ is given by

$$\langle \partial_* \varphi^t(v; z), w \rangle = \int_\Omega a(t, x, (\mathcal{L}v)(t, x)) \nabla z(x) \cdot \nabla w(x) dx \tag{11.21}$$

for all $t \in [0, T]$, $v \in D_0$, and $z, \ w \in V$.

(ii) (B2') holds. In fact, assuming that $\{v_n\}_{n \in \mathbb{N}} \subset D_0$, $\sup_{n \in \mathbb{N}} \int_0^T \psi^t(v_n'(t)) dt < \infty$, and $v_n \to v$ in $C([0, T]; H)$ as $n \to \infty$, we observe from (11.8) and (11.21) that:

$$\begin{aligned} & |\langle \partial_* \varphi^t(v_n; z) - \partial_* \varphi^t(v; z), w \rangle| \\ \leq & \int_\Omega |a(t, x, (\mathcal{L}v_n)(t, x)) - a(t, x, (\mathcal{L}v)(t, x))| |\nabla z(x)| |\nabla w(x)| dx \\ \leq & L_a \left(\int_\Omega |(\mathcal{L}v_n)(t, x) - (\mathcal{L}v)(t, x)|^2 |\nabla z(x)|^2 dx \right)^{\frac{1}{2}} |w|_V, \\ & \qquad \forall v_n, v \in D_0, \ \forall z, w \in V, \ \forall t \in [0, T]. \end{aligned} \tag{11.22}$$

Here, note from (11.16) and (11.17) that:

$$\begin{aligned} & |(\mathcal{L}v_n)(t, x) - (\mathcal{L}v)(t, x)| \\ \leq & \int_\Omega |\ell(t, x, \xi, v_n(t, \xi)) - \ell(t, x, \xi, v(t, \xi))| d\xi \\ \leq & \int_\Omega L_\ell |v_n(t, \xi) - v(t, \xi)| d\xi \\ \leq & L_\ell |\Omega|^{\frac{1}{2}} |v_n(t) - v(t)|_H, \quad \forall v_n, v \in D_0, \ \forall (t, x) \in [0, T] \times \Omega, \end{aligned} \tag{11.23}$$

where $|\Omega|$ denotes the volume of Ω. Therefore, from (11.17), (11.18), (11.22), (11.23), and the Lebesgue dominated convergence theorem, we infer that (11.22) converges to 0 as $n \to \infty$. Hence, we conclude that $\partial_* \varphi^t(v_n; z) \to \partial_* \varphi^t(v; z)$ in V^* as $n \to \infty$. Thus, (B2') holds.

(iii) Condition (B4') is verified using (11.8), (11.13), (11.16)–(11.18), and (11.20) as follows:

$$
\begin{aligned}
& |\varphi^t(v;z) - \varphi^s(v;z)| \\
\le\ & \frac{1}{2}\int_\Omega |a(t,x,(\mathcal{L}v)(t,x)) - a(s,x,(\mathcal{L}v)(s,x))||\nabla z(x)|^2 dx \\
\le\ & \frac{1}{2}\int_\Omega \int_s^t |a_\tau(\tau,x,(\mathcal{L}v)(\tau,x)) + a_r(\tau,x,(\mathcal{L}v)(\tau,x))(\mathcal{L}v)_\tau(\tau,x)||\nabla z(x)|^2 d\tau dx \\
\le\ & \frac{1}{a_*}(L_a + L_a L_\ell(|\Omega|(1+\bar{\rho}^*)+1))|t-s| \cdot \frac{1}{2}\int_\Omega a(s,x,(\mathcal{L}v)(s,x))|\nabla z(x)|^2 dx \\
=\ & \frac{1}{a_*} L_a(1 + L_\ell(|\Omega|(1+\bar{\rho}^*)+1))|t-s|\varphi^s(v;z), \\
& \qquad \forall v \in D_0,\ \forall z \in V,\ \forall s,t \in [0,T],
\end{aligned}
$$

where $a_\tau := \frac{\partial}{\partial \tau} a(\tau,x,r)$, $a_r := \frac{\partial}{\partial r} a(\tau,x,r)$, and $(\mathcal{L}v)_\tau := \frac{\partial}{\partial \tau}(\mathcal{L}v)(\tau,x)$.

