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RELATIVISTIC ENTROPY INEQUALITY

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Abstract. In this paper we apply the entropy principle to the relativistic version of the differential equations describing a standard fluid flow, that is, the equations for mass, momentum, and a system for the energy matrix. These are the second order equations which have been introduced in [3]. Since the principle also says that the entropy equation is a scalar equation, this implies, as we show, that one has to take a trace in the energy part of the system. Thus one arrives at the relativistic mass-momentum-energy system for the fluid. In the procedure we use the well-known Liu-Müller sum [10] in order to deduce the Gibbs relation and the residual entropy inequality.

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1 Introduction

It has been a long history for the entropy principle to come up to the essential differential inequality

$$\sigma := \partial_t\eta + \mathrm{div}_x\psi \geq 0 \tag{1.1}$$

in classical coordinates (t, x), $x = (x_1, \ldots, x_n)$ where $n = 3$ is the physical case. Here η is the entropy and ψ the entropy flux, hence $\underline{\eta} = (\eta, \psi)$ are the total entropy quantities. This principle has been successfully applied to the mass-momentum-energy system in many physical examples. The history started $\approx$ 150 years ago and one can find this principle in many books, among them Prigogine [12, Chapter III], DeGroot & Mazur [4, Chapter III], Truesdell & Noll [14, D.II], Truesdell [13], Ingo Müller [9, Kapitel IV], to mention a few, which were all published in the period 1954-1973. It is part of the entropy principle that the differential equation $\sigma = \partial_t\eta + \mathrm{div}_x\psi$ is an objective scalar equation, see [2, Sec II.3], by which we mean that for the weak equation

$$\int (\partial_t\zeta \cdot \eta + \nabla\zeta \bullet \psi + \zeta \cdot \sigma)\,\mathrm{dL}^{n+1} = 0$$

the test function ζ is an objective scalar, that is $\zeta \circ Y = \zeta^*$, where Y is the observer transformation. This is satisfied, see [2, Sec I.5], if η is an objective scalar, that is $\eta \circ Y = \eta^*$, and ψ satisfies $\psi \circ Y = \eta^* \dot{X} + Q\psi^*$.

In the relativistic case one formulates the entropy principle in the form

$$\sigma := \textstyle\sum_{j\geq 0} \partial_{y_j}\underline{\eta} \geq 0 \tag{1.2}$$

with 4-dimensional coordinates $y = (y_0, y_1, \ldots, y_n)$, again $n = 3$ in the physical case. As postulate we assume that the weak version

$$\int \Big(\sum_{j\geq 0} \partial_{y_j}\zeta \cdot \underline{\eta} + \zeta \cdot \sigma \Big)\,\mathrm{dL}^{n+1} = 0 \tag{1.3}$$

is satisfied for objective test functions ζ, that is $\zeta \circ Y = \zeta^*$. Here Y is a relativistic observer transformation. This is satisfied, see [2, Sec I.5], if the 4-entropy vector $\underline{\eta}$ satisfies $\underline{\eta} \circ Y = \mathrm{D}Y\,\underline{\eta}^*$, that is, $\underline{\eta}$ is a contravariant vector (see the definition below). The relativistic case one finds also in sections of the books of Ingo Müller [10] and Müller & Ruggeri [11].

Here we take advantage of this principle (1.2) and apply it to the relativistic system [3, (10.2)]

$$\sum_{j\geq 0} \partial_{y_j} T_{\alpha j} = g_\alpha \quad \text{for } \alpha \in \{0, \ldots, n\}^N \tag{1.4}$$

which we have developed in [3]. But here we will take it only for $N = 2$, that is, we write $\alpha = (k, l)$ with $k, l \geq 0$, and we use a representation which is made for gases and fluids

$$T_{klj} = (\varrho \underline{v}_k \underline{v}_l + E_{kl})\underline{v}_j + \widetilde{\mathrm{Q}}_{klj} \tag{1.5}$$

where $\underline{v}$ is the four dimensional fluid velocity, see the definition in [3, 5.2], and with the assumptions (4.3) on E and $\widetilde{\mathrm{Q}}$. It should be said that the right-hand side g_{kl} of this system contains the Coriolis coefficients and of course external or internal forces.

Altogether, this system includes the mass-momentum system and a system describing the energy matrix E. The entropy principle for gases and fluids, see section 5 and 6, forces us to perform a trace of the energy matrix equation in order to have an entropy η which is an objective scalar. This method is even new for the classical fluid case. You will find the result in the final theorem s4.7. It contains the statement that the residual inequality $\sigma \geq 0$, that is, the entropy production (4.10) is non-negative. Also as a consequence of the entropy principle there are some important identities. So the entropy $\eta = \widehat{\eta}(\varrho, \varepsilon)$ is a function of the density ϱ and the internal energy

$$\varepsilon = \frac{1}{2}(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P}){\bullet}E\,.$$

And the system (1.4) is specified by two equations, the mass-momentum and the energy equation, see 4.4(2) and 4.5,

$$\sum_{j\geq 0}\partial_{y_j}\left(\varrho\underline{v}_k\underline{v}_j + \underline{v}_k\underline{\mathbf{J}}_j + \underline{\Pi}_{kj}\right) = g_k\,,$$
$$\sum_{j\geq 0}\partial_{y_j}\Big(\big(\frac{\varrho}{2}\,\underline{v}{\bullet}(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})\underline{v} + \varepsilon\big)\,\underline{v}_j + \widetilde{q}_j\Big) = g\,,$$

with

$$\underline{\Pi} := p\,(\mathrm{P}\mathrm{G}\,\mathrm{P}^{\mathrm{T}}) - \underline{S}\,,$$
$$\widetilde{q} := \frac{\varrho}{2}\,(\underline{v}{\bullet}(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})\underline{v})\,\mathbf{J} + \underline{v}(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})\underline{\Pi} + \underline{q}\,,$$

where the inequality restrictions are in the residual inequality $\sigma \geq 0$, see 4.7 for details about the entropy production σ and the total entropy $\underline{\eta}$.

Both, the mass-momentum system and the energy equation are reductions of the equations we started with. The statement 4.7 is the entropy principle in the simplest case. More complicated versions one expects in the case that η might, for example, depend on gradients as in the classical case, or on other vectorial quantities, because the whole system then is more complicated.

Notation: The definition of a contravariant M-tensor $T = (T_{k_1\cdots k_M})_{k_1,\ldots,k_M}$ is

$$T_{k_1\cdots k_M}\circ Y = \sum_{\bar{k}_1,\ldots,\bar{k}_M\geq 0} Y_{k_1\,\prime\bar{k}_1}\cdots Y_{k_M\,\prime\bar{k}_M} T^{*}_{\bar{k}_1\cdots\bar{k}_M}\,, \tag{1.6}$$

and the definition of a covariant M-tensor $T = (T_{k_1\cdots k_M})_{k_1,\ldots,k_M}$

$$T^{*}_{\bar{k}_1\cdots\bar{k}_M} = \sum_{k_1,\ldots,k_M\geq 0} Y_{k_1\,\prime\bar{k}_1}\cdots Y_{k_M\,\prime\bar{k}_M} T_{k_1\cdots k_M}\circ Y\,. \tag{1.7}$$

Here $y = Y(y^{*})$ is the observer transformation.

2 General moments

The version of moments of order less or equal N is

$$\sum_{j\geq 0} \partial_{y_j} T_{\alpha j} = g_\alpha \quad \text{for } \alpha \in \{0,\dots,n\}^N \tag{2.1}$$

with a tensor $T = (T_\beta)_{\beta\in\{0,\dots,n\}^{N+1}}$ which has to be symmetric only in the first N components of the multiindex $\beta = (\beta_1,\dots,\beta_M)$, $M := N+1$, that is, setting $\beta = (\alpha, j)$ as in the equations, $T_{\alpha j}$ and g_α are symmetric in the components of α. Here $y \in \mathcal{U} \subset \mathbb{R}^{n+1}$ where $y = (y_0,\dots,y_n)$ and $n = 3$ in the physical situation. See [3, 10 Higher moments] for more information. System (2.1) is equivalent to the weak version

$$\sum_\alpha \int_{\mathcal{U}} \Big(\sum_{j\geq 0} \partial_{y_j}\zeta_\alpha \cdot T_{\alpha j} + \zeta_\alpha g_\alpha \Big) = 0 \quad \text{for } \zeta_\alpha \in C_0^\infty(\mathcal{U}), \tag{2.2}$$

where the physical type of the system is defined by the fact that the test function $\zeta := (\zeta_\alpha)_\alpha$ is a covariant N-tensor, that is it satisfies the transformation rule

$$\zeta^*_{\bar\alpha} = \sum_\alpha Y_{\alpha_1\,'\bar\alpha_1} \cdots Y_{\alpha_N\,'\bar\alpha_N} \zeta_\alpha \circ Y \,. \tag{2.3}$$

This is satisfied, see [2, Chap I §5], if T satisfies the transformation rule

$$T_\beta \circ Y = \sum_{\bar\beta} Y_{\beta_1\,'\bar\beta_1} \cdots Y_{\beta_M\,'\bar\beta_M} T^*_{\bar\beta} \tag{2.4}$$

and g the transformation rule

$$g_\alpha \circ Y = \sum_{j\geq 0,\bar\alpha} \big(Y_{\alpha_1\,'\bar\alpha_1} \cdots Y_{\alpha_N\,'\bar\alpha_N} \big)_{,j} T^*_{\bar\alpha j} + \sum_{\bar\alpha} Y_{\alpha_1\,'\bar\alpha_1} \cdots Y_{\alpha_N\,'\bar\alpha_N} g^*_{\bar\alpha} \,. \tag{2.5}$$

Here $Y : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is any observer transformation, that is with determinant 1. Do to the special rule (2.5) we define the "Coriolis coefficients" C^β_α by the identity (see [3], for $N = 1$ they are identical with the negative Christoffel symbols)

$$g_\alpha = \underline{\mathbf{f}}_\alpha + \sum_{\beta\in\{0,\dots,n\}^{N+1}} \mathrm{C}^\beta_\alpha T_\beta \quad \text{for } \alpha \in \{0,\dots,n\}^N \tag{2.6}$$

satisfying for all $(\alpha, \bar\gamma, \bar j)$ the transformation rule

$$\begin{aligned} &\sum_{\gamma j} Y_{\gamma_1\,'\bar\gamma_1} \cdots Y_{\gamma_N\,'\bar\gamma_N} Y_{j\,'\bar j}\, \mathrm{C}^{\gamma j}_{\alpha_1\cdots\alpha_N} \circ Y \\ &= \sum_{\bar\alpha} Y_{\alpha_1\,'\bar\alpha_1} \cdots Y_{\alpha_N\,'\bar\alpha_N} \mathrm{C}^{*\bar\gamma\bar j}_{\bar\alpha} + \big(Y_{\alpha_1\,'\bar\gamma_1} \cdots Y_{\alpha_N\,'\bar\gamma_N} \big)_{,\bar j} \,, \end{aligned} \tag{2.7}$$

so that the so-called "force" $\underline{\mathbf{f}} = (\underline{\mathbf{f}}_\alpha)_\alpha$ satisfies the transformation rule

$$\underline{\mathbf{f}}_\alpha \circ Y = \sum_{\bar\alpha} Y_{\alpha_1\,'\bar\alpha_1} \cdots Y_{\alpha_N\,'\bar\alpha_N} \underline{\mathbf{f}}^*_{\bar\alpha} \,. \tag{2.8}$$

Here, as said above, Y is any observer transformation $Y : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. With this the system (2.1) reads

$$\sum_{j\geq 0} \partial_{y_j} T_{\alpha j} - \sum_{\beta\in\{0,\dots,3\}^{N+1}} \mathrm{C}_\alpha^\beta T_\beta = \underline{\mathbf{f}}_\alpha \quad \text{for } \alpha \in \{0,\dots,n\}^N \tag{2.9}$$

were now T and $\underline{\mathbf{f}}$ by (2.4) and (2.8) are contravariant tensors, and the the Coriolis coefficients satisfy (2.7). This is the general form of the system of N-moments. In [3, 10 Higher moments] the following reduction has been performed.

2.1 Reduction. If $\boldsymbol{e}$ is the "time vector", the $(N-1)$-moments system

$$\sum_{j\geq 0} \partial_{y_j} T_{\gamma j} = g_\gamma \quad \text{for } \gamma \in \{0,\dots,n\}^{N-1}$$

is fulfilled for

$$T_{\gamma j} := \sum_{i\geq 0} \boldsymbol{e}_i T_{\gamma ij}\,, \quad g_\gamma := \sum_{i,j\geq 0} \partial_{y_j}\boldsymbol{e}_i \cdot T_{\gamma ij} + \sum_{i\geq 0} \boldsymbol{e}_i g_{\gamma i}$$

This gives also a reduction of the Coriolis coefficients.

Proof. Define the test function of the N-moments system as

$$\zeta_\alpha = \zeta_\gamma \boldsymbol{e}_i \quad \text{for } \alpha = (\gamma, i)\,.$$

That is, if $(\zeta_\gamma)_\gamma$ is a covariant tensor then $(\zeta_\alpha)_\alpha$ is an allowed covariant test function since $\boldsymbol{e}$ is a covariant vector. Then

$$\begin{aligned} 0 &= \int_{\mathbb{R}^4} \sum_\alpha \Big(\sum_j \partial_{y_j}\zeta_\alpha \cdot T_{\alpha j} + \zeta_\alpha\, g_\alpha \Big)\,\mathrm{dL}^4 \\ &= \int_{\mathbb{R}^4} \sum_{\gamma i} \Big(\sum_j \partial_{y_j}(\zeta_\gamma \boldsymbol{e}_i)\cdot T_{\gamma ij} + \zeta_\gamma \boldsymbol{e}_i g_{\gamma i} \Big)\,\mathrm{dL}^4 \\ &= \int_{\mathbb{R}^4} \sum_\gamma \Big(\sum_j \partial_{y_j}\zeta_\gamma \sum_i \boldsymbol{e}_i T_{\gamma ij} + \zeta_\gamma \sum_i \Big(\sum_j \partial_{y_j}\boldsymbol{e}_i \cdot T_{\gamma ij} + \boldsymbol{e}_i g_{\gamma i} \Big)\Big)\,\mathrm{dL}^4\,, \end{aligned}$$

which is the weak reduced equation. □

Due to examples we obtain the following form of the tensor T.

2.2 Special form of $\boldsymbol{T}$. The usual representation of the tensor T is, see for example [3, 10 Higher moments],

$$T_\beta = \varrho \underline{v}_{\beta_1} \cdots \underline{v}_{\beta_M} + \widetilde{\Pi}_\beta\,. \tag{2.10}$$

Here the "4-velocity" $\underline{v}$ is defined as in [3, 5.2 Velocity], that is, as a contravariant vector $\underline{v}$ satisfying

$$\underline{v}_i \circ Y = \sum_{\bar{i}\geq 0} Y_{i'\bar{i}}\, \underline{v}^*_{\bar{i}} \quad \text{for } i \geq 0$$

with the normalization that, with $\boldsymbol{e}$ being the covariant "time vector",

$$\sum_{i\geq 0} \boldsymbol{e}_i \underline{v}_i = 1\,.$$

And ϱ is defined as a "spacetime mass density", which is an objective scalar satisfying $\varrho \circ Y = \varrho^*$. Then the tensor T satisfies (2.4), if $\widetilde{\Pi}$ is also a contravariant tensor.

Here nothing special about $\widetilde{\Pi}$ is said, see e.g. the form in (4.2).