Therefore, by Theorem 6.1, we see that (11.19) admits a solution on $[0,T]$. □

In addition, using calculations similar to (11.22) with (11.23), we observe that:

$$
\begin{aligned}
& |\langle \partial_*\varphi^t(v_1;z) - \partial_*\varphi^t(v_2;z), w\rangle| \\
\le\ & \int_\Omega |a(t,x,(\mathcal{L}v_1)(t,x)) - a(t,x,(\mathcal{L}v_2)(t,x))||\nabla z(x)||\nabla w(x)|dx \\
\le\ & L_a \left(\int_\Omega |(\mathcal{L}v_1)(t,x) - (\mathcal{L}v_2)(t,x)|^2 |\nabla z(x)|^2 dx\right)^{\frac{1}{2}} |w|_V \\
\le\ & L_a L_\ell |\Omega|^{\frac{1}{2}} |v_1(t) - v_2(t)|_H |z|_V |w|_V, \\
& \qquad \forall v_i \in D_0\ (i=1,2),\ \forall z,w \in V,\ \forall t \in [0,T].
\end{aligned}
$$

Therefore, (B5') holds.

Note that, in general, problem (11.19) has multiple solutions on $[0,T]$, so the corresponding optimal control problem is of singular type. We can discuss this by applying Theorem 8.1 with $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$ (cf. Proposition 11.4).

For simplicity, assume $\tau = 0$. In this case, the approximate procedure to (11.19) is performed using the following approximate problems (cf. Theorem 9.1):

$$
\left.
\begin{aligned}
& u_t(t) \in K(t) \quad \text{for a.a. } t \in [0,T], \\
& \varepsilon \int_\Omega \nabla u_t(t,x)\cdot\nabla(u_t(t,x) - v(x))dx \\
& \qquad + \int_\Omega a(t,x,(\mathcal{L}u)(t,x))\nabla u(t,x)\cdot\nabla(u_t(t,x)-v(x))dx \\
& \qquad + \int_\Omega g(t,u(t,x))(u_t(t,x)-v(x))dx \\
& \le \int_\Omega f(t)(u_t(t,x)-v(x))dx + \varepsilon\int_\Omega \nabla h(t)\cdot\nabla(u_t(t,x)-v(x))dx \\
& \qquad \text{for all } v \in K(t) \text{ and a.a. } t \in (0,T), \\
& \qquad u(0,x) = u_0(x), \quad x \in \Omega,
\end{aligned}
\right\} \tag{11.24}
$$

where h is given in $L^2(0,T;V)$ as well as f in $L^2(0,T;H)$ and $u_0 \in V$.

On account of Theorem 9.2, given the admissible control set $\mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ (see Section 5 for the detailed formulation), the approximate optimal control problem $(\mathrm{OP})_{\mathrm{QV},\varepsilon}$ has at least one solution $\{f_\varepsilon^*, h_\varepsilon^*\} \in \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ for each $\varepsilon > 0$ and any weak limit f^* of $\{f_\varepsilon^*\}$ in $L^2(0,T;V^*)$ (as $\varepsilon \to 0$) is an optimal control of $(\mathrm{OP})_{\mathrm{QV}}$.

11.4 Parameter-dependent variational inequality with constraint

In this subsection, we present an application of Theorems 10.1, 10.2, and 10.3.

Let D_0 be the set defined by (11.10). Then, for each $w \in D_0$, we consider the following parameter-dependent variational inequality:

$$\left.\begin{aligned} &u_t(t) \in K(t) \quad \text{for a.a. } t \in (0,T),\\ &\tau \int_\Omega u_t(t,x)(u_t(t,x) - v(x))dx\\ &\quad + \int_\Omega a(t,x,w(t,x))\nabla u(t,x) \cdot \nabla(u_t(t,x) - v(x))dx\\ &\quad + \int_\Omega g(t,u(t,x))(u_t(t,x) - v(x))dx\\ &\le \int_\Omega f(t)(u_t(t,x) - v(x))dx \qquad \text{for all } v \in K(t) \text{ and a.a. } t \in (0,T),\\ &u(0,x) = u_0(x), \quad x \in \Omega, \end{aligned}\right\} \tag{11.25}$$

where $\tau \ge 0$, $K(\cdot)$, $g(\cdot,\cdot)$, f, and u_0 are the same as in Subsection 11.1, the obstacle function ρ satisfies (11.1), and the function $a(t,x,r)$ satisfies (11.8).

A function $u : [0,T] \to V$ is called a solution to (11.25) on $[0,T]$ if $u \in W^{1,2}(0,T;V)$ and all of the properties required in (11.25) are satisfied.

On account of Theorem 10.1, we have the following existence–uniqueness result for problem (11.25).

Proposition 11.6 (cf. Theorem 10.1). *Let $\tau \ge 0$ be a constant. Then, for each $w \in D_0$, $f \in L^2(0,T;H)$, and $u_0 \in V$, problem* (11.25) *admits at least one solution u on* $[0,T]$. *Moreover, if $\tau > 0$, then the solution to* (11.25) *is unique.*

Proof. For each $t \in [0,T]$, let ψ^t be the proper l.s.c., and convex function on V defined by (11.3). Additionally, for each $w \in D_0$, we define the proper l.s.c., and convex function $\varphi^t(w;\cdot)$ on V by

$$\varphi^t(w;z) := \frac{1}{2}\int_\Omega a(t,x,w(t,x))|\nabla z(x)|^2 dx, \quad \forall t \in [0,T],\ \forall w \in D_0,\ \forall z \in V. \tag{11.26}$$

Then, we easily observe that (11.25) can be reformulated in the abstract form $(\mathrm{DP};w,f,u_0)$. Additionally, using arguments similar to Proposition 11.3 (cf. (11.11)–(11.12)), *Assumptions (A), (B)*, and (2.1) are verified. Moreover, if $\tau > 0$, we observe from (11.5) that $\partial_* \psi^t$ is strictly monotone, namely, (2.10) holds.

Therefore, by Theorem 10.1 with Remark 2.4, we have shown that problem (11.25) has a solution on $[0,T]$, and if $\tau > 0$, then it is unique. □

Note that, in the case $\tau = 0$, problem (11.25) generally has multiple solutions, so the corresponding optimal control problem may be of singular type.

We now discuss the singular optimal control problem associated with state (11.25) in the case $\tau = 0$. By applying Theorem 10.2 with $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$, we can consider the singular optimal control problem $(\widetilde{\mathrm{OP}})$ for parameter-dependent variational inequality with constraint (11.25) and control space $\mathcal{W}_{M'} \times \mathcal{F}_M$ (see Section 10 for the detailed formulation).

Proposition 11.7 (cf. Theorem 10.2)**.** *Let u_{ad} be a given function in $L^2(0,T;V)$, $u_0 \in V$, and let $M > 0$, $M' > 0$ be given constants. Assume $\tau = 0$. Then, $(\widetilde{\mathrm{OP}})$ has at least one optimal control $(w^*, f^*) \in \mathcal{W}_{M'} \times \mathcal{F}_M$ such that*

$$\widetilde{J}(w^*, f^*) = \inf_{(w,f)\in\mathcal{W}_{M'}\times\mathcal{F}_M} \widetilde{J}(w, f),$$

where $\widetilde{J}(\cdot,\cdot)$ is the cost functional defined by (10.4).

Using an approach similar to that employed in Sections 11.1 and 11.2, it is possible to propose an approximate parameter-dependent variational inequality for (11.25) and the corresponding approximate optimal control problems. The approximate state problem has a parameter $\varepsilon > 0$, a form denoted by (DP;$w, f + \varepsilon F h, u_0)_\varepsilon$:

$$\left.\begin{array}{c} u_t(t) \in K(t) \quad \text{for a.a. } t \in (0,T), \\ \displaystyle \varepsilon \int_\Omega \nabla u_t(t,x) \cdot \nabla(u_t(t,x) - v(x))dx \\ \displaystyle \quad + \int_\Omega a(t,x,w(t,x))\nabla u(t,x) \cdot \nabla(u_t(t,x) - v(x))dx \\ \displaystyle \quad + \int_\Omega g(t,u(t,x))(u_t(t,x) - v(x))dx \\ \displaystyle \le \int_\Omega f(t)(u_t(t,x) - v(x))dx + \varepsilon \int_\Omega \nabla h(t) \cdot \nabla(u_t(t,x) - v(x))dx \\ \text{for all } v \in K(t) \text{ and a.a. } t \in (0,T), \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{array}\right\} \tag{11.27}$$