3 Lagrange multipliers

The aim is to derive an entropy inequality. Therefore following Liu & Müller, see the article [6] and the book [9] or the books [10] or [11], and also [2, III §3], we have to find multipliers Λ_α for $\alpha \in \{0,\dots,n\}^N$ which satisfy for "all functions" (that means for a larger set $\mathcal{P}'$ than the set $\mathcal{P}$ of solutions of (2.1))

$$\sum_{j\geq 0} \partial_j \underline{\eta}_j - \sigma = \sum_\alpha \Lambda_\alpha \big(\sum_{j\geq 0} \partial_j T_{\alpha j} - g_\alpha \big) \,, \tag{3.1}$$

where η is the 4-entropy and σ the entropy production. It is part of the entropy principle that $\sum_{j\geq 0} \partial_j \underline{\eta}_j - \sigma$ is an objective scalar, hence in order to have the equation (3.1) it is necessary to state the following lemma. This lemma and the following is true for all values of $(\Lambda_\alpha)_\alpha$.

3.1 Lemma. For the sum

$$\sum_\alpha \Lambda_\alpha \big(\sum_{j\geq 0} \partial_j T_{\alpha j} - g_\alpha \big)$$

being an objective scalar it is sufficient that $(\Lambda_\alpha)_\alpha$ is a covariant N-tensor. *Remark:* Here we make use of (2.6), that is the splitting of g_α.

Proof. Let $(\Lambda_\alpha)_\alpha$ be a covariant N-tensor, that is

$$\Lambda^*_{\bar\alpha} = \sum_\alpha Y_{\alpha_1{}'\bar\alpha_1} \cdots Y_{\alpha_m{}'\bar\alpha_N} \Lambda_\alpha \circ Y \,.$$

We use the splitting in (2.6). Since $(\underline{\mathbf{f}}_\alpha)_\alpha$ is a contravariant N-tensor it follows immediately that

$$\sum_\alpha \Lambda_\alpha \underline{\mathbf{f}}_\alpha$$

is an objective scalar. By (2.6) it remains to consider

$$h_\alpha := \sum_{j\geq 0} \partial_j T_{\alpha j} - \sum_\beta \mathrm{C}^\beta_\alpha T_\beta \,,$$

that is, we have to show that

$$\Big(\sum_\alpha \Lambda_\alpha h_\alpha \Big) \circ Y = \sum_{\bar\alpha} \Lambda^*_{\bar\alpha} h^*_{\bar\alpha} \,. \tag{3.2}$$

If ζ is an objective scalar, that is $\zeta \circ Y = \zeta^*$ hence $\partial_{\bar j} \zeta^* = \sum_j Y_{j\,'\bar j} \, (\partial_j \zeta) \circ Y$, with compact support then

$$\begin{aligned} -\int \zeta \sum_\alpha \Lambda_\alpha h_\alpha \, \mathrm{dL}^4 &= \int \sum_\alpha \Big(\sum_j \partial_j (\zeta \Lambda_\alpha) T_{\alpha j} + \zeta \sum_\beta \Lambda_\alpha \mathrm{C}^\beta_\alpha T_\beta \Big) \, \mathrm{dL}^4 \\ &= \int \Big(\sum_j \partial_j \zeta \cdot \sum_\alpha \Lambda_\alpha T_{\alpha j} + \zeta \Big(\sum_{\alpha j} \partial_j \Lambda_\alpha \cdot T_{\alpha j} + \sum_{\alpha\beta} \Lambda_\alpha \mathrm{C}^\beta_\alpha T_\beta \Big) \Big) \, \mathrm{dL}^4 \,. \end{aligned}$$

First let us treat the last term

$$\sum_{\alpha j} \partial_j \Lambda_\alpha \cdot T_{\alpha j} + \sum_{\alpha\beta} \Lambda_\alpha \mathrm{C}^\beta_\alpha T_\beta \,.$$

Since $(\Lambda_\alpha)_\alpha$ is a covariant N-tensor, we compute for the derivatives

$$\partial_{\bar{j}}\Lambda^*_{\bar{\alpha}} = \sum_\alpha \partial_{\bar{j}}(Y_{\alpha_1{}'\bar{\alpha}_1}\cdots Y_{\alpha_N{}'\bar{\alpha}_N})\Lambda_\alpha\circ Y + \sum_{\alpha j} Y_{\alpha_1{}'\bar{\alpha}_1}\cdots Y_{\alpha_N{}'\bar{\alpha}_N} Y_{j\,'\bar{j}}\partial_j\Lambda_\alpha\circ Y\,.$$

Now, using (2.7) for the Coriolis coefficients,

$$\begin{aligned}\sum_{\bar{\alpha}\bar{\gamma}\bar{j}} \Lambda^*_{\bar{\alpha}}\mathrm{C}^{*\bar{\gamma}\bar{j}}_{\bar{\alpha}} T^*_{\bar{\gamma}\bar{j}} &= \sum_{\alpha\bar{\alpha}\bar{\gamma}\bar{j}} \Lambda_\alpha\circ Y\, Y_{\alpha_1{}'\bar{\alpha}_1}\cdots Y_{\alpha_N{}'\bar{\alpha}_N}\mathrm{C}^{*\bar{\gamma}\bar{j}}_{\bar{\alpha}} T^*_{\bar{\gamma}\bar{j}}\\ &= \sum_{\alpha\bar{\gamma}\bar{j}\gamma j} \Lambda_\alpha\circ Y Y_{\gamma_1{}'\bar{\gamma}_1}\cdots Y_{\gamma_N{}'\bar{\gamma}_N} Y_{j\,'\bar{j}}\,\mathrm{C}^{\gamma j}_{\alpha_1\cdots\alpha_N}\circ Y\, T^*_{\bar{\gamma}\bar{j}}\\ &\qquad - \sum_{\alpha\bar{\gamma}\bar{j}} \Lambda_\alpha\circ Y\partial_{\bar{j}}(Y_{\alpha_1{}'\bar{\gamma}_1}\cdots Y_{\alpha_N{}'\bar{\gamma}_N})T^*_{\bar{\gamma}\bar{j}}\,,\end{aligned}$$

and therefore, using that T is an contravariant $(N+1)$-tensor,

$$\begin{aligned}&\sum_{\bar{\alpha}\bar{j}} \partial_{\bar{j}}\Lambda^*_{\bar{\alpha}}\cdot T^*_{\bar{\alpha}\bar{j}} + \sum_{\bar{\alpha}\bar{\gamma}\bar{j}} \Lambda^*_{\bar{\alpha}}\mathrm{C}^{*\bar{\gamma}\bar{j}}_{\bar{\alpha}} T^*_{\bar{\gamma}\bar{j}}\\ &= \sum_{\bar{\alpha}\bar{j}\alpha j} Y_{\alpha_1{}'\bar{\alpha}_1}\cdots Y_{\alpha_N{}'\bar{\alpha}_N} Y_{j\,'\bar{j}}\partial_j\Lambda_\alpha\circ Y\, T^*_{\bar{\alpha}\bar{j}}\\ &\qquad + \sum_{\alpha\bar{\alpha}\bar{j}} \Lambda_\alpha\circ Y\partial_{\bar{j}}(Y_{\alpha_1{}'\bar{\alpha}_1}\cdots Y_{\alpha_N{}'\bar{\alpha}_N})T^*_{\bar{\alpha}\bar{j}} + \sum_{\bar{\alpha}\bar{\gamma}\bar{j}} \Lambda^*_{\bar{\alpha}}\mathrm{C}^{*\bar{\gamma}\bar{j}}_{\bar{\alpha}} T^*_{\bar{\gamma}\bar{j}}\\ &= \big(\sum_{\alpha j}\partial_j\Lambda_\alpha\cdot T_{\alpha j}\big)\circ Y\\ &\qquad + \sum_{\alpha\bar{\gamma}\bar{j}\gamma j} \Lambda_\alpha\circ Y Y_{\gamma_1{}'\bar{\gamma}_1}\cdots Y_{\gamma_N{}'\bar{\gamma}_N} Y_{j\,'\bar{j}}\,\mathrm{C}^{\gamma j}_{\alpha_1\cdots\alpha_N}\circ Y\, T^*_{\bar{\gamma}\bar{j}}\\ &= \big(\sum_{\alpha j}\partial_j\Lambda_\alpha\cdot T_{\alpha j} + \sum_{\alpha\gamma j}\Lambda_\alpha \mathrm{C}^{\gamma j}_\alpha T_{\gamma j}\big)\circ Y\,,\end{aligned}$$

The term with the derivative of the test function is obviously

$$\begin{aligned}\sum_{\bar{j}}\partial_{\bar{j}}\zeta^*\sum_{\bar{\alpha}}\Lambda^*_{\bar{\alpha}}T^*_{\bar{\alpha}\bar{j}} &= \sum_{j\bar{j}} Y_{j\,'\bar{j}}\partial_j\zeta\circ Y\sum_{\bar{\alpha}}\Lambda^*_{\bar{\alpha}}T^*_{\bar{\alpha}\bar{j}}\\ &= \sum_j \partial_j\zeta\circ Y\sum_{\bar{\alpha}} Y_{j\,'\bar{j}}\Lambda^*_{\bar{\alpha}}T^*_{\bar{\alpha}\bar{j}} = \Big(\sum_j\partial_j\zeta\sum_\alpha\Lambda_\alpha T_{\alpha j}\Big)\circ Y\,,\end{aligned}$$

so that altogether

$$\int\zeta\sum_\alpha\Lambda_\alpha h_\alpha\,\mathrm{dL}^4 = \int\zeta^*\sum_{\bar{\alpha}}\Lambda^*_{\bar{\alpha}}h^*_{\bar{\alpha}}\,\mathrm{dL}^4$$

hence (3.2) is satisfied. □

We now use the elements of the dual basis

$$\{e'_0(y), e'_1(y), \ldots, e'_n(y)\}\subset\mathbb{R}^{n+1}\,,\ \text{it is } \boldsymbol{e} = e'_0,$$
$$\{e_0(y), e_1(y), \ldots, e_n(y)\}\subset\mathbb{R}^{n+1} \text{ with } e'_k\bullet e_l = \delta_{kl}.$$

It is known that $\{e_1(y), \ldots, e_n(y)\} = \boldsymbol{W}(y) = \{e'_0(y)\}^\perp$, see [3, 3 Time and space]. General physical statements about fluids depend only on the vector $\boldsymbol{e}(y) = e'_0(y)$ or $\boldsymbol{W}(y)$ and not on single vectors $e_i(y)$, $i \geq 1$ (as for example the space directions of crystals or the director of liquid crystals). But we are allowed to use these vectors in proofs. In doing so we introduce values $(\lambda_\gamma)_\gamma$:

3.2 Definition. We define

$$\lambda_\gamma = \sum_\alpha \Lambda_\alpha e_{\gamma_1\alpha_1} \cdots e_{\gamma_N\alpha_N} \quad \text{or} \quad \Lambda_\alpha = \sum_\gamma \lambda_\gamma e'_{\gamma_1\alpha_1} \cdots e'_{\gamma_N\alpha_N} \,.$$

The new values λ_γ are objective scalars.

Proof. By this definition $(\lambda_\gamma)_\gamma$ and $(\Lambda_\alpha)_\alpha$ are equivalent quantities. If Λ_α are given as stated we conclude

$$\begin{aligned}\sum_\alpha \Lambda_\alpha e_{\delta_1\alpha_1} \cdots e_{\delta_N\alpha_N} &= \sum_{\alpha,\gamma} \lambda_\gamma e'_{\gamma_1\alpha_1} e_{\delta_1\alpha_1} \cdots e'_{\gamma_N\alpha_N} e_{\delta_N\alpha_N} \\ &= \sum_\gamma \lambda_\gamma e'_{\gamma_1} \bullet e_{\delta_1} \cdots e'_{\gamma_N} \bullet e_{\delta_N} = \sum_\gamma \lambda_\gamma \delta_{\gamma_1,\delta_1} \cdots \delta_{\gamma_N,\delta_N} = \lambda_\delta \,.\end{aligned}$$

Similar the other way around. □

Since we are in the proof of the main theorem we introduce an equivalent system to the given one presented by the terms $\sum_j \partial_{y_j} T_{\alpha j} - g_\alpha$. The new system is given by the terms $\sum_j \partial_{y_j} T'_{\gamma j} - \mathbf{r}'_\gamma$.

3.3 Equivalent system. For any vectors $(\Lambda_\alpha)_\alpha$ or $(\lambda_\gamma)_\gamma$ as in 3.2

$$\sum_\alpha \Lambda_\alpha \big(\sum_{j\geq 0} \partial_j T_{\alpha j} - g_\alpha \big) = \sum_\gamma \lambda_\gamma \big(\sum_{j\geq 0} \partial_j T'_{\gamma j} - \mathbf{r}'_\gamma \big) \,,$$

$$T'_{\gamma j} := \sum_\alpha e'_{\gamma_1\alpha_1} \cdots e'_{\gamma_N\alpha_N} T_{\alpha j} \quad \text{and} \quad \mathbf{r}'_\gamma := \sum_\alpha e'_{\gamma_1\alpha_1} \cdots e'_{\gamma_N\alpha_N} \underline{\mathbf{f}}_\alpha \,.$$

For each γ the vector $\big(T'_{\gamma j}\big)_{j\geq 0}$ is a covariant vector and $\mathbf{r}'_\gamma$ is an objective scalar.

Since only $\underline{\mathbf{f}}_\alpha$ enter in the definition of $\mathbf{r}'_\gamma$, it means that during the process of computation in the Liu & Müller sum the fictitious forces drop out, that is, they do not enter the entropy principle.