Moreover, the approximate optimal control problem $(\widetilde{\mathrm{OP}})_\varepsilon$ is formulated as follows:

Problem $(\widetilde{\mathrm{OP}})_\varepsilon$**:** Find a control $(w_\varepsilon^*, f_\varepsilon^*, h_\varepsilon^*)$ such that

$$\widetilde{J}_\varepsilon(w_\varepsilon^*, f_\varepsilon^*, h_\varepsilon^*) = \inf_{(w,f,h)\in\mathcal{W}_{M'}\times\mathcal{F}_M\times\mathcal{H}_N^\varepsilon} \widetilde{J}_\varepsilon(w, f, h).$$

We see that problem $(\widetilde{\mathrm{OP}})$ is approximated by $(\widetilde{\mathrm{OP}})_\varepsilon$ (see Section 10 for details).

In the rest of this subsection, we consider an application of Theorem 10.3. We define the following functional on $\widehat{\mathcal{W}}_{M'}(u_0) \times \mathcal{F}_M$ for each $\delta \in (0,1]$:

$$\widehat{E}_\delta(w,f) := \inf_{u\in\mathcal{S}(w,f)} \widehat{\pi}^\delta_{(w,f)}(u), \quad \forall (w,f) \in \widehat{\mathcal{W}}_{M'}(u_0) \times \mathcal{F}_M, \tag{11.28}$$

where $\mathcal{S}(w,f)$ is the set of all solutions to (11.25) with $\tau=0$ and, for any $u\in\mathcal{S}(w,f)$,

$$\widehat{\pi}^{\delta}_{(w,f)}(u) := \frac{1}{2}\int_0^T |u(t)-u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f(t)|_{V^*}^2 dt + \frac{1}{2\delta}\int_0^T |u(t)-w(t)|_V^2 dt. \quad (11.29)$$

Now, we denote by $(\widehat{\mathrm{DP}})_\delta$ the optimal control problem for state problem (11.25) with $\tau=0$ and cost functional (11.28) defined on the control space $\widehat{\mathcal{W}}_{M'}(u_0)\times\mathcal{F}_M$, $M'=N_5(|u_0|_V^2+M^2+1)$ (cf. (10.2)). Let (w_δ^*, f_δ^*) be an optimal control of $(\widehat{\mathrm{DP}})_\delta$ for every $\delta\in(0,1]$, and let (w^*,f^*) be any weak limit of $\{(w_\delta^*,f_\delta^*)\}$ as $\delta\downarrow 0$. Then, by Theorem 10.3, f^* is an optimal control of $(\mathrm{OP})_{\mathrm{QV}}$, w^* is the corresponding optimal state, and its optimal value is $\lim_{\delta\downarrow 0}\widehat{E}_\delta(w_\delta^*,f_\delta^*)$. Moreover, w^* is a solution to (11.7) with $\tau=0$.

11.5 Unilateral obstacle problem with time-dependent non-local effect

In this final subsection, we consider a unilateral obstacle problem with a time-dependent non-local effect, which is of a different type from those in Subsections 11.1–11.4.

Let k be a prescribed obstacle function in $C(\overline{Q})$ such that

$$k(t,x)\le 0, \quad \forall (t,x)\in\overline{Q}, \quad (11.30)$$

and, for each $t\in[0,T]$, define a convex constraint set $K_k(t)$ in V by

$$K_k(t) := \{z\in V\ ;\ z(x)\ge k(t,x),\ \text{a.a. } x\in\Omega\}.$$

Additionally, we define the operator $\mathcal{R}: L^2(0,T;H)\to L^2(0,T;H)$ by

$$(\mathcal{R}v)(t,x) := \int_0^t\int_\Omega q(t,x,\tau,\xi,v(\tau,\xi))d\xi d\tau + q_0(t,x) \ \text{ for } v\in L^2(0,T;H) \quad (11.31)$$

with prescribed functions $q\in C(\overline{Q}\times\overline{Q}\times\mathbb{R})$ and $q_0\in C(\overline{Q})$ satisfying the following conditions for positive constants q^* and L_q:

$$\begin{cases} |q(t,x,\tau,\xi,r)|\le q^*, \quad \forall (t,x,\tau,\xi,r)\in\overline{Q}\times\overline{Q}\times\mathbb{R}, \\ |q(t_1,x,\tau,\xi,r_1)-q(t_2,x,\tau,\xi,r_2)|\le L_q\left(|t_1-t_2|+|r_1-r_2|\right), \\ \forall t_i\in[0,T],\ \forall r_i\in\mathbb{R},\ i=1,2,\ \forall (x,\tau,\xi)\in\overline{\Omega}\times\overline{Q}, \end{cases} \quad (11.32)$$

and

$$|q_0(t_1,x)-q_0(t_2,x)|\le L_q|t_1-t_2|, \quad \forall t_i\in[0,T],\ i=1,2,\ \forall x\in\overline{\Omega}. \quad (11.33)$$

Now, consider the following unilateral obstacle problem with a time-dependent non-local effect:

$$\left.\begin{aligned} &u_t(t) \in K_k(t) \quad \text{for a.a. } t \in (0,T), \\ &\int_\Omega \nabla u_t(t,x) \cdot \nabla(u_t(t,x) - v(x))dx \\ &\quad + \int_\Omega a(t,x,(\mathcal{R}u)(t,x))\nabla u(t,x) \cdot \nabla(u_t(t,x) - v(x))dx \\ &\quad + \int_\Omega g(t,u(t,x))(u_t(t,x) - v(x))dx \\ &\le \int_\Omega f(t)(u_t(t,x) - v(x))dx \quad \text{for all } v \in K_k(t) \text{ and a.a. } t \in (0,T), \\ &u(0,x) = u_0(x), \quad x \in \Omega, \end{aligned}\right\} \tag{11.34}$$

where $g(\cdot,\cdot)$ is the same as in Subsection 11.1 and the function $a(t,x,r)$ satisfies (11.8).

A function $u : [0,T] \to V$ is called a solution to (11.34) on $[0,T]$ if $u \in W^{1,2}(0,T;V)$ and all of the properties required in (11.34) are satisfied.

From Theorem 6.1, we obtain the following result for the existence of solutions to problem (11.34).

Proposition 11.8 (cf. Theorem 6.1). *For each $f \in L^2(0,T;H)$ and $u_0 \in V$, problem* (11.34) *admits at least one solution u on* $[0,T]$.

Proof. For each $t \in [0,T]$, we define the proper l.s.c., and convex function ψ^t on V by

$$\psi^t(z) := \frac{1}{2}\int_\Omega |\nabla z(x)|^2 dx + I_{K_k(t)}(z), \quad \forall z \in V \tag{11.35}$$

as well as the (t,v)-dependent functional $\varphi^t(v;\cdot)$ by

$$\varphi^t(v;z) := \frac{1}{2}\int_\Omega a(t,x,(\mathcal{R}v)(t,x))|\nabla z(x)|^2 dx, \quad \forall t \in [0,T],\ \forall v \in D_0,\ \forall z \in V, \tag{11.36}$$

where D_0 is the set defined by (11.10) with $K(t)$ replaced by $K_k(t)$.

In this case, (11.34) can be reformulated in the abstract form (QP; f, u_0) and, as in [22, Lemma 2], we can check *Assumption (A)*. In fact, we have the following:

(i) $z^* \in \partial_* \psi^t(z)$ if and only if $z^* \in V^*$, $z \in K_k(t)$, and

$$\int_\Omega \nabla z(x) \cdot \nabla(z(x) - v(x))dx + \langle -z^*, z - v\rangle \le 0, \quad \forall v \in K_k(t). \tag{11.37}$$

(ii) Assumption (A1) holds. Indeed, because $k \in C(\overline{Q})$, a similar approach as for the proof of [22, Lemma 2] shows that, for any sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ with $t_n \to t$ (as $n \to \infty$),

$$\psi^{t_n} \text{ converges to } \psi^t \text{ on } V \text{ in the sense of Mosco [26] as } n \to \infty.$$

(iii) Assumption (A2) holds by the Poincaré inequality. Moreover, condition (A3) is verified by (11.30).