Proof. The definition 3.2 and the definition of $\mathbf{r}'_\gamma$ implies

$$\sum_\alpha \Lambda_\alpha \underline{\mathbf{f}}_\alpha = \sum_{\alpha,\gamma} \lambda_\gamma e'_{\gamma_1\alpha_1} \cdots e'_{\gamma_N\alpha_N} \underline{\mathbf{f}}_\alpha = \sum_\gamma \lambda_\gamma \mathbf{r}'_\gamma \,.$$

And it is

$$\begin{aligned}\mathbf{r}'_\gamma \circ Y &= \sum_\alpha e'_{\gamma_1\alpha_1} \circ Y \cdots e'_{\gamma_N\alpha_N} \circ Y \, \mathbf{f}_\alpha \circ Y \\ &= \sum_{\alpha,\bar\alpha} e'_{\gamma_1\alpha_1} \circ Y Y_{\alpha_1{}'\bar\alpha_1} \cdots e'_{\gamma_N\alpha_N} \circ Y Y_{\alpha_N{}'\bar\alpha_N} \, \mathbf{f}^*_{\bar\alpha} \\ &= \sum_{\bar\alpha} e'^*_{\gamma_1\bar\alpha_1} \cdots e'^*_{\gamma_N\bar\alpha_N} \mathbf{f}^*_{\bar\alpha} = \mathbf{r}'^*_\gamma \,.\end{aligned}$$

Therefore, by (2.6), with

$$h_\alpha := \sum_j \partial_j T_{\alpha j} - \sum_\beta \mathrm{C}^\beta_\alpha T_\beta \quad \text{and} \quad h'_\gamma := \sum_j \partial_j T'_{\gamma j}$$

we have to show that

$$\sum_\alpha \Lambda_\alpha h_\alpha = \sum_\gamma \lambda_\gamma h'_\gamma\,. \tag{3.3}$$

Now

$$\begin{aligned}\sum_{\alpha,j} \Lambda_\alpha \partial_j T_{\alpha j} &= \sum_{\alpha,\gamma,j} \lambda_\gamma e'_{\gamma_1\alpha_1}\cdots e'_{\gamma_N\alpha_N}\partial_j T_{\alpha j}\\ &= \sum_{\gamma,j}\lambda_\gamma \partial_j T'_{\gamma j} - \sum_{\alpha,\gamma,j}\lambda_\gamma \partial_j(e'_{\gamma_1\alpha_1}\cdots e'_{\gamma_N\alpha_N})T_{\alpha j}\end{aligned}$$

and

$$\sum_{\alpha,\beta}\Lambda_\alpha \mathrm{C}^\beta_\alpha T_\beta = \sum_{\alpha,\delta,j}\Lambda_\delta \mathrm{C}^{\alpha j}_\delta T_{\alpha j} = \sum_{\alpha,\gamma,j}\lambda_\gamma\Big(\sum_\delta e'_{\gamma_1\delta_1}\cdots e'_{\gamma_N\delta_N}\mathrm{C}^{\alpha j}_\delta\Big)T_{\alpha j},$$

therefore

$$\begin{aligned}\sum_\alpha \Lambda_\alpha\Big(\sum_j \partial_j T_{\alpha j} - \sum_\beta \mathrm{C}^\beta_\alpha T_\beta\Big) &= \sum_{\alpha,j}\Lambda_\alpha\partial_j T_{\alpha j} - \sum_{\alpha,\beta}\Lambda_\alpha \mathrm{C}^\beta_\alpha T_\beta\\ = \sum_{\gamma,j}\lambda_\gamma\partial_j T'_{\gamma j} - \sum_{\alpha,\gamma,j}\lambda_\gamma\Big(\partial_j(e'_{\gamma_1\alpha_1}\cdots e'_{\gamma_N\alpha_N}) &+ \sum_\delta e'_{\gamma_1\delta_1}\cdots e'_{\gamma_N\delta_N}\mathrm{C}^{\alpha j}_\delta\Big)T_{\alpha j}\end{aligned}$$

and for all (α,γ,j)

$$\partial_j(e'_{\gamma_1\alpha_1}\cdots e'_{\gamma_N\alpha_N}) + \sum_\delta e'_{\gamma_1\delta_1}\cdots e'_{\gamma_N\delta_N}\mathrm{C}^{\alpha j}_\delta = 0\,, \tag{3.4}$$

since by the following theorem 3.4

$$\begin{aligned}&-\sum_\delta e'_{\gamma_1\delta_1}\cdots e'_{\gamma_N\delta_N}\mathrm{C}^{\alpha j}_\delta\\ &= \sum_{\delta,\beta} e'_{\gamma_1\delta_1}e_{\beta_1\delta_1}\cdots e'_{\gamma_N\delta_N}e_{\beta_N\delta_N}\partial_j(e'_{\beta_1\alpha_1}\cdots e'_{\beta_N\alpha_N})\\ &= \sum_\beta \delta_{\gamma,\beta}\partial_j(e'_{\beta_1\alpha_1}\cdots e'_{\beta_N\alpha_N}) = \partial_j(e'_{\gamma_1\alpha_1}\cdots e'_{\gamma_N\alpha_N})\,,\end{aligned}$$

□

3.4 Theorem. For every (α,γ,j)

$$\mathrm{C}^{\gamma j}_\alpha = -\sum_\beta e_{\beta_1\alpha_1}\cdots e_{\beta_N\alpha_N}\partial_j(e'_{\beta_1\gamma_1}\cdots e'_{\beta_N\gamma_N})\,,$$

since this is true for at least one observer.
Remark: Usually true for "inertial systems".

Proof. The transformation rule for $B^{\gamma j}_\alpha := -\mathrm{C}^{\gamma j}_\alpha$ is according to (2.7)

$$\begin{aligned}\sum_{\bar\alpha} Y_{\alpha_1\,'\bar\alpha_1}\cdots Y_{\alpha_N\,'\bar\alpha_N}B^{*\bar\gamma\bar j}_{\bar\alpha} = \sum_{\gamma,j} & Y_{\gamma_1\,'\bar\gamma_1}\cdots Y_{\gamma_N\,'\bar\gamma_N}Y_{j\,'\bar j}\,B^{\gamma j}_{\alpha_1\cdots\alpha_N}\circ Y\\ &+\big(Y_{\alpha_1\,'\bar\gamma_1}\cdots Y_{\alpha_N\,'\bar\gamma_N}\big)_{,\bar j}\,.\end{aligned} \tag{3.5}$$

Now set

$$B^{\gamma j}_\alpha := \sum_\beta e_{\beta_1\alpha_1}\cdots e_{\beta_N\alpha_N}\partial_j(e'_{\beta_1\gamma_1}\cdots e'_{\beta_N\gamma_N})\,.$$

Since, see [3, 4 Change of observer] and 3.5 below,

$$e_{kl}\circ Y = \textstyle\sum_{\bar l\ge 0} Y_{l\,'\bar l}\, e^{*}_{k\bar l} \quad \text{for } k,l\ge 0, \tag{3.6}$$

$$e'^{*}_{k\bar l} = \textstyle\sum_{l\ge 0} Y_{l\,'\bar l}\, e'^{*}_{kl}\circ Y \quad \text{for } k,\bar l\ge 0, \tag{3.7}$$

$$\textstyle\sum_{m\ge 0} e_{mk} e'_{ml} = \delta_{k,l} \quad \text{for } k,l\ge 0, \tag{3.8}$$

we compute for $(\alpha,\bar\gamma,\bar j)$

$$\begin{aligned}
&\sum_{\bar\alpha} Y_{\alpha_1\,'\bar\alpha_1}\cdots Y_{\alpha_N\,'\bar\alpha_N} B^{*\bar\gamma\bar j}_{\bar\alpha}\\
&= \sum_{\bar\alpha,\beta} Y_{\alpha_1\,'\bar\alpha_1} e^{*}_{\beta_1\bar\alpha_1}\cdots Y_{\alpha_N\,'\bar\alpha_N} e^{*}_{\beta_N\bar\alpha_N}\partial_{\bar j}(e'^{*}_{\beta_1\bar\gamma_1}\cdots e'^{*}_{\beta_N\bar\gamma_N})\\
&= \sum_{\beta} e_{\beta_1\alpha_1}\circ Y\cdots e_{\beta_N\alpha_N}\circ Y\,\partial_{\bar j}(e'^{*}_{\beta_1\bar\gamma_1}\cdots e'^{*}_{\beta_N\bar\gamma_N}) && \text{(see (3.6))}\\
&= \sum_{\beta,\gamma} e_{\beta_1\alpha_1}\circ Y\cdots e_{\beta_N\alpha_N}\circ Y\cdot\\
&\qquad\cdot\,\partial_{\bar j}\big(Y_{\gamma_1\,'\bar\gamma_1} e'_{\beta_1\gamma_1}\circ Y\cdots Y_{\gamma_N\,'\bar\gamma_N} e'_{\beta_N\gamma_N}\circ Y\big) && \text{(see (3.7))}\\
&= \sum_{\beta,\gamma} e_{\beta_1\alpha_1}\circ Y\cdots e_{\beta_N\alpha_N}\circ Y\, Y_{\gamma_1\,'\bar\gamma_1}\cdots Y_{\gamma_N\,'\bar\gamma_N}\partial_{\bar j}(e'_{\beta_1\gamma_1}\circ Y\cdots e'_{\beta_N\gamma_N}\circ Y)\\
&\quad+\sum_{\beta,\gamma} e_{\beta_1\alpha_1}\circ Y\, e'_{\beta_1\gamma_1}\circ Y\cdots e_{\beta_N\alpha_N}\circ Y\, e'_{\beta_N\gamma_N}\circ Y\,\partial_{\bar j}(Y_{\gamma_1\,'\bar\gamma_1}\cdots Y_{\gamma_N\,'\bar\gamma_N})\\
&= \sum_{\beta,\gamma} Y_{\gamma_1\,'\bar\gamma_1}\cdots Y_{\gamma_N\,'\bar\gamma_N}\, e_{\beta_1\alpha_1}\circ Y\cdots e_{\beta_N\alpha_N}\circ Y\,\partial_{\bar j}(e'_{\beta_1\gamma_1}\circ Y\cdots e'_{\beta_N\gamma_N}\circ Y)\\
&\qquad\qquad+\sum_{\gamma}\delta_{\alpha,\gamma}\,\partial_{\bar j}(Y_{\gamma_1\,'\bar\gamma_1}\cdots Y_{\gamma_N\,'\bar\gamma_N}) && \text{(see (3.8))}\\
&= \sum_{\gamma,j} Y_{\gamma_1\,'\bar\gamma_1}\cdots Y_{\gamma_N\,'\bar\gamma_N} Y_{j\,'\bar j}\, B^{\gamma j}_{\alpha_1\cdots\alpha_N}\circ Y + \partial_{\bar j}\big(Y_{\alpha_1\,'\bar\gamma_1}\cdots Y_{\alpha_N\,'\bar\gamma_N}\big)\,,
\end{aligned}$$

since

$$\partial_{\bar j}(e'_{\beta_1\gamma_1}\circ Y\cdots e'_{\beta_N\gamma_N}\circ Y) = \sum_j Y_{j\,'\bar j}\partial_j(e'_{\beta_1\gamma_1}\cdots e'_{\beta_N\gamma_N})\circ Y\,,$$

hence B satisfies (3.5). Therefore the difference

$$\widetilde B^{\gamma j}_{\alpha} := \mathrm{C}^{\gamma j}_{\alpha} + \sum_{\beta} e_{\beta_1\alpha_1}\cdots e_{\beta_N\alpha_N}\partial_j(e'_{\beta_1\gamma_1}\cdots e'_{\beta_N\gamma_N})$$

satisfies the transformation rule

$$\sum_{\bar\alpha} Y_{\alpha_1\,'\bar\alpha_1}\cdots Y_{\alpha_N\,'\bar\alpha_N}\widetilde B^{*\bar\gamma\bar j}_{\bar\alpha} = \sum_{\gamma,j} Y_{\gamma_1\,'\bar\gamma_1}\cdots Y_{\gamma_N\,'\bar\gamma_N} Y_{j\,'\bar j}\,\widetilde B^{\gamma j}_{\alpha_1\cdots\alpha_N}\circ Y$$

which is homogeneous and therefore we can choose $\widetilde B = 0$. □

3.5 Lemma. Because $\{e_k\ ;\ k\ge 0\}$ and $\{e'_k\ ;\ k\ge 0\}$ are dual basis we know that $\delta_{k,l} = e'_k\bullet e_l = \sum_m e'_{km}e_{lm}$. It also implies that $\sum_m e'_{mk}e_{ml} = \delta_{k,l}$.

Proof. Define $E_{mk} := e_{mk} = e_m \bullet \mathbf{e}_k$ and $E'_{lm} = e'_{lm} = e'_l \bullet \mathbf{e}_m$. Then

$$\delta_{k,l} = e'_k \bullet e_l = \sum_m e'_{km} e_{lm} = (E' E^{\mathrm{T}})_{kl} ,$$

hence $E' E^{\mathrm{T}} = \mathrm{Id}$ and thus $E'(E^{\mathrm{T}} E' - \mathrm{Id}) = (E' E^{\mathrm{T}})E' - E' = E' - E' = 0$. Therefore, since E' is bijective, $E^{\mathrm{T}} E' - \mathrm{Id} = 0$, that is $E^{\mathrm{T}} E' = \mathrm{Id}$, which means

$$\delta_{l,k} = (E^{\mathrm{T}} E')_{lk} = \sum_m e_{ml} e'_{mk} ,$$

which is the assertion. □

4 The entropy theorem

We start with the general system (2.1) in the special case $N = 2$

$$\sum_{j \geq 0} \partial_{y_j} T_{klj} = g_{kl} \quad \text{for } k, l \geq 0, \quad g_{kl} := \underline{\mathbf{f}}_{kl} + \sum_\beta \mathrm{C}^\beta_{kl} T_\beta \tag{4.1}$$

by writing the multiindex $\alpha = (k, l)$ for $k, l \geq 0$, and where all quantities are symmetric in k and l. The system (4.1) has by definition covariant test functions, and this is satisfied if T, $\mathbf{f}$, and C satisfy the transformation rules which we have stated in (2.4), (2.8), and (2.7). We shall consider a simple fluid which is defined by the following representation of the tensor components T_{klj} for $k, l, j \geq 0$

$$T_{klj} = \varrho \underline{v}_k \underline{v}_l \underline{v}_j + E_{kl} \underline{v}_j + \widetilde{\mathrm{Q}}_{klj} , \tag{4.2}$$

see 2.2, where also the properties of the mass density ϱ and the 4-velocity $\underline{v}$ are stated. The terms in (4.2) are independent fron each other by assuming that with the "time vector" $\boldsymbol{e}$

$$\sum_{k \geq 0} \boldsymbol{e}_k E_{kl} = 0 , \quad \sum_{j \geq 0} \boldsymbol{e}_j \widetilde{\mathrm{Q}}_{klj} = 0 . \tag{4.3}$$

The usage of the time vector $\boldsymbol{e}$ says that the "time component" of E is zero and that $\widetilde{\mathrm{Q}}$ has no "time derivative". The system (4.1) therefore can be considered as the mass-momentum-energymatrix system.

In 2.1 we have defined a reduced system of (4.1) via the covariant vector $\boldsymbol{e}$. This reduced system is the mass-momentum system

$$\begin{gathered} \sum_{j \geq 0} \partial_{y_j} T_{kj} = g_k \quad \text{for } k \geq 0 , \\ T_{kj} := \sum_{l \geq 0} \boldsymbol{e}_l T_{klj} = \varrho \underline{v}_k \underline{v}_j + \widetilde{\Pi}_{kj} , \quad \widetilde{\Pi}_{kj} := \sum_{l \geq 0} \boldsymbol{e}_l \widetilde{\mathrm{Q}}_{klj} , \\ g_k := \sum_{l,j \geq 0} \partial_{y_j} \boldsymbol{e}_l \cdot T_{klj} + \sum_{l \geq 0} \boldsymbol{e}_l g_{kl} . \end{gathered} \tag{4.4}$$

Similarly, defined as a reduction of (4.4) there is the mass equation

$$\begin{gathered} \sum_{j \geq 0} \partial_{y_j} T_j = g , \\ T_j := \sum_{k \geq 0} \boldsymbol{e}_k T_{kj} = \varrho \underline{v}_j + \underline{\mathbf{J}}_j , \quad \underline{\mathbf{J}}_j := \sum_{k \geq 0} \boldsymbol{e}_k \widetilde{\Pi}_{kj} , \\ g := \sum_{k,j \geq 0} \partial_{y_j} \boldsymbol{e}_k \cdot T_{kj} + \sum_{k \geq 0} \boldsymbol{e}_k g_k . \end{gathered} \tag{4.5}$$

Realize that we can also write

$$T_j = \sum_{k,l\geq 0} \boldsymbol{e}_k\boldsymbol{e}_l T_{klj}\,, \quad g = \sum_{k,l,j\geq 0} \partial_{y_j}(\boldsymbol{e}_k\boldsymbol{e}_l)\cdot T_{klj} + \sum_{k,l\geq 0} \boldsymbol{e}_k\boldsymbol{e}_l g_{kl}\,,$$

and that assumption (4.3) for $\widetilde{\mathrm{Q}}$ implies that $\underline{\mathbf{J}}$ and $\widetilde{\Pi}$ satisfy

$$\sum_{j\geq 0} \boldsymbol{e}_j \mathbf{J}_j = 0\,, \quad \sum_{j\geq 0} \boldsymbol{e}_j \widetilde{\Pi}_{kj} = 0 \text{ for all } k\geq 0.$$

What is left from (4.1), after one has determined the reduced mass-momentum system (4.4), is an equation

$$\sum_{j\geq 0} \partial_j T^E_{klj} = g^E_{kl} \quad \text{for } k,l\geq 0\,, \tag{4.6}$$

which is given in the next statement where the vector e_0 satisfies

$$\mathrm{G}e_0' = -\frac{1}{\mathrm{c}^2}e_0\,, \quad \boldsymbol{e} = e_0'\,, \tag{4.7}$$

see [3, Theorem 3.4].

4.1 Remaining system. If we define the in k and l symmetric terms by

$$\begin{aligned} T^E_{klj} &:= T_{klj} - e_{0k}T_{lj} - e_{0l}T_{kj} + e_{0k}e_{0l}T_j\,,\\ g^E_{kl} &:= g_{kl} - \sum_{j\geq 0}\partial_j(e_{0k}T_{lj} + e_{0l}T_{kj}) + \sum_{j\geq 0}\partial_j(e_{0k}e_{0l}T_j)\,. \end{aligned}$$

then system (4.6) is fulfilled. For these system the reduction is zero.
Remark: There are also different representations for g^E_{kl}, see the proof.

Proof. We have

$$\begin{aligned} \sum_{j\geq 0}\partial_j T^E_{klj} &= \sum_{j\geq 0}\partial_j T_{klj} - \sum_{j\geq 0}\partial_j(e_{0k}T_{lj} + e_{0l}T_{kj}) + \sum_{j\geq 0}\partial_j(e_{0k}e_{0l}T_j)\\ &= g_{kl} - \sum_{j\geq 0}\partial_j(e_{0k}T_{lj} + e_{0l}T_{kj}) + \sum_{j\geq 0}\partial_j(e_{0k}e_{0l}T_j) = g^E_{kl}\,, \end{aligned}$$

so that (4.6) is satisfied. Now, since $\boldsymbol{e} = e_0'$ and $e_0'{\bullet}e_0 = 1$ it follows

$$\sum_{k\geq 0} e'_{0k}T^E_{klj} = \Big(\sum_{k\geq 0} e'_{0k}T_{klj} - T_{lj}\Big) - e_{0l}\Big(\sum_{k\geq 0} e'_{0k}T_{kj} - T_j\Big) = 0\,,$$

because by the above reduction

$$\sum_{k\geq 0} e'_{0k}T_{klj} - T_{lj} = 0\,, \quad \sum_{k\geq 0} e'_{0k}T_{kj} - T_j = 0\,.$$

If we now show that for any k

$$\sum_{l,j\geq 0}\partial_j e'_{0l}\cdot T^R_{klj} + \sum_{l\geq 0} e'_{0l}g^R_{kl} \tag{4.8}$$

is equal to 0, it follows that the reduction of (4.6) vanishes. To prove this we write the above identity for g^E as

$$\begin{aligned} g^E_{kl} &= g_{kl} - e_{0l}g_k - e_{0k}g_l + e_{0k}e_{0l}g \\ &\quad - \sum_{j\geq 0}\partial_j e_{0l}\cdot T_{kj} - \sum_{j\geq 0}\partial_j e_{0k}\cdot T_{lj} + \sum_{j\geq 0}\partial_j(e_{0k}e_{0l})\,T_j\,. \end{aligned}$$

Using this and the above identity for T^E_{klj}, making use of $e'_0{\bullet}e_0 = 1$, we obtain for the term in (4.8)

$$\begin{aligned} \sum_{l,j\geq 0}\partial_j e'_{0l}\cdot T^R_{klj} + \sum_{l\geq 0} e'_{0l}g^R_{kl} &= \sum_{l,j\geq 0}\partial_j e'_{0l}\cdot T_{klj} + \sum_{l\geq 0} e'_{0l}(g_{kl} - e_{0l}g_k) \\ &\quad - \sum_{l,j\geq 0} e_{0k}\partial_j e'_{0l}T_{lj} - \sum_{l\geq 0} e_{0k}e'_{0l}(g_l - e_{0l}g) \\ &\quad - \sum_{l,j\geq 0}(\partial_j e'_{0l}\cdot e_{0l}T_{kj} + \partial_j e_{0l}\cdot e'_{0l}T_{kj}) \\ &\quad + \sum_{l,j\geq 0}\big(\partial_j e'_{0l}\cdot e_{0k}e_{0l}T_j + e'_{0l}\partial_j(e_{0k}e_{0l})T_j\big) - \sum_{l,j\geq 0} e'_{0l}\partial_j e_{0k}\cdot T_{lj} \\ &= \Big(\sum_{l,j\geq 0}\partial_j e'_{0l}\cdot T_{klj} + \sum_{l\geq 0} e'_{0l}g_{kl} - g_k\Big) - e_{0k}\Big(\sum_{l,j\geq 0}\partial_j e'_{0l}T_{lj} + \sum_{l\geq 0} e'_{0l}g_l - g\Big) \\ &\quad -\partial_j\Big(\sum_{l\geq 0} e'_{0l}e_{0l}\Big)\cdot T_{kj} + \sum_{l,j\geq 0}\partial_j(e'_{0l}e_{0k}e_{0l})\cdot T_j - \sum_{l,j\geq 0}\partial_j e_{0k}\cdot e'_{0l}T_{lj} \\ &= \Big(\sum_{l,j\geq 0}\partial_j e'_{0l}\cdot T_{klj} + \sum_{l\geq 0} e'_{0l}g_{kl} - g_k\Big) - e_{0k}\Big(\sum_{l,j\geq 0}\partial_j e'_{0l}T_{lj} + \sum_{l\geq 0} e'_{0l}g_l - g\Big) \\ &\quad + \sum_{j\geq 0}\partial_j e_{0k}\Big(T_j - \sum_{l\geq 0} e'_{0l}T_{lj}\Big) = 0\,. \end{aligned}$$

Hence the reduction of (4.6) vanishes. □

This is a general lemma, that is, it holds without assumption (4.2). With this assumption we perform in the next sections 5 and 6 the entropy principle to system (4.1) and the outcome will be that the physical system we derive finally will consist of

- the reduced mass-momentum system (4.4),
- a trace of the remaining system, which will be the operation $\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P}$.

Here the map P is defined in the following lemma and it is important that it depends only on G and $\boldsymbol{e}$.

4.2 Lemma. We define a linear projection $\mathrm{P}:\mathbb{R}^4 \to \boldsymbol{W} := \{\boldsymbol{e}\}^\perp$ by

$$\mathrm{P} = \mathrm{Id} \text{ on } \boldsymbol{W}, \qquad \mathrm{P}(\mathrm{G}\boldsymbol{e}) = 0\,. \tag{4.9}$$

By this definition P depends only on G and $\boldsymbol{e}$. It follows

$$\mathrm{P} = \sum_{i\geq 1} e_i{\otimes}e'_i\,, \quad \text{also} \quad \mathrm{P}' = \sum_{i\geq 1} e'_i{\otimes}e_i$$

if we define $\mathrm{P}' := \mathrm{P}^{\mathrm{T}}$. Moreover,

(1) the matrix $\mathrm{P^T\,G^{-1}\,P}$ is

$$\mathrm{P^T\,G^{-1}\,P} = \sum_{i\ge 1} e_i' \otimes e_i' \,.$$

(2) the matrix $\mathrm{PG\,P^T}$ is

$$\mathrm{PG\,P^T} = \sum_{i\ge 1} e_i \otimes e_i \,.$$

Remark: In [3, Sec.5] we have defined $\mathrm{G^{sp} = PG\,P^T}$.

Proof. Since $\boldsymbol{W} = \operatorname{span}\{e_i\,;\ i \ge 1\}$ we have by definition $\mathrm{P}e_i = e_i$ for $i \ge 1$. And $\mathrm{P}e_0 = 0$ since $\mathrm{G}e_0'$ and e_0 are proportional by (4.7). Since $\{e_k'\,;\ k \ge 0\}$ is the dual basis we conclude

$$\mathrm{P} = \sum_{i\ge 1} e_i \otimes e_i' \,.$$

Since $\mathrm{G}^{-1}\, e_i = e_i'$ for $i \ge 1$, see [3, Theorem 3.4], we obtain

$$\mathrm{P^T\,G^{-1}\,P} = \Big(\sum_{i\ge 1} e_i' \otimes e_i\Big)\,\mathrm{G}^{-1}\,\Big(\sum_{i\ge 1} e_i \otimes e_i'\Big) = \sum_{i\ge 1} e_i' \otimes e_i' \,,$$

and

$$\mathrm{PG\,P^T} = \Big(\sum_{i\ge 1} e_i \otimes e_i'\Big)\mathrm{G}\Big(\sum_{i\ge 1} e_i' \otimes e_i\Big) = \sum_{i\ge 1} e_i \otimes e_i$$

since the same reads $\mathrm{G}e_i' = e_i$ for $i \ge 1$. □

4.3 Transformation formula of P. It holds

$$\mathrm{P}\circ Y\,\mathrm{D}Y = \mathrm{D}Y\mathrm{P}^* \,.$$

The matrix $\mathrm{P^T\,G^{-1}\,P}$ is covariant, and $\mathrm{PG\,P^T}$ is contravariant.

Proof. Consider the linear map $(\mathrm{D}Y)^{-1}\,\mathrm{P}\circ Y\,\mathrm{D}Y$. If a point $z^* \in \boldsymbol{W}^*$ then the point $z\circ Y := \mathrm{D}Y z^*$ satisfies

$$(z \bullet \boldsymbol{e})\circ Y = (\mathrm{D}Y z^*)\bullet(\boldsymbol{e}\circ Y) = z^*\bullet(\mathrm{D}Y^{\mathrm{T}}\,\boldsymbol{e}\circ Y) = z^*\bullet\boldsymbol{e}^* = 0$$

that is $z \in \boldsymbol{W}$. Hence $\mathrm{P}z = z$ and therefore

$$(\mathrm{D}Y)^{-1}\,\mathrm{P}\circ Y\,\mathrm{D}Y z^* = (\mathrm{D}Y)^{-1}\,(\mathrm{P}z)\circ Y = (\mathrm{D}Y)^{-1}\,z\circ Y = z^* \,.$$

Moreover, since $e_0\circ Y = \mathrm{D}Y\,e_0^*$, it follows from $\mathrm{P}e_0 = 0$

$$(\mathrm{D}Y)^{-1}\,\mathrm{P}\circ Y\,\mathrm{D}Y e_0^* = (\mathrm{D}Y)^{-1}\,(\mathrm{P}e_0)\circ Y = 0 \,.$$

Since the linear map is determined by these two properties it follows $(\mathrm{D}Y)^{-1}\,\mathrm{P}\circ Y\,\mathrm{D}Y = \mathrm{P}^*$. The matrix $\mathrm{P^T\,G^{-1}\,P}$ is covariant since

$$\begin{aligned} \mathrm{P^{*T}\,G^{*-1}\,P^*} &= \mathrm{P^{*T}}\,\mathrm{D}Y^{\mathrm{T}}\,\mathrm{G}^{-1}\circ Y\mathrm{D}Y\mathrm{P}^* \\ &= (\mathrm{P}\circ Y\mathrm{D}Y)^{\mathrm{T}}\,\mathrm{G}^{-1}\circ Y\mathrm{P}\circ Y\mathrm{D}Y = \mathrm{D}Y^{\mathrm{T}}\,(\mathrm{P^T\,G^{-1}\,P})\circ Y\,\mathrm{D}Y \end{aligned}$$

and the matrix $\mathrm{PG\,P^T}$ is contravariant since

$$\begin{aligned} (\mathrm{PG\,P^T})\circ Y &= \mathrm{P}\circ Y\mathrm{D}Y\mathrm{G}^*\,\mathrm{D}Y^{\mathrm{T}}\,(\mathrm{P}\circ Y)^{\mathrm{T}} \\ &= \mathrm{P}\circ Y\mathrm{D}Y\mathrm{G}^*\,(\mathrm{P}\circ Y\mathrm{D}Y)^{\mathrm{T}} = \mathrm{D}Y\mathrm{P}^*\mathrm{G}^*\,\mathrm{P}^{*\mathrm{T}}\,\mathrm{D}Y^{\mathrm{T}} \end{aligned}$$

for every observer transformation Y. □

For the reduced mass-momentum equation we obtain

4.4 Theorem. If for (4.1) with (4.2), (4.3) the entropy principle is valid then

(1) the reduced mass equation becomes

$$\sum_{j\geq 0} \partial_{y_j} T_j = g\,, \quad T_j := \varrho \underline{v}_j + \underline{\mathbf{J}}_j\,.$$

(2) the reduced mass-momentum system becomes for $k \geq 0$

$$\sum_{j\geq 0} \partial_{y_j} T_{kj} = g_k\,, \quad T_{kj} := \varrho \underline{v}_k \underline{v}_j + \underline{v}_k \underline{\mathbf{J}}_j + \underline{\Pi}_{kj}\,,$$
$$\underline{\Pi}_{kj} = p\,(\mathrm{P}\mathrm{G}\,\mathrm{P}^{\mathrm{T}})_{kj} - \underline{S}_{kj}$$

The fluxes $\underline{\mathbf{J}}$, $\underline{\Pi}$ and $\underline{S}$ have the property

$$\sum_{k\geq 0} \boldsymbol{e}_k \underline{\Pi}_{kj} = 0\,, \quad \sum_{k\geq 0} \boldsymbol{e}_k \underline{S}_{kj} = 0\,,$$
$$\sum_{j\geq 0} \boldsymbol{e}_j \underline{\mathbf{J}}_j = 0\,, \quad \sum_{j\geq 0} \boldsymbol{e}_j \underline{\Pi}_{kj} = 0\,, \quad \sum_{j\geq 0} \boldsymbol{e}_j \underline{S}_{kj} = 0\,.$$

The right-hand sides g and g_k are as in (4.5) and (4.4), we do not say more here about these terms. The mass equation is, of course, contained in the mass-momentum system.