Additionally, we observe that *Assumption (B')* and (2.1) hold. Indeed, using calculations similar to those in the proof of Proposition 11.5, we have the following:

(i) The subdifferential $\partial_*\varphi^t(v;\cdot)$ of $\varphi^t(v;\cdot)$ is given by

$$\langle \partial_*\varphi^t(v;z), w\rangle = \int_\Omega a(t,x,(\mathcal{R}v)(t,x))\nabla z(x)\cdot\nabla w(x)dx \tag{11.38}$$

for all $t\in[0,T]$, $v\in D_0$, and $z,w\in V$.

(ii) (B2') holds. In fact, assuming $\{v_n\}_{n\in\mathbb{N}}\subset D_0$, $\sup_{n\in\mathbb{N}}\int_0^T\psi^t(v_n'(t))dt<\infty$, and $v_n\to v$ in $C([0,T];H)$ as $n\to\infty$, we observe from (11.8) and (11.38) that:

$$\begin{aligned}
&|\langle \partial_*\varphi^t(v_n;z)-\partial_*\varphi^t(v;z),w\rangle|\\
\le&\int_\Omega |a(t,x,(\mathcal{R}v_n)(t,x))-a(t,x,(\mathcal{R}v)(t,x))||\nabla z(x)||\nabla w(x)|dx\\
\le& L_a\left(\int_\Omega |(\mathcal{R}v_n)(t,x)-(\mathcal{R}v)(t,x)|^2|\nabla z(x)|^2dx\right)^{\frac12}|w|_V, \qquad (11.39)\\
&\qquad \forall v_n,v\in D_0,\ \forall z,w\in V,\ \forall t\in[0,T].
\end{aligned}$$

Here, note from (11.31) and (11.32) that:

$$\begin{aligned}
&|(\mathcal{R}v_n)(t,x)-(\mathcal{R}v)(t,x)|\\
\le&\int_0^t\int_\Omega |q(t,x,\tau,\xi,v_n(\tau,\xi))-q(t,x,\tau,\xi,v(\tau,\xi))|d\xi d\tau\\
\le&\int_0^t\int_\Omega L_q|v_n(\tau,\xi)-v(\tau,\xi)|d\xi d\tau\\
\le& tL_q|\Omega|^{\frac12}|v_n-v|_{C([0,T];H)},\quad \forall v_n,v\in D_0,\ \forall (t,x)\in[0,T]\times\Omega. \qquad (11.40)
\end{aligned}$$

Therefore, (11.32), (11.33), (11.39), (11.40), and the Lebesgue dominated convergence theorem imply that the integral part in (11.39) converges to 0 as $n\to\infty$. Hence, we conclude that $\partial_*\varphi^t(v_n;z)\to\partial_*\varphi^t(v;z)$ in V^* as $n\to\infty$ for all $z\in V$. Thus, (B2') holds.

(iii) Condition (B4') is verified using (11.8), (11.31)–(11.33), and (11.36). Indeed, note from (11.31)–(11.33) that

$$\begin{aligned}
&|(\mathcal{R}v)(t,x)-(\mathcal{R}v)(s,x)|\\
\le\ &\int_s^t\int_\Omega |q(t,x,\tau,\xi,v(\tau,\xi))|d\xi d\tau\\
&\quad+\int_0^s\int_\Omega |q(t,x,\tau,\xi,v(\tau,\xi))-q(s,x,\tau,\xi,v(\tau,\xi))|d\xi d\tau+|q_0(t,x)-q_0(s,x)|\\
\le\ &(q^*|\Omega|+L_qT|\Omega|+L_q)\,|t-s|,\qquad \forall t,s\in[0,T] \text{ with } s\le t,\ \forall x\in\Omega.
\end{aligned}$$