Proof. See section 6, here only this: The reduction (4.4) implies that

$$T_{kj} = \varrho \underline{v}_k \underline{v}_j + \widetilde{\Pi}_{kj}\,, \quad \widetilde{\Pi}_{kj} := \sum_{l\geq 0} e'_{0l} \widetilde{\mathrm{Q}}_{klj}\,,$$

a definition which is also made in section 6, see (6.9). And the reduction (4.5) implies that

$$T_j = \varrho \underline{v}_j + \underline{\mathbf{J}}_j\,, \quad \underline{\mathbf{J}}_j := \sum_{k\geq 0} e'_{0k} \widetilde{\Pi}_{kj} = \sum_{k,l\geq 0} e'_{0k} e'_{0l} \widetilde{\mathrm{Q}}_{klj}\,.$$

Now if one defines $\underline{\Pi}_{kj} := \widetilde{\Pi}_{kj} - \underline{v}_k \underline{\mathbf{J}}_j$ to have the correct formula in (2), see the formula (6.11). And one derives

$$\sum_{k\geq 0} e'_{0k} \underline{\Pi}_{kj} = \sum_{k\geq 0} e'_{0k} (\widetilde{\Pi}_{kj} - \underline{v}_k \underline{\mathbf{J}}_j) = \underline{\mathbf{J}}_j - \sum_{k\geq 0} e'_{0k} \underline{v}_k \underline{\mathbf{J}}_j = 0\,.$$

This proves the assertion, since also

$$\sum_{k\geq 0} e'_{0k} ((\mathrm{P}\mathrm{G}\,\mathrm{P}^{\mathrm{T}}))_{kj} = \sum_{\bar{k},\bar{l}\geq 0} \mathrm{G}_{\bar{k}\bar{l}} \sum_{k\geq 0} e'_{0k} \mathrm{P}_{k\bar{k}} \mathrm{P}_{j\bar{l}} = 0\,,$$
$$\sum_{j\geq 0} e'_{0j} ((\mathrm{P}\mathrm{G}\,\mathrm{P}^{\mathrm{T}}))_{kj} = \sum_{\bar{k},\bar{l}\geq 0} \mathrm{G}_{\bar{k}\bar{l}} \sum_{j\geq 0} e'_{0j} \mathrm{P}_{k\bar{k}} \mathrm{P}_{j\bar{l}} = 0$$

by the form of P in 4.2. That $\underline{\mathbf{J}}$ and $\underline{\Pi}$ have no "time derivative", that is,

$$\sum_{j\geq 0} \boldsymbol{e}_j \underline{\mathbf{J}}_j = 0\,, \quad \sum_{j\geq 0} \boldsymbol{e}_j \underline{\Pi}_{kj} = 0\,,$$

follows from (4.3) for $\widetilde{\mathrm{Q}}_{klj}$. □

Now we perform a trace of the remaining system, namely we multiply by the matrix $\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P}$. This gives, since $\mathrm{P}e_0 = 0$ and $e_i' \bullet e_0 = 0$ for $i \geq 1$,

$$(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P}) : \big(T^E_{klj}\big)_{k,l\geq 0} = \sum_{i\geq 1} (e_i' \otimes e_i') : \big(T^E_{klj}\big)_{k,l\geq 0}$$

$$= \sum_{i\geq 1} (e_i' \otimes e_i') : \big(T_{klj}\big)_{k,l\geq 0} = \sum_{i\geq 1} \sum_{k,l\geq 0} e_{ik}' e_{il}' T_{klj} \,.$$

Therefore the multiplication of the remaining tensor T^E with $\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P}$ is the same as multiplying the original tensor T with the same matrix. We obtain

4.5 Theorem. Multiplying the system (4.1) by the matrix $H := \frac{1}{2}\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P}$ leads to the differential equation

$$\sum_{j\geq 0} \partial_j \big(H : (T_{klj})_{kl}\big) = g \,, \quad g := \sum_{k,l\geq 0} \Big(\sum_{j\geq 0} \partial_j H_{kl} \cdot T_{klj} + H_{kl} g_{kl} \Big) \,.$$

If the assumption (4.2) holds, the "total energy 4-flux" is

$$H : (T_{klj})_{kl} = \Big(\frac{\varrho}{2}\, \underline{v} \bullet (\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P}) \underline{v} + \varepsilon \Big)\, \underline{v}_j + \widetilde{q}_j \quad \text{for } j \geq 0,$$

where in analogy to section 6 the "internal energy" ε is

$$\varepsilon := \frac{1}{2} (\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P}) : E = \frac{1}{2} \mathrm{G}^{-1} : E \,.$$

and the 4-flux $\widetilde{q}$

$$\widetilde{q}_j = H : \Big(\widetilde{\mathrm{Q}}_{klj} \Big)_{kl} = \frac{1}{2} \sum_{i\geq 1} \sum_{k,l\geq 0} e_{ik}' e_{il}' \widetilde{\mathrm{Q}}_{klj}$$

with $\sum_{j\geq 0} \boldsymbol{e}_j \widetilde{q}_j = 0$, hence $\widetilde{q}$ has no time derivative.

Proof. For a scalar test function ζ let $\zeta_{kl} := \zeta H_{kl}$ consist of the test function for the system (4.1). It follows from 4.3 that H is a covariant tensor, hence the test function is allowed. Then

$$0 = \sum_{k,l\geq 0} \int_{\mathbb{R}^4} \Big(\sum_{j\geq 0} \partial_j \zeta_{kl} \cdot T_{klj} + \zeta_{kl} g_{kl} \Big)$$

$$= \sum_{k,l\geq 0} \int_{\mathbb{R}^4} \Big(\sum_{j\geq 0} \partial_j (\zeta H_{kl}) \cdot T_{klj} + \zeta H_{kl} g_{kl} \Big)$$

$$= \int_{\mathbb{R}^4} \Big(\sum_{j\geq 0} \partial_j \zeta \cdot \underbrace{\sum_{k,l\geq 0} H_{kl} T_{klj}}_{= H : (T_{klj})_{kl}} + \zeta \underbrace{\Big(\sum_{j\geq 0} \partial_j H_{kl} \cdot T_{klj} + \sum_{k,l\geq 0} H_{kl} g_{kl} \Big)}_{=: g} \Big) \,,$$

hence the new differential equation is

$$\sum_{j\geq 0} \partial_j (H : (T_{klj})_{kl}) = g \,, \quad g = \sum_{k,l\geq 0} \Big(\sum_{j\geq 0} \partial_j H_{kl} \cdot T_{klj} + H_{kl} g_{kl} \Big) \,,$$

where here we do not take care about g in detail. Instead we focus here on the 4-field $\big(H\boldsymbol{:}(T_{klj})_{kl}\big)_{j\geq 0}$. It is under the assumption (4.2)

$$H\boldsymbol{:}(T_{klj})_{kl} = \Big(\varrho H\boldsymbol{:}(\underline{v}\otimes\underline{v}) + H\boldsymbol{:}E\Big)\underline{v}_j + \sum_{k,l\geq 0} H_{kl}\widetilde{\mathrm{Q}}_{klj}\,,$$

where, since $e_0'\bullet\underline{v} = 1$ and $\mathrm{G}^{-1}\,e_0 = -\mathrm{c}^2 e_0'$,

$$\begin{aligned}
&2H\boldsymbol{:}(\underline{v}\otimes\underline{v}) = \mathrm{G}^{-1}\boldsymbol{:}(\mathrm{P}\underline{v}\otimes\mathrm{P}\underline{v}) = \mathrm{P}\underline{v}\bullet\mathrm{G}^{-1}\,\mathrm{P}\underline{v}\\
&= (\underline{v}-e_0)\bullet\mathrm{G}^{-1}\,(\underline{v}-e_0) = \underline{v}\bullet\mathrm{G}^{-1}\,\underline{v} - 2\underline{v}\bullet\mathrm{G}^{-1}\,e_0 + e_0\bullet\mathrm{G}^{-1}\,e_0\\
&= \underline{v}\bullet\mathrm{G}^{-1}\,\underline{v} + 2\mathrm{c}^2\underline{v}\bullet e_0' - \mathrm{c}^2 e_0\bullet e_0'\\
&= \underline{v}\bullet\mathrm{G}^{-1}\,\underline{v} + \mathrm{c}^2 = \underline{v}\bullet(\mathrm{G}^{-1} + \mathrm{c}^2\boldsymbol{e}\otimes\boldsymbol{e})\underline{v}\,,
\end{aligned}$$

just to have a few representations of this term. Therefore one calls the following term the "kinetic energy"

$$\varrho H\boldsymbol{:}(\underline{v}\otimes\underline{v}) = \frac{\varrho}{2}\underline{v}\bullet(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})\underline{v} = \frac{\varrho}{2}\mathrm{P}\underline{v}\bullet\mathrm{G}^{-1}\,\mathrm{P}\underline{v}$$

and, since $H = \frac{1}{2}\sum_{i\geq 1} e_i'\otimes e_i'$ by 4.2(1), the "internal energy"

$$\begin{aligned}
\varepsilon &:= H\boldsymbol{:}E = \frac{1}{2}(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})\boldsymbol{:}E = \frac{1}{2}\sum_{i\geq 1}\sum_{k,l\geq 0} e_{ik}'e_{il}'E_{kl}\\
&= \frac{1}{2}\mathrm{G}^{-1}\boldsymbol{:}(\mathrm{P}E\,\mathrm{P}^{\mathrm{T}}) = \frac{1}{2}\mathrm{G}^{-1}\boldsymbol{:}E\,,
\end{aligned}$$

since assumption (4.2) implies $\mathrm{P}E = E$. Finally for the 4-flux

$$\begin{aligned}
\widetilde{q}_j &:= \sum_{k,l\geq 0} H_{kl}\widetilde{\mathrm{Q}}_{klj}\\
&= \frac{1}{2}\sum_{i\geq 1}\sum_{k,l\geq 0}(e_i'\otimes e_i)_{kl}\widetilde{\mathrm{Q}}_{klj} = \frac{1}{2}\sum_{i\geq 1}\sum_{k,l\geq 0} e_{ik}'e_{il}'\widetilde{\mathrm{Q}}_{klj}\,.
\end{aligned}$$

For more about $\widetilde{q}$ see the next statement. □

For the following lemma we need some formulas from section 5.

4.6 Heat flux. The entropy principle implies that the 4-flux $\widetilde{q}$ of the previous theorem has the following representation

$$\widetilde{q}_j = \frac{\varrho}{2}\,\underline{v}\bullet(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})\underline{v}\,\mathbf{J}_j + \sum_{\bar{k},k\geq 0}\underline{v}_{\bar{k}}(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})_{\bar{k}k}\underline{\Pi}_{kj} + \underline{q}_j\,,$$

where $\underline{q}$ is the "heat flux" occurring in the entropy production.

Proof. From the last theorem

$$\begin{aligned}
\widetilde{q}_j &:= \frac{1}{2}\sum_{i\geq 1}\sum_{k,l\geq 0} e_{ik}'e_{il}'\widetilde{\mathrm{Q}}_{klj} = \frac{1}{2}\sum_{i\geq 1}\widetilde{\mathrm{Q}}_{iij}' \quad \text{(by (5.6))}\\
&= \frac{1}{2}\sum_{i\geq 1}\big(v_i'v_i'\underline{\mathbf{J}}_j + 2\underline{\Pi}_{ij}'v_i' + \underline{\mathrm{Q}}_{iij}'\big) \quad \text{(by (5.8))}\\
&= \frac{1}{2}\sum_{i\geq 1}|v_i'|^2\mathbf{J}_j + \sum_{i\geq 1}\underline{\Pi}_{ij}'v_i' + \frac{1}{2}\sum_{i\geq 1}\underline{\mathrm{Q}}_{iij}\,,
\end{aligned}$$

where the heat flux is

$$\underline{q}_j := \frac{1}{2}\sum_{i\geq 1} \underline{\mathrm{Q}}_{iij}$$

and, by the first equation in (5.6),

$$\sum_{i\geq 1} |v_i'|^2 = \sum_{i\geq 1}\sum_{k,l\geq 0} e_{ik}' e_{il}' \underline{v}_k \underline{v}_l = \underline{v}\bullet(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})\underline{v}\,.$$

To handle the middle term we derive from (6.10) for $i \geq 1$

$$\sum_{k\geq 0} e_{ik}' \underline{\Pi}_{kj} = \sum_{k\geq 0} e_{ik}' \sum_{\bar{i}\geq 1} e_{\bar{i}k} \underline{\Pi}_{\bar{i}j}' = \sum_{\bar{i}\geq 1} \big(\sum_{k\geq 0} e_{ik}' e_{\bar{i}k}\big) \underline{\Pi}_{\bar{i}j}' = \underline{\Pi}_{ij}'$$

and therefore

$$\begin{aligned}
&\sum_{i\geq 1} \underline{\Pi}_{ij}' v_i' = \sum_{i\geq 1} \big(\sum_{k\geq 0} e_{ik}' \underline{\Pi}_{kj}\big)\big(\sum_{\bar{k}\geq 0} e_{i\bar{k}}' \underline{v}_{\bar{k}}\big) \\
&= \big(\sum_{i\geq 1} e_i' \otimes e_i'\big) \mathbin{:} \big(\underline{\Pi}_{kj} \underline{v}_{\bar{k}}\big)_{k\bar{k}} = \sum_{\bar{k},k\geq 0} \underline{v}_{\bar{k}} (\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})_{\bar{k}k} \underline{\Pi}_{kj}\,.
\end{aligned}$$

□

Altogether the main theorem is the

4.7 Entropy theorem. Consider the system (4.1), (4.2), (4.3). The application of the entropy principle

$$\sigma := \sum_{j\geq 0} \partial_j \underline{\eta}_j \geq 0$$

leads to the "mass-momentum-energy system". This system consists of the "mass-momentum equation" in 4.4(2), and of the the "energy equation" in 4.5, which with 4.6 is

$$\begin{aligned}
&\sum_{j\geq 0} \partial_j \Big(\Big(\frac{\varrho}{2} \mathrm{P}\underline{v}\bullet \mathrm{G}^{-1}\,\mathrm{P}\underline{v} + \varepsilon \Big) \underline{v}_j + \widetilde{q}_j \Big) = g\,, \\
&\widetilde{q}_j = \frac{\varrho}{2}\, \underline{v}\bullet(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})\underline{v}\,\mathbf{J}_j + \sum_{\bar{k},k\geq 0} \underline{v}_{\bar{k}} (\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P})_{\bar{k}k} \underline{\Pi}_{kj} + \underline{q}_j\,.
\end{aligned}$$

Here the entropy and entropy 4-flux are

$$\eta := \widehat{\eta}(\varrho,\varepsilon)\,, \quad \underline{\eta} := \eta\underline{v} + \eta'_{\varrho}\underline{\mathbf{J}} + \eta'_{\varepsilon}\underline{q}\,, \quad \eta = \boldsymbol{e}\bullet\underline{\eta}$$

and the entropy production is

$$\begin{aligned}
0 \leq \sigma = &\sum_{k,j\geq 0} \big(\sum_{i\geq 1} e_{ik}' \partial_j (e_i' \bullet \underline{v})\big) \underline{S}_{kj} \\
&+ \sum_{j\geq 0} \partial_j \widehat{\eta}'_{\varrho} \cdot \underline{\mathbf{J}}_j + \sum_{j\geq 0} \partial_j \widehat{\eta}'_{\varepsilon} \cdot \underline{q}_j + \widehat{\eta}'_{\varrho} \mathbf{r}^{\varrho} + \widehat{\eta}'_{\varepsilon} \cdot \mathbf{r}^{e}\,.
\end{aligned} \tag{4.10}$$

Proof. The proof of this theorem is contained in section 5 and 6. The splitting of the mass-momentum-energymatrix equation into mass-momentum and energy equation is contained in the statements 4.4 to 4.6. □

The pressure tensor $\underline{\Pi}$ is by 4.4(2)

$$\underline{\Pi} = p\,\mathrm{P}\mathrm{G}\mathrm{P}^{\mathrm{T}} - \underline{S}$$

In the case of a gas $\underline{\Pi} = p\,\mathrm{PGP}^{\mathrm{T}}$ and $\underline{S} = 0$, therefore the first term of the entropy production vanishes. For fluids the stress tensor $\underline{S}$ has to be chosen so that the entropy production is non-negative. This term in the entropy production is

$$\sum_{k,j\geq 0} \Big(\sum_{i\geq 1} e'_{ik} \partial_j (e'_i \bullet \underline{v}) \Big) \underline{S}_{kj}$$

and we show in section 7 that in the classical limit it converges to the well known expression

$$\sum_{k,j\geq 1} \partial_{x_j} v_k \cdot S_{kj}\,,$$

if $\underline{S}$ is given by a symmetric matrix S.

5 Evaluation of the Liu & Müller sum

In this section we consider the relativistic moments of up to second order, that is $N = 2$ in (2.1) and $\alpha = (k, l)$,

$$\sum_{j\geq 0} \partial_{y_j} T_{klj} - g_{kl} = 0 \quad \text{for } k,l \geq 0, \quad g_{kl} := \underline{\mathbf{f}}_{kl} + \sum_{\beta\in\{0,1,2,3\}^3} \mathrm{C}^{\beta}_{kl} T_\beta\,, \tag{5.1}$$

where we have set $n = 3$ (the physical case). We consider fluid equations, therefore

$$T_{klj} = \underline{\varrho}\underline{v}_k\underline{v}_l\underline{v}_j + E_{kl}\underline{v}_j + \widetilde{\mathrm{Q}}_{klj}\,, \tag{5.2}$$

$$\sum_{k\geq 0} \boldsymbol{e}_k E_{kl} = 0\,, \quad \sum_{j\geq 0} \boldsymbol{e}_j \widetilde{\mathrm{Q}}_{klj} = 0\,. \tag{5.3}$$

The first step in exploiting the entropy principle is to multiply the differential operators $\sum_{j\geq 0} \partial_{y_j} T_{\alpha j} - g_\alpha$ by certain factors, which Liu & Müller call Lagrange multipliers $(\Lambda_\alpha)_\alpha$, see section 3, and then sum up these expressions to get

$$\sum_\alpha \Lambda_\alpha \Big(\sum_{j\geq 0} \partial_j T_{\alpha j} - g_\alpha \Big)\,.$$

Now we apply 3.3, that is, we replace these sum by an equivalent system of differential operators

$$L_\gamma := \sum_{j\geq 0} \partial_j T'_{\gamma j} - \mathbf{r}'_\gamma\,, \tag{5.4}$$

we obtain a new representation

$$\sum_\alpha \Lambda_\alpha \Big(\sum_{j\geq 0} \partial_j T_{\alpha j} - g_\alpha \Big) = \sum_\gamma \lambda_\gamma \Big(\sum_{j\geq 0} \partial_j T'_{\gamma j} - \mathbf{r}'_\gamma \Big) = \sum_\gamma \lambda_\gamma L_\gamma\,,$$

where the quantities of the new relation are defined by

$$\begin{aligned} T'_{\gamma j} &:= \sum_\alpha e'_{\gamma_1\alpha_1} e'_{\gamma_2\alpha_2} T_{\alpha j}\,, \quad \mathbf{r}'_\gamma := \sum_\alpha e'_{\gamma_1\alpha_1} e'_{\gamma_2\alpha_2} \mathbf{f}_\alpha\,, \\ \lambda_\gamma &= \sum_\alpha \Lambda_\alpha e_{\gamma_1\alpha_1} e_{\gamma_2\alpha_2} \quad \text{or} \quad \Lambda_\alpha = \sum_\gamma \lambda_\gamma e'_{\gamma_1\alpha_1} e'_{\gamma_2\alpha_2}\,. \end{aligned} \tag{5.5}$$

Now the differential operators L_γ have no Coriolis coefficients, therefore "fictitious forces" do not appear in the entropy equation. The representation of T in (5.2) transforms by (5.5) in the following identities for $k, l \geq 1$ and $j \geq 0$, if one uses the assumptions in (5.3),

$$\begin{aligned} T'_{00j} &= \varrho \underline{v}_j + \underline{\mathbf{J}}_j\,, \\ T'_{k0j} &= \varrho v'_k \underline{v}_j + \widetilde{\Pi}'_{kj}\,, \\ T'_{klj} &= (\varrho v'_k v'_l + E'_{kl})\underline{v}_j + \widetilde{\mathrm{Q}}'_{klj}\,, \end{aligned}$$

where

$$\begin{aligned} v'_k &:= \sum_{\bar k \geq 0} e'_{k\bar k} \underline{v}_{\bar k} \text{ for } k \geq 1\,, \\ E'_{kl} &:= \sum_{\bar k, \bar l \geq 0} e'_{k\bar k} e'_{l\bar l} E_{\bar k \bar l} \text{ for } k, l \geq 1\,, \\ \widetilde{\mathrm{Q}}'_{klj} &:= \sum_{\bar k, \bar l \geq 0} e'_{k\bar k} e'_{l\bar l} \widetilde{\mathrm{Q}}_{\bar k \bar l j} \text{ for } k, l, j \geq 0\,, \\ \widetilde{\Pi}'_{kj} &:= \widetilde{\mathrm{Q}}'_{k0j} \text{ for } k \geq 1\,, \quad \underline{\mathbf{J}}_j := \widetilde{\mathrm{Q}}'_{00j}\,. \end{aligned} \tag{5.6}$$

In a second step we show that the system (5.1) is equivalent to the system given by $(L^\varrho, (L^v_k)_{k\geq 1}, (L^e_{kl})_{k,l\geq 1}) = 0$, where

$$\begin{aligned} L_{00} &= L^\varrho\,, \\ L_{0k} &= L_{k0} = L^v_k + v'_k L^\varrho\,, \\ L_{kl} &= L^e_{kl} + v'_k L^v_l + v'_l L^v_k + v'_k v'_l L^\varrho \end{aligned} \tag{5.7}$$

for $k, l \geq 1$. These new operators are defined in the following theorem.

5.1 Theorem. Define for $k, l \geq 1$

$$\begin{aligned} L^\varrho &:= \sum_{j\geq 0} \partial_j(\varrho \underline{v}_j + \underline{\mathbf{J}}_j) - \mathbf{r}^\varrho\,, \\ L^v_k &:= \varrho \sum_{j\geq 0} \underline{v}_j \partial_j v'_k + \sum_{j\geq 0} \partial_j \underline{\Pi}'_{kj} + \sum_{j\geq 0} \underline{\mathbf{J}}_j \partial_j v'_k - \mathbf{r}^v_k\,, \\ L^e_{kl} &:= \sum_{j\geq 0} \partial_j(E'_{kl}\underline{v}_j) + \sum_{j\geq 0} \partial_j \underline{\mathrm{Q}}'_{klj} + \sum_{j\geq 0} (\underline{\Pi}'_{lj}\partial_j v'_k + \underline{\Pi}'_{kj}\partial_j v'_l) - \mathbf{r}^e_{kl}\,. \end{aligned}$$

Then the equations (5.7) are satisfied, if for $k, l \geq 1$

$$\begin{aligned} \widetilde{\Pi}'_{kj} &= v'_k \underline{\mathbf{J}}_j + \underline{\Pi}'_{kj}\,, \quad \widetilde{\mathrm{Q}}'_{klj} = v'_k v'_l \underline{\mathbf{J}}_j + \underline{\Pi}'_{lj} v'_k + \underline{\Pi}'_{kj} v'_l + \underline{\mathrm{Q}}'_{klj}\,, \\ \mathbf{r}^v_k &:= \mathbf{r}'_{k0} - v'_k \mathbf{r}^\varrho\,, \quad \mathbf{r}^e_{kl} := \mathbf{r}'_{kl} - (v'_k \mathbf{r}^v_l + v'_l \mathbf{r}^v_k) - v'_k v'_l \mathbf{r}^\varrho\,, \end{aligned} \tag{5.8}$$

and of course $\mathbf{r}^\varrho := \mathbf{r}'_{00}$.

This follows by the same procedure as in the classical case.

Proof. That $L_{00} = L^\varrho$ is evident. For the velocity part

$$\begin{aligned} L^v_k + v'_k L^\varrho &= \varrho \sum_{j\geq 0} \underline{v}_j \partial_j v'_k + v'_k \sum_{j\geq 0} \partial_j(\varrho \underline{v}_j + \underline{\mathbf{J}}_j) + \sum_{j\geq 0} \underline{\mathbf{J}}_j \partial_j v'_k \\ &\quad + \sum_{j\geq 0} \partial_j \underline{\Pi}'_{kj} - (\mathbf{r}^v_k + v'_k \mathbf{r}^\varrho) \\ &= \sum_{j\geq 0} \partial_j \big(\varrho v'_k \underline{v}_j + v'_k \underline{\mathbf{J}}_j + \underline{\Pi}'_{kj} \big) - (\mathbf{r}^v_k + v'_k \mathbf{r}^\varrho) = L_{0k} \end{aligned}$$

if for $k \geq 1$

$$\widetilde{\Pi}_{kj} = v'_k \underline{\mathbf{J}}_j + \underline{\Pi}'_{kj}\,, \quad \mathbf{r}'_{k0} = \mathbf{r}^v_k + v'_k \mathbf{r}^\varrho\,.$$

The energy part is

$$\begin{aligned} & L^e_{kl} + v'_k L^v_l + v'_l L^v_k + v'_k v'_l L^\varrho \\ &= \sum_{j\geq 0} \partial_j (E'_{kl} \underline{v}_j) + \sum_{j\geq 0} \partial_j \underline{\mathrm{Q}}'_{klj} + \sum_{j\geq 0} (\underline{\Pi}'_{lj} \partial_j v'_k + \underline{\Pi}'_{kj} \partial_j v'_l) - \mathbf{r}^e_{kl} \\ &\quad + \varrho \sum_{j\geq 0} v'_k \underline{v}_j \partial_j v'_l + \sum_{j\geq 0} v'_k \partial_j \underline{\Pi}'_{lj} + \sum_{j\geq 0} v'_k \underline{\mathbf{J}}_j \partial_j v'_l - v'_k \mathbf{r}^v_l \\ &\quad + \varrho \sum_{j\geq 0} v'_l \underline{v}_j \partial_j v'_k + \sum_{j\geq 0} v'_l \partial_j \underline{\Pi}'_{kj} + \sum_{j\geq 0} v'_l \underline{\mathbf{J}}_j \partial_j v'_k - v'_l \mathbf{r}^v_k \\ &\quad + v'_k v'_l \sum_{j\geq 0} \partial_j (\varrho \underline{v}_j) + v'_k v'_l \sum_{j\geq 0} \underline{\mathbf{J}}_j - v'_k v'_l \mathbf{r}^\varrho \\ &= \partial_j (\varrho v'_k v'_l \underline{v}_j) + \partial_j (E'_{kl} \underline{v}_j + v'_k v'_l \underline{\mathbf{J}}_j + \underline{\Pi}'_{lj} v'_k + \underline{\Pi}'_{kj} v'_l + \underline{\mathrm{Q}}'_{klj}) \\ &\quad - (\mathbf{r}^e_{kl} + v'_k \mathbf{r}^v_l + v'_l \mathbf{r}^v_k + v'_k v'_l \mathbf{r}^\varrho) = L_{kl} \end{aligned}$$

if for $k, l \geq 1$

$$\begin{aligned} \widetilde{\mathrm{Q}}'_{klj} &= v'_k v'_l \underline{\mathbf{J}}_j + \underline{\Pi}'_{lj} v'_k + \underline{\Pi}'_{kj} v'_l + \underline{\mathrm{Q}}'_{klj}\,, \\ \mathbf{r}'_{kl} &= \mathbf{r}^e_{kl} + v'_k \mathbf{r}^v_l + v'_l \mathbf{r}^v_k + v'_k v'_l \mathbf{r}^\varrho\,. \end{aligned}$$

□

Thus following the procedure of Liu & Müller we have for all functions

$$\begin{aligned} \sum_\alpha \Lambda_\alpha \big(\sum_{j\geq 0} \partial_{y_j} T_{\alpha j} - g_\alpha \big) &= \sum_\gamma \lambda_\gamma \big(\sum_{j\geq 0} \partial_j T'_{\gamma j} - \mathbf{r}'_\gamma \big) \\ &= \sum_\gamma \lambda_\gamma L_\gamma = \lambda_{00} L_{00} + \sum_{k\geq 1} 2\lambda_{k0} L_{k0} + \sum_{k,l\geq 1} \lambda_{kl} L_{kl} \\ &= \lambda_{00} L^\varrho + \sum_{k\geq 1} 2\lambda_{k0} (L^v_k + v'_k L^\varrho) \\ &\quad + \sum_{k,l\geq 1} \lambda_{kl} (L^e_{kl} + v'_k L^v_l + v'_l L^v_k + v'_k v'_l L^\varrho) \\ &= \lambda^\varrho L^\varrho + \sum_{k\geq 1} \lambda^v_k L^v_k + \sum_{k,l\geq 1} \lambda^e_{kl} L^e_{kl} \end{aligned}$$

where the new set of parameters is given by

$$\lambda^\varrho := \lambda_{00} + 2\sum_{k\geq 1} v'_k\lambda_{k0} + \sum_{k,l\geq 1} v'_k v'_l \lambda_{kl}\,,$$
$$\lambda^v_k := 2\lambda_{k0} + 2\sum_{l\geq 1} v'_l\lambda_{kl}\,,$$
$$\lambda^e_{kl} := \lambda_{kl}\,.$$

for $k, l \geq 1$. Now we compute

$$\begin{aligned}
&\lambda^\varrho L^\varrho + \sum_{k\geq 1}\lambda^v_k L^v_k + \sum_{k,l\geq 1}\lambda^e_{kl}L^e_{kl}\\
&= \lambda^\varrho\Big(\sum_{j\geq 0}\partial_j(\varrho\underline{v}_j + \underline{\mathbf{J}}_j) - \mathbf{r}^\varrho\Big)\\
&\qquad + \sum_{k\geq 1}\lambda^v_k\Big(\varrho\sum_{j\geq 0}\underline{v}_j\partial_j v'_k + \sum_{j\geq 0}(\partial_j\underline{\Pi}'_{kj} + \underline{\mathbf{J}}_j\partial_j v'_k) - \mathbf{r}^v_k\Big)\\
&+ \sum_{k,l\geq 1}\lambda^e_{kl}\Big(\sum_{j\geq 0}((\partial_j(E'_{kl}\underline{v}_j) + \partial_j\underline{\mathrm{Q}}'_{klj}) + \sum_{j\geq 0}(\underline{\Pi}'_{lj}\partial_j v'_k + \underline{\Pi}'_{kj}\partial_j v'_l) - \mathbf{r}^e_{kl}\Big)\\
&= \lambda^\varrho\sum_{j\geq 0}\partial_j(\varrho\underline{v}_j) + \sum_{k\geq 1}\lambda^v_k\varrho\sum_{j\geq 0}\underline{v}_j\partial_j v'_k + \sum_{k,l\geq 1}\lambda^e_{kl}\sum_{j\geq 0}\partial_j(E'_{kl}\underline{v}_j)\\
&\qquad + \sum_{j\geq 0,k\geq 1}\partial_j v'_k\cdot\Big(\lambda^v_k\underline{\mathbf{J}}_j + 2\sum_{l\geq 1}\lambda^e_{kl}\underline{\Pi}'_{lj}\Big)\\
&\qquad + \sum_{j\geq 0}\lambda^\varrho\partial_j\underline{\mathbf{J}}_j + \sum_{j\geq 0,k\geq 1}\lambda^v_k\partial_j\underline{\Pi}'_{kj} + \sum_{j\geq 0,k,l\geq 1}\lambda^e_{kl}\partial_j\underline{\mathrm{Q}}'_{klj}\\
&\qquad -\lambda^\varrho\mathbf{r}^\varrho - \sum_{k\geq 1}\lambda^v_k\mathbf{r}^v_k - \sum_{k,l\geq 1}\lambda^e_{kl}\mathbf{r}^e_{kl}\,,
\end{aligned}$$

where for the first line on the right-hand side we prove

5.2 Lemma. We can write for every function h

$$\sum_{j\geq 0}\partial_j(h\underline{v}_j) = \sum_{j\geq 0}\underline{v}_j\partial_j h + \sum_{j\geq 0,k\geq 0}\partial_j v'_k\cdot(he_{kj})$$

Basic expression: For each $k \geq 0$ we have the following equality $\underline{\operatorname{div}}\, e_k = 0$. This is true since the situation is connected to the standard one.