Taking account of the above inequality, we have:

$$
\begin{aligned}
& |\varphi^t(v;z) - \varphi^s(v;z)| \\
\le\ & \frac{1}{2}\int_\Omega |a(t,x,(\mathcal{R}v)(t,x)) - a(s,x,(\mathcal{R}v)(s,x))||\nabla z(x)|^2 dx \\
\le\ & \frac{1}{2}\int_\Omega |a(t,x,(\mathcal{R}v)(t,x)) - a(s,x,(\mathcal{R}v)(t,x))||\nabla z(x)|^2 dx \\
& \qquad + \frac{1}{2}\int_\Omega |a(s,x,(\mathcal{R}v)(t,x)) - a(s,x,(\mathcal{R}v)(s,x))||\nabla z(x)|^2 dx \\
\le\ & \frac{1}{2}\int_\Omega L_a|t-s||\nabla z(x)|^2 dx + \frac{1}{2}\int_\Omega L_a|(\mathcal{R}v)(t,x) - (\mathcal{R}v)(s,x)||\nabla z(x)|^2 dx \\
\le\ & \frac{L_a}{a_*}\left(1 + q^*|\Omega| + L_q T|\Omega| + L_q\right)|t-s| \cdot \frac{1}{2}\int_\Omega a(s,x,(\mathcal{R}v)(s,x))|\nabla z(x)|^2 dx \\
=\ & \frac{L_a}{a_*}\left(1 + q^*|\Omega| + L_q T|\Omega| + L_q\right)|t-s|\varphi^s(v;z), \\
& \qquad\qquad \forall v \in D_0,\ \forall z \in V,\ \forall s,t \in [0,T].
\end{aligned}
$$

Thus, (B4') holds.

Therefore, by Theorem 6.1, we see that (11.34) admits a solution on $[0,T]$. Thus, the proof of Proposition 11.8 is complete. □

Note that problem (11.34) generally has multiple solutions on $[0,T]$, so the corresponding optimal control problem may be of singular type. We can discuss the singular optimal control problem associated with state (11.34) by applying Theorem 8.1 with $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

However, we cannot show (B5') for $\varphi^t(v;z)$ defined by (11.36) (cf. (11.23), (11.40)). Thus, the approximate problems for (11.34) and its optimal control problem are still open, as we need the uniqueness of solutions to the approximate state systems.

References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, Second edition, Pure and Applied Mathematics (Amsterdam), Vol. **140**, Elsevier/Academic Press, Amsterdam, 2003.

[2] G. Akagi, Doubly nonlinear evolution equations with non-monotone perturbations in reflexive Banach spaces, J. Evol. Equ., **11** (2011), 1–41.

[3] T. Arai, On the existence of the solution for $\partial\varphi(u'(t)) + \partial\psi(u(t)) \ni f(t)$, J. Fac. Sci. Univ. Tokyo Sec. IA Math., **26** (1979), 75–96.

[4] M. Aso, M. Frémond and N. Kenmochi, Phase change problems with temperature dependent constraints for the volume fraction velocities, Nonlinear Anal., **60** (2005), 1003–1023.

[5] M. Aso and N. Kenmochi, Quasivariational evolution inequalities for a class of reaction-diffusion systems, Nonlinear Anal., **63** (2005), e1207–e1217.

[6] H. Attouch, *Variational Convergence for Functions and Operators,* Pitman Advanced Publishing Program, Boston-London-Melbourne, 1984.

[7] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach spaces*, Noordhoff, Leyden, 1976.

[8] V. Barbu, *Optimal control of variational inequalities*, Research Notes in Mathematics, **100**, Pitman, London, 1984.

[9] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer Monographs in Mathematics, 2010.

[10] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam-London-New York, 1973.

[11] P. Colli, On some doubly nonlinear evolution equations in Banach spaces, Japan J. Indust. Appl. Math., **9** (1992), 181–203.

[12] P. Colli and A. Visintin, On a class of doubly nonlinear evolution equations, Comm. Partial Differential Equations, **15** (1990), 737–756.

[13] M. H. Farshbaf-Shaker and N. Yamazaki, Optimal control of doubly nonlinear evolution equations governed by subdifferentials without uniqueness of solutions, submitted.

[14] K.-H. Hoffmann, M. Kubo and N. Yamazaki, Optimal control problems for elliptic-parabolic variational inequalities with time-dependent constraints, Numer. Funct. Anal. Optim., **27** (2006), 329–356.