Proof. It is for every function h

$$\sum_{j\geq 0}\partial_j(h\underline{v}_j) = \sum_{j\geq 0}\underline{v}_j\partial_j h + h\sum_{j\geq 0}\partial_j\underline{v}_j\,. \tag{5.9}$$

Now since by (5.6)

$$\underline{v} = \sum_{k\geq 0} v'_k e_k\,,\quad v'_k = e'_k\bullet\underline{v}\,,$$

we get

$$\begin{aligned}
\sum_{j\geq 0}\partial_j\underline{v}_j &= \underline{\operatorname{div}}\,\underline{v} = \underline{\operatorname{div}}\Big(\sum_{k\geq 0} v'_k e_k\Big) = \sum_{j\geq 0,k\geq 0}\partial_j(v'_k e_{kj})\\
&= \sum_{j\geq 0,k\geq 0}\partial_j v'_k\cdot e_{kj} + \sum_{k\geq 0} v'_k\sum_{j\geq 0}\partial_j e_{kj} = \sum_{j\geq 0,k\geq 0}\partial_j v'_k\cdot e_{kj}\,,
\end{aligned}$$

since as we show now $\underline{\operatorname{div}}\, e_k = 0$. □

Proof of basic expression. This follows since $e_k \circ Y = \mathrm{D}Y e_k^*$, that is e_k is a contravariant vector, and therefore $(\underline{\operatorname{div}}\, e_k)\circ Y = \underline{\operatorname{div}}\, e_k^*$. Since the situation is connected to the standard one we can choose Y such that $e_k^* = \mathbf{e}_k = \text{const}$. □

From 5.2, with h equals ϱ and $E'_{\bar{k}\bar{l}}$, we obtain for the first line on the right-hand side of our expression

$$\begin{aligned}
&\lambda^\varrho \sum_{j\geq 0} \partial_j(\varrho \underline{v}_j) + \sum_{k\geq 1} \lambda_k^v \varrho \sum_{j\geq 0} \underline{v}_j \partial_j v'_k + \sum_{\bar{k},\bar{l}\geq 1} \lambda^e_{\bar{k}\bar{l}} \sum_{j\geq 0} \partial_j (E'_{\bar{k}\bar{l}} \underline{v}_j)\\
&= \lambda^\varrho \sum_{j\geq 0} \underline{v}_j \partial_j \varrho + \sum_{k\geq 1} \lambda_k^v \varrho \sum_{j\geq 0} \underline{v}_j \partial_j v'_k + \sum_{\bar{k},\bar{l}\geq 1} \lambda^e_{\bar{k}\bar{l}} \sum_{j\geq 0} \underline{v}_j \partial_j E'_{\bar{k}\bar{l}}\\
&\qquad + \sum_{j\geq 0, k\geq 0} \partial_j v'_k \cdot \Big(\lambda^\varrho \varrho + \sum_{\bar{k},\bar{l}\geq 1} \lambda^e_{\bar{k}\bar{l}} E'_{\bar{k}\bar{l}}\Big) e_{kj}\,.
\end{aligned}$$

Now we can write the first three terms on the right-hand side as a derivative of a function η, which is later the entropy, if we let

$$\begin{aligned}
\eta &= \widetilde{\eta}\big(\varrho, (v'_k)_{k\geq 1}\,, (E'_{kl})_{k,l\geq 1}\big)\,,\\
\lambda^\varrho := \widetilde{\eta}'_\varrho\,, \quad \lambda_k^v &:= \frac{1}{\varrho} \widetilde{\eta}'_{v'_k}\,, \quad \lambda^e_{kl} := \widetilde{\eta}'_{E'_{kl}}\,,
\end{aligned} \tag{5.10}$$

since then by the chain rule

$$\begin{aligned}
&\sum_{j\geq 0} \underline{v}_j \partial_j \eta = \widetilde{\eta}'_\varrho \sum_{j\geq 0} \underline{v}_j \partial_j \varrho + \sum_{k\geq 1} \widetilde{\eta}'_{v'_k} \sum_{j\geq 0} \underline{v}_j \partial_j v'_k + \sum_{\bar{k},\bar{l}\geq 1} \widetilde{\eta}'_{E'_{\bar{k}\bar{l}}} \sum_{j\geq 0} \underline{v}_j \partial_j E'_{\bar{k}\bar{l}}\\
&= \lambda^\varrho \sum_{j\geq 0} \underline{v}_j \partial_j \varrho + \sum_{k\geq 1} \lambda_k^v \varrho \sum_{j\geq 0} \underline{v}_j \partial_j v'_k + \sum_{\bar{k},\bar{l}\geq 1} \lambda^e_{\bar{k}\bar{l}} \sum_{j\geq 0} \underline{v}_j \partial_j E'_{\bar{k}\bar{l}}\,,
\end{aligned}$$

which are the first three terms on the right-hand side. And it follows also from 5.2, with h equals η,

$$\sum_{j\geq 0} \underline{v}_j \partial_j \eta = \sum_{j\geq 0} \partial_j(\eta \underline{v}_j) - \sum_{j\geq 0, k\geq 0} \partial_j v'_k \cdot (\eta e_{kj})\,.$$

Altogether we infer that

$$\begin{aligned}
&\sum_\alpha \Lambda_\alpha \big(\sum_{j\geq 0} \partial_j T_{\alpha j} - g_\alpha\big) = \sum_\gamma \lambda_\gamma \big(\sum_{j\geq 0} \partial_j T'_{\gamma j} - \mathbf{r}'_\gamma\big)\\
&= \lambda^\varrho L^\varrho + \sum_{k\geq 1} \lambda_k^v L_k^v + \sum_{k,l\geq 1} \lambda^e_{kl} L^e_{kl}\\
&= \sum_{j\geq 0} \partial_j(\eta \underline{v}_j)\\
&+ \sum_{j\geq 0, k\geq 1} \partial_j v'_k \Big(\big(\lambda^\varrho \varrho + \sum_{\bar{k},\bar{l}\geq 1} \lambda^e_{\bar{k}\bar{l}} E'_{\bar{k}\bar{l}} - \eta\big) e_{kj} + \lambda_k^v \underline{\mathbf{J}}_j + 2 \sum_{l\geq 1} \lambda^e_{kl} \underline{\Pi}'_{lj}\Big)\\
&\qquad + \sum_{j\geq 0} \lambda^\varrho \partial_j \underline{\mathbf{J}}_j + \sum_{j\geq 0, k\geq 1} \lambda_k^v \partial_j \underline{\Pi}'_{kj} + \sum_{j\geq 0, k,l\geq 1} \lambda^e_{kl} \partial_j \underline{\mathbf{Q}}'_{klj}\\
&\qquad\qquad - \lambda^\varrho \mathbf{r}^\varrho - \sum_{k\geq 1} \lambda_k^v \mathbf{r}_k^v - \sum_{k,l\geq 1} \lambda^e_{kl} \mathbf{r}^e_{kl}
\end{aligned}$$

$$= \sum_{j\geq 0} \partial_j \Big(\eta \underline{v}_j + \lambda^\varrho \underline{\mathbf{J}}_j + \sum_{k\geq 1} \lambda^v_k \underline{\Pi}'_{kj} + \sum_{k,l\geq 1} \lambda^e_{kl} \underline{Q}'_{klj} \Big)$$
$$+ \sum_{j\geq 0, k\geq 1} \partial_j v'_k \Big(\big(\lambda^\varrho \varrho + \sum_{\bar{k},\bar{l}\geq 1} \lambda^e_{\bar{k}\bar{l}} E'_{\bar{k}\bar{l}} - \eta \big) e_{kj} + \lambda^v_k \underline{\mathbf{J}}_j + 2 \sum_{l\geq 1} \lambda^e_{kl} \underline{\Pi}'_{lj} \Big)$$
$$- \sum_{j\geq 0} \partial_j \lambda^\varrho \cdot \underline{\mathbf{J}}_j - \sum_{j\geq 0, k\geq 1} \partial_j \lambda^v_k \cdot \underline{\Pi}'_{kj} - \sum_{j\geq 0, k,l\geq 1} \partial_j \lambda^e_{kl} \cdot \underline{Q}'_{klj}$$
$$- \lambda^\varrho \mathbf{r}^\varrho - \sum_{k\geq 1} \lambda^v_k \mathbf{r}^v_k - \sum_{k,l\geq 1} \lambda^e_{kl} \mathbf{r}^e_{kl}$$
$$= \sum_{j\geq 0} \partial_j \underline{\eta}_j - \sigma \,,$$

if for $j \geq 0$

$$\eta = \widetilde{\eta}\big(\varrho, (v'_k)_{k\geq 1}, (E'_{kl})_{k,l\geq 1}\big) \,,$$
$$\underline{\eta}_j := \eta \underline{v}_j + \lambda^\varrho \underline{\mathbf{J}}_j + \sum_{k\geq 1} \lambda^v_k \underline{\Pi}'_{kj} + \sum_{k,l\geq 1} \lambda^e_{kl} \underline{Q}'_{klj} \,, \tag{5.11}$$

and

$$\sigma :=$$
$$- \sum_{j\geq 0, k\geq 1} \partial_j v'_k \Big(\big(\lambda^\varrho \varrho + \sum_{\bar{k},\bar{l}\geq 1} \lambda^e_{\bar{k}\bar{l}} E'_{\bar{k}\bar{l}} - \eta \big) e_{kj} + \lambda^v_k \underline{\mathbf{J}}_j + 2 \sum_{l\geq 1} \lambda^e_{kl} \underline{\Pi}'_{lj} \Big)$$
$$+ \sum_{j\geq 0} \partial_j \lambda^\varrho \cdot \underline{\mathbf{J}}_j + \sum_{j\geq 0, k\geq 1} \partial_j \lambda^v_k \cdot \underline{\Pi}'_{kj} + \sum_{j\geq 0, k,l\geq 1} \partial_j \lambda^e_{kl} \cdot \underline{Q}'_{klj} \tag{5.12}$$
$$+ \lambda^\varrho \mathbf{r}^\varrho + \sum_{k\geq 1} \lambda^v_k \mathbf{r}^v_k + \sum_{k,l\geq 1} \lambda^e_{kl} \mathbf{r}^e_{kl} \,.$$

Therefore for solutions of (5.1)

$$\sum_{j\geq 0} \partial_j \underline{\eta}_j - \sigma = \sum_\alpha \Lambda_\alpha \big(\sum_{j\geq 0} \partial_j T_{\alpha j} - g_\alpha \big) = 0 \,,$$

if the entropy quantities are given as in (5.11) and if σ consists of the quantities in (5.12). For consequences see the next section.

6 Entropy as objective scalar

Here we deal with system (5.1) and the assumption (5.2) and (5.3). In this situation we have derived in the previous section, that for solutions of (5.1)

$$\sum_{j\geq 0} \partial_j \underline{\eta}_j = \sigma \tag{6.1}$$

where the entropy 4-flux $\underline{\eta}$ satisfies (5.11) and the entropy production σ satisfies (5.12). And the entropy principle $\sigma \geq 0$ is required. It is also a postulate of the entropy principle that the equation (6.1) has to be a scalar differential equation, which is satisfied if $\underline{\eta}$ is a contravariant vector and σ an objective scalar. Now, the first term on the right-hand side of $\underline{\eta}$ in (5.11) is $\eta \underline{v}$ where $\underline{v}$ is a contravariant vector, therefore, if η is an objective scalar this term is a contravariant vector. Remember that in (5.11) we have made a constitutive relation for η depending on ϱ, $v'_k = e'_k \bullet \underline{v}$ and E'_{kl}. These quantities are all objective scalars, but they depend on single basis vectors e'_i for $i \geq 1$. This would be a non-isotropic behaviour if η depends really on one of these vectors. Such a dependence one would not allow for a simple fluid. Therefore we come to the conclusion that η depends only on ϱ and the trace of E', that is

$$\varepsilon := \frac{1}{2}\sum_{k\geq 1} E'_{kk} = \frac{1}{2}\sum_{i\geq 1}\sum_{\bar k,\bar l\geq 0} e'_{i\bar k}e'_{i\bar l}E_{\bar k\bar l} = \frac{1}{2}\Big(\sum_{i\geq 1} e'_i\otimes e'_i\Big) : E\,,$$

which is the "internal energy" and which of course is an objective scalar as the sum of the objective scalars E'_{kk}. It has been proved in 4.2 that ε is depending on $E = (E_{kl})_{k,l\geq 0}$, the energy matrix in definition (5.2), and apart from this only on G and $\boldsymbol{e}$, that is

$$\varepsilon = \frac{1}{2}\Big(\sum_{i\geq 1} e'_i\otimes e'_i\Big) : E = \frac{1}{2}(\mathrm{P}^{\mathrm{T}}\,\mathrm{G}^{-1}\,\mathrm{P}) : E = \frac{1}{2}\mathrm{G}^{-1} : E\,, \tag{6.2}$$

where the last equality holds by assumption (5.3) on E. Thus, if the entropy η depends only on ϱ and ε, then η is an allowed objective scalar. Therefore we assume

$$\eta = \widehat{\eta}(\varrho, \varepsilon)\,. \tag{6.3}$$

Consequently we have for the function $\widetilde{\eta}$ in (5.11)

$$\widetilde{\eta}\big(\varrho, (v'_k)_{k\geq 1}, (E'_{kl})_{k,l\geq 1}\big) = \eta = \widehat{\eta}\Big(\varrho, \frac{1}{2}\sum_{k\geq 1} E'_{kk}\Big)\,,$$

and it follows from (5.10) that

$$\lambda^{\varrho} = \widetilde{\eta}'_{\varrho} = \widehat{\eta}'_{\varrho}\,, \quad \lambda^v_k = \frac{1}{\varrho}\widetilde{\eta}'_{v'_k} = 0\,,$$
$$\lambda^e_{kl} = \widetilde{\eta}'_{E'_{kl}} = \frac{\lambda^e}{2}\delta_{k,l}\,, \quad \lambda^e := \widehat{\eta}'_{\varepsilon}\,.$$

With these identities and

$$\underline{q}_j := \frac{1}{2}\sum_{k\geq 1}\underline{\mathrm{Q}}'_{kkj}\,, \quad \mathbf{r}^e := \frac{1}{2}\sum_{k\geq 1}\mathbf{r}^e_{kk}$$

the formula (5.11) for the entropy equation becomes

$$\underline{\eta}_j = \eta\underline{v}_j + \widehat{\eta}'_{\varrho}\underline{\mathbf{J}}_j + \widehat{\eta}'_{\varepsilon}\underline{q}_j\,, \quad \boldsymbol{e}\bullet\underline{\eta} = \eta\,, \tag{6.4}$$

where the last equation follows from the assumption on $\widetilde{Q}$ in (5.3). Besides this the entropy production (5.12) becomes

$$\sigma = -\sum_{j\geq 0,k\geq 1}\partial_j v'_k\Big(\big(\widehat{\eta}'_{\varrho}\varrho + \widehat{\eta}'_{\varepsilon}\varepsilon - \eta\big)e_{kj} + \widehat{\eta}'_{\varepsilon}\underline{\Pi}'_{kj}\Big)$$
$$+\sum_{j\geq 0}\partial_j\widehat{\eta}'_{\varrho}\cdot\underline{\mathbf{J}}_j + \sum_{j\geq 0}\partial_j\widehat{\eta}'_{\varepsilon}\cdot\underline{q}_j$$
$$+\widehat{\eta}'_{\varrho}\mathbf{r}^{\varrho} + \widehat{\eta}'_{\varepsilon}\cdot\mathbf{r}^e\,.$$

To proceed further let us assume that, in analogy to the classical case,

$$\frac{1}{\theta} := \widehat{\eta}_\varepsilon(\varrho, \varepsilon) > 0\,, \quad \frac{\mu}{\theta} := \widehat{\eta}_\varrho(\varrho, \varepsilon)\,. \tag{6.5}$$