[15] S. Hu and N. S. Papageorgiou, *Time-dependent subdifferential evolution inclusions and optimal control*, Mem. Am. Math. Soc., No.**632**, 1998.

[16] A. Kadoya and N. Kenmochi, Optimal Shape Design in Elliptic-Parabolic Equations, Bull. Fac. Education, Chiba Univ., **41** (1993), 1–20.

[17] A. Kadoya, Y. Murase and N. Kenmochi, A class of nonlinear parabolic systems with environmental constraints, Adv. Math. Sci. Appl., **20** (2010), 281–313.

[18] N. Kenmochi, Some nonlinear parabolic variational inequalities, Israel J. Math., **22** (1975), 304–331.

[19] N. Kenmochi, Solvability of nonlinear evolution equations with time-dependent constraints and applications, Bull. Fac. Education, Chiba Univ., **30** (1981), 1–87.

[20] N. Kenmochi, Monotonicity and compactness methods for nonlinear variational inequalities, *Handbook of Differential Equations, Stationary Partial Differential Equations, Vol.* **4**, ed. M. Chipot, Chapter 4, 203–298, North Holland, Amsterdam, 2007.

[21] N. Kenmochi and I. Pawlow, *A class of nonlinear elliptic-parabolic equations with time-dependent constraints,* Nonlinear Anal., **10** (1986), 1181–1202.

[22] N. Kenmochi, K. Shirakawa and N. Yamazaki, New class of doubly nonlinear evolution equations governed by time-dependent subdifferentials, *Solvability, Regularity, Optimal Control of Boundary Value Problems for PDEs*, P. Colli, A. Favini, E. Rocca, G. Schimperna, J. Sprekels (eds.), Springer INdAM Series, (In Press).

[23] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Gouthiers-Villars, Paris, 1969.

[24] J.-L. Lions, *Optimal control of systems governed by partial differential equations*, Springer-Verlag, New York-Berlin, 1971.

[25] J.-L. Lions, *Contrôle des systèmes distribués singuliers,* Méthodes Mathématiques de l'Informatique, No.**13**, Gauthier-Villars, Montrouge, 1983.

[26] U. Mosco, Convergence of convex sets and of solutions variational inequalities, *Advances Math.,* **3** (1969), 510–585.

[27] Y. Murase, A. Kadoya and N. Kenmochi, Optimal control problems for quasi-variational inequalities and its numerical approximation, *Dynamical Systems, Differential Equations and Applications, 8th AIMS Conference,* Discrete Contin. Dyn. Syst. 2011, Suppl. Vol. II, 1131–1140.

[28] P. Neittaanmäki and D. Tiba, *Optimal control of nonlinear parabolic systems, Theory, algorithms, and applications*, Monographs and Textbooks in Pure and Applied Mathematics, No. **179**, Marcel Dekker, Inc., New York, 1994.

[29] P. Neittaanmäki, J. Sprekels and D. Tiba, *Optimization of Elliptic Systems: Theory and Applications*, Springer Monographs in Mathematics, Springer, New York, 2006.

[30] M. Ôtani, Nonlinear evolution equations with time-dependent constraints, Adv. Math. Sci. Appl., **3** (1993/94), Special Issue, 383–399.

[31] N. S. Papageorgiou, On parametric evolution inclusions of the subdifferential type with applications to optimal control problems, Trans. Am. Math. Soc., **347** (1995), 203–231.

[32] T. Senba, On some nonlinear evolution equation, Funkcial Ekvac., **29** (1986), 243–257.

[33] A. A. Tolstonogov, Continuity in the parameter of the minimum value of an integral functional over the solutions of an evolution control system, Nonlinear Anal., **75** (2012), 4711–4727.

[34] T. Tonn, K. Urban and S. Volkwein, Optimal control of parameter-dependent convection-diffusion problems around rigid bodies, SIAM J. Sci. Comput., **32** (2010), 1237–1260.

[35] Y. Yamada, On evolution equations generated by subdifferential operators, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **23** (1976), 491–515.

[36] N. Yamazaki, Convergence and optimal control problems of nonlinear evolution equations governed by time-dependent operator, Nonlinear Anal., **70** (2009), 4316–4331.