Here θ is the "absolute temperature" and μ the "chemical potential". We define the preliminary version $\big(\underline{S}'_{kj}\big)_{j\geq 0, k\geq 1}$ of the stress tensor by

$$\underline{S}'_{kj} := \theta\big(\big(\eta - \widehat{\eta}'_\varrho \varrho - \widehat{\eta}'_\varepsilon \varepsilon\big)e_{kj} - \widehat{\eta}'_\varepsilon \underline{\Pi}'_{kj}\big) = \big(\theta\eta - \mu\varrho - \varepsilon\big)e_{kj} - \underline{\Pi}'_{kj}$$

for $k \geq 1$, so that one gets for the entropy production the final version

$$\begin{aligned} 0 \leq \sigma = \widehat{\eta}'_\varepsilon \sum_{j\geq 0, k\geq 1} \partial_j v'_k \underline{S}'_{kj} + \sum_{j\geq 0} \partial_j \widehat{\eta}'_\varrho \cdot \underline{\mathbf{J}}_j + \sum_{j\geq 0} \partial_j \widehat{\eta}'_\varepsilon \cdot \underline{q}_j \\ + \widehat{\eta}'_\varrho \mathbf{r}^\varrho + \widehat{\eta}'_\varepsilon \cdot \mathbf{r}^e \end{aligned} \tag{6.6}$$

where $\sigma \geq 0$ by the entropy principle. If we now define the "pressure" p by

$$p := \theta\eta - \mu\varrho - \varepsilon\,, \tag{6.7}$$

which is Gibbs relation, the above definition takes the common form

$$\underline{\Pi}'_{kj} = p\, e_{kj} - \underline{S}'_{kj} \quad \text{for } k \geq 1 \tag{6.8}$$

We have to write this in terms of the reduced mass-momentum system (4.4)

$$\begin{aligned} \sum_{j\geq 0} \partial_{y_j} T_{kj} = g_k \quad \text{for } k \geq 0\,, \\ T_{kj} := \sum_{l\geq 0} \boldsymbol{e}_l T_{klj} = \varrho \underline{v}_k \underline{v}_j + \widetilde{\Pi}_{kj}\,, \quad \widetilde{\Pi}_{kj} := \sum_{l\geq 0} \boldsymbol{e}_l \widetilde{\mathrm{Q}}_{klj}\,. \end{aligned} \tag{6.9}$$

Now, by (5.6), for $k \geq 0$

$$\widetilde{\mathrm{Q}}'_{k0j} = \sum_{\bar{k},\bar{l}\geq 0} e'_{k\bar{k}} e'_{0\bar{l}} \widetilde{\mathrm{Q}}_{\bar{k}\bar{l}j} = \sum_{\bar{k}\geq 0} e'_{k\bar{k}} \widetilde{\Pi}_{\bar{k}j}$$

or, by renaming k as $\bar{k}$ and vice versa,

$$\widetilde{\mathrm{Q}}'_{\bar{k}0j} = \sum_{k\geq 0} e'_{\bar{k}k} \widetilde{\Pi}_{kj}$$

hence for $k \geq 0$, making use of 3.5,

$$\begin{aligned} \widetilde{\Pi}_{kj} &= \sum_{\bar{k}\geq 0} e_{\bar{k}k} \widetilde{\mathrm{Q}}'_{\bar{k}0j} = e_{0k} \underline{\mathbf{J}}_j + \sum_{i\geq 1} e_{ik} \widetilde{\Pi}'_{ij} \quad \text{(using (5.6))} \\ &= e_{0k} \underline{\mathbf{J}}_j + \sum_{i\geq 1} e_{ik} (v'_i \underline{\mathbf{J}}_j + \underline{\Pi}'_{ij}) \quad \text{(using (5.8))} \\ &= \Big(e_{0k} + \sum_{i\geq 1} e_{ik} v'_i\Big) \underline{\mathbf{J}}_j + \sum_{i\geq 1} e_{ik} \underline{\Pi}'_{ij} = \underline{v}_k \underline{\mathbf{J}}_j + \sum_{i\geq 1} e_{ik} \underline{\Pi}'_{ij}\,. \end{aligned}$$

Therefore, if we define for $k \geq 0$ the "pressure tensor" and the "stress tensor" by

$$\underline{\Pi}_{kj} := \sum_{i\geq 1} e_{ik}\underline{\Pi}'_{ij} \quad \text{and} \quad \underline{S}_{kj} := \sum_{i\geq 1} e_{ik}\underline{S}'_{ij}\,, \tag{6.10}$$

we have shown

$$\widetilde{\Pi}_{kj} = \underline{v}_k \underline{\mathbf{J}}_j + \underline{\Pi}_{kj} \text{ for } k \geq 0, \tag{6.11}$$

and the identity (6.8) becomes

$$\begin{aligned}\underline{\Pi}_{kj} &:= \sum_{i\geq 1} e_{ik}\underline{\Pi}'_{ij} = \sum_{i\geq 1} e_{ik}\big(p\, e_{ij} - \underline{S}'_{ij}\big) \\ &= p\sum_{i\geq 1} e_{ik}e_{ij} - \underline{S}_{kj} = p\,(\mathrm{P}\mathrm{G}\,\mathrm{P}^{\mathrm{T}})_{kj} - \underline{S}_{kj} \quad (\text{using } 4.2(2))\,,\end{aligned}$$

that is, the well known formula

$$\underline{\Pi} = p\,\mathrm{P}\mathrm{G}\,\mathrm{P}^{\mathrm{T}} - \underline{S}\,. \tag{6.12}$$

This shows 4.4(2), and therefore the statements about the reduced mass-momentum system are proved. We come back to the entropy production σ in (6.6), which is not so final since it contains the term

$$\sum_{j\geq 0, k\geq 1} \partial_j v'_k \underline{S}'_{kj} = \sum_{j\geq 0}\Big(\sum_{i\geq 1} \partial_j v'_i \underline{S}'_{ij}\Big)$$

depending on $\underline{S}' = \big(\underline{S}'_{ij}\big)_{i\geq 1, j\geq 0}$ and not on the stress tensor $\underline{S} = \big(\underline{S}_{kj}\big)_{k,j\geq 0}$. Now, we get from the definition (6.10)

$$\sum_{k\geq 0} e'_{ik}\underline{S}_{kj} = \sum_{\bar{i}\geq 1}\sum_{k\geq 0} e'_{ik}e_{\bar{i}k}\underline{S}'_{\bar{i}j} = \sum_{\bar{i}\geq 1} \delta_{i,\bar{i}}\underline{S}'_{\bar{i}j} = \underline{S}'_{ij}$$

and thus

$$\begin{aligned}\sum_{j\geq 0}\Big(\sum_{i\geq 1}\partial_j v'_i \underline{S}'_{ij}\Big) &= \sum_{k,j\geq 0}\Big(\sum_{i\geq 1} e'_{ik}\partial_j v'_i\Big)\underline{S}_{kj} \\ &= \sum_{k,j\geq 0}\Big(\sum_{i\geq 1} e'_{ik}\partial_j (e'_i \bullet \underline{v})\Big)\underline{S}_{kj}\,.\end{aligned} \tag{6.13}$$

That this is the generalization of the term in the classical case is shown in the next session.

7 Constitutive equation for fluids

We deal with the term (6.13) for the stress tensor $\underline{S} = \big(\underline{S}_{kj}\big)_{k,j\geq 0}$

$$\sum_{k,j\geq 0} \big(\sum_{i\geq 1} e'_{ik}\partial_j(e'_i\bullet\underline{v})\big)\underline{S}_{kj}\,. \tag{7.1}$$

which is part of the entropy inequality $\sigma \geq 0$ in 4.7. We show that this expression converges as $\mathrm{c}\to\infty$ to the well known term of the Navier-Stokes limit. By this limit we mean that $e'_k \to \overline{e}'_k$ and $e_k \to \overline{e}_k$ as $\mathrm{c}\to\infty$, where the limit basis are given as usual:

7.1 Limit basis. We obtain in the standard case the limits

$$\overline{e}'_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \overline{e}_0 = \begin{bmatrix} 1 \\ \mathbf{V} \end{bmatrix}, \quad \overline{e}_i = \begin{bmatrix} 0 \\ \mathbf{Q}\mathbf{e}_i \end{bmatrix}, \quad \overline{e}'_i = \begin{bmatrix} -\mathbf{V}\bullet\mathbf{Q}\mathbf{e}_i \\ \mathbf{Q}\mathbf{e}_i \end{bmatrix} \qquad \text{for } i \geq 1$$

where $\mathrm{D}_x\mathbf{V}$ is antisymmetric and $\mathbf{Q}$ depends only on t.

Proof. We consider the standard case, that is, we assume that $|\overline{e}'_0| = 1$. Then

$$\boldsymbol{e} = e'_0 \to \overline{\boldsymbol{e}} = \overline{e}'_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{with} \quad \boldsymbol{W} = \{e'_0\}^\perp \to \{\overline{e}'_0\}^\perp =: \overline{\boldsymbol{W}}\,,$$

which implies, since $\overline{e}'_0\bullet\overline{e}_0 = 1$, that

$$e_0 \to \overline{e}_0 = \begin{bmatrix} 1 \\ \mathbf{V} \end{bmatrix} =: \underline{\mathbf{V}}$$

which is the definition of the vector $\mathbf{V}$. The elements $\{\overline{e}_i\,;\ i\geq 1\}$ are an orthonormal set of $\overline{\boldsymbol{W}}$, that is

$$\overline{e}_i = \begin{bmatrix} 0 \\ \mathbf{Q}\mathbf{e}_i \end{bmatrix} \text{ for } i \geq 1$$

which is the definition of the orthonormal matrix $\mathbf{Q}$. Then the representation of the elements $\overline{e}'_i$ follow easily. Now with Y being a Newton transformation, $\mathbf{Q}$ satisfies the transformation rule

$$\begin{bmatrix} 0 \\ \mathbf{Q}\circ Y\,\mathbf{e}_i \end{bmatrix} = \overline{e}_i\circ Y = \mathrm{D}Y\overline{e}^*_i = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{Q}^*\mathbf{e}_i \end{bmatrix} = \begin{bmatrix} 0 \\ Q\,\mathbf{Q}^*\mathbf{e}_i \end{bmatrix}$$

that is $\mathbf{Q}\circ Y = Q\,\mathbf{Q}^*$. Hence if $\mathbf{Q}^*$ is the Identity for at least one *-observer, then $\mathbf{Q}\circ Y$ is a function of t^* only and so $\mathbf{Q}$ is independent of x. Similarly, $\mathbf{V}$ satifies the transformation rule

$$\begin{bmatrix} 1 \\ \mathbf{V}\circ Y \end{bmatrix} = \overline{e}_0\circ Y = \mathrm{D}Y\overline{e}^*_0 = \begin{bmatrix} 1 & 0 \\ \dot{X} & Q \end{bmatrix}\begin{bmatrix} 1 \\ \mathbf{V}^* \end{bmatrix} = \begin{bmatrix} 1 \\ \dot{X} + Q\,\mathbf{V}^* \end{bmatrix}$$

that is $\mathbf{V}\circ Y = \dot{X} + Q\,\mathbf{V}^*$, and therefore

$$\sum_{j\geq 1} Q_{j\bar{j}}(\partial_{x_j}\mathbf{V}_i)\circ Y = \partial_{x^*_{\bar{j}}}(\mathbf{V}_i\circ Y) = \dot{Q}_{i\bar{j}} + \sum_{\bar{i}\geq 1} Q_{i\bar{i}}\partial_{x^*_{\bar{j}}}\mathbf{V}^*_{\bar{i}}\,,$$

hence

$$(\partial_{x_j}\mathbf{V}_i)\circ Y = (\dot{Q}\,Q^{\mathrm{T}})_{ij} + \sum_{\bar{i},\bar{j}\geq 1} Q_{i\bar{i}}Q_{j\bar{j}}\partial_{x^*_{\bar{j}}}\mathbf{V}^*_{\bar{i}}\,.$$

It follows that if $\mathbf{V}^*$ is zero for at least one *-observer, then $\big(\partial_{x_j}\mathbf{V}_i\big)_{ij}$ is antisymmetric.

□

Since

$$\sum_{k\geq 0} e'_{0k}\underline{S}_{kj} = 0\,, \quad \sum_{j\geq 0} e'_{0j}\underline{S}_{kj} = 0\,,$$

we have also in the classical limit

$$0 = \sum_{k\geq 0} \overline{e}'_{0k}\underline{S}_{kj} = \underline{S}_{0j}\,, \quad 0 = \sum_{j\geq 0} \overline{e}'_{0j}\underline{S}_{kj} = \underline{S}_{k0}\,,$$

therefore

$$\underline{S} = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}.$$

Having this in mind we compute, since $\underline{v} = (1, v)$ and

$$\begin{bmatrix} -\mathbf{V}\bullet\mathbf{Q}\mathbf{e}_i \\ \mathbf{Q}\mathbf{e}_i \end{bmatrix} \bullet \underline{v} = (v - \mathbf{V})\bullet\mathbf{Q}\mathbf{e}_i = \big(\mathbf{Q}^{\mathrm{T}}(v - \mathbf{V})\big)_i\,,$$

and since $\mathbf{Q}$ depends only on t,

$$\begin{aligned}
&\sum_{k,j\geq 0}\big(\sum_{i\geq 1} e'_{ik}\partial_j(e'_i\bullet\underline{v})\big)\underline{S}_{kj} \longrightarrow \sum_{k,j\geq 0}\big(\sum_{i\geq 1} \overline{e}'_{ik}\partial_j(\overline{e}'_i\bullet\underline{v})\big)\underline{S}_{kj} \\
&= \Big(\sum_{i\geq 1}\begin{bmatrix} -\mathbf{V}\bullet\mathbf{Q}\mathbf{e}_i \\ \mathbf{Q}\mathbf{e}_i \end{bmatrix}\otimes\underline{\nabla}\Big(\begin{bmatrix} -\mathbf{V}\bullet\mathbf{Q}\mathbf{e}_i \\ \mathbf{Q}\mathbf{e}_i \end{bmatrix}\bullet\underline{v}\Big)\Big):\underline{S} \\
&= \Big(\sum_{i\geq 1}\begin{bmatrix} -\mathbf{V}\bullet\mathbf{Q}\mathbf{e}_i \\ \mathbf{Q}\mathbf{e}_i \end{bmatrix}\otimes\underline{\nabla}\big(\big(\mathbf{Q}^{\mathrm{T}}(v - \mathbf{V})\big)_i\big)\Big):\underline{S} \\
&= \Big(\sum_{i\geq 1}(\mathbf{Q}\mathbf{e}_i)\otimes\nabla\big(\big(\mathbf{Q}^{\mathrm{T}}(v - \mathbf{V})\big)_i\big)\Big):S \\
&= \sum_{k,j\geq 1}\sum_{i\geq 1}(\mathbf{Q}\mathbf{e}_i)_k\partial_{x_j}\big(\big(\mathbf{Q}^{\mathrm{T}}(v - \mathbf{V})\big)_i S_{kj} \\
&= \sum_{k,j\geq 1}\sum_{i\geq 1}\mathbf{Q}_{ki}\partial_{x_j}\big(\sum_{l\geq 1}\mathbf{Q}_{li}(v - \mathbf{V})_l\big)S_{kj} \\
&= \sum_{k,j\geq 1}\sum_{l\geq 1}\big(\sum_{i\geq 1}\mathbf{Q}_{ki}\mathbf{Q}_{li}\big)\partial_{x_j}(v - \mathbf{V})_l\cdot S_{kj} = \sum_{k,j\geq 1}\partial_{x_j}(v - \mathbf{V})_k\cdot S_{kj} \\
&= \sum_{k,j\geq 1}\big(\partial_{x_j}v_k - \partial_{x_j}\mathbf{V}_k\big)S_{kj} = \sum_{k,j\geq 1}\partial_{x_j}v_k\cdot S_{kj}\,,
\end{aligned}$$

if S is symmetric. This is true since $\big(\partial_{x_j}\mathbf{V}_k\big)_{jk}$ is antisymmetric.

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