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MEASURABILITY OF A SOLUTION OF A FREE BOUNDARY PROBLEM DESCRIBING ADSORPTION PHENOMENON

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Abstract. In this paper we consider infinite number of one dimensional free boundary problems as a mathematical model describing adsorption phenomena in holes of a porous material. Here, we denote by $\mathrm{P}(x, u_0(x), h(x))$ the free boundary problem for $x \in \Omega$, where x is a parameter taking a value in Ω and $u_0(x)$ and $h(x)$ are the initial data and the boundary data.

In [8] the problem was studied and we obtain the continuous property of the solution with respect to x, when u_0 and h are continuous. The main purpose of this paper is to establish the measurability of the solution with respect to x under relaxed assumptions given in [8] for u_0 and h.

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1 Introduction

In this paper, we consider the following free boundary problem in one dimensional domain for each $x \in \Omega$:

$$\rho_v u_t(x) - k u_{zz}(x) = 0 \text{ on } (s(x)(t), L) \text{ for } t \in [0,T], \tag{1.1}$$

$$u(x)(t, L) = h(x,t) \text{ for } 0 \le t \le T, \tag{1.2}$$

$$k u_z(x)(t, s(x)(t)) = (\rho_w - \rho_v u(x)(t, s(x)(t)))\, s_t(x)(t) \text{ for } t \in [0,T], \tag{1.3}$$

$$s_t(x)(t) = a\,(u(x)(t, s(x)(t))) - \varphi(s(x)(t)) \text{ for } t \in [0,T], \tag{1.4}$$

$$s(x)(0) = s_0(x), u(x)(0,z) = u_0(x,z) \text{ for } z \in [s_0(x), L], \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^3$, L, ρ_v, ρ_w, k and a are given positive constants, h is a given function on $\Omega \times (0,T)$, φ is a given continuous function on $\mathbb{R}$ and s_0 and u_0 are also given functions on Ω, and on $Q_{s_0}(\Omega) := \{(x,z) : x \in \Omega, s_0(x) < z < L\}$, respectively.

This model is proposed by Sato-Aiki-Murase-Shirakawa [7, 9] and represents the relationship between the relative humidity u and the degree of saturation s in the porous material. More precisely, $s = s(x)$ is a function on $[0,T]$ and $x \in \Omega$ so that $s(x) = s(x)(t)$ for $t \in [0,T]$ and $u = u(x) = u(x)(t,z)$ is a function on $Q_{s(x)}(T)$ given by

$$Q_{s(x)}(T) := \{(t,z) : 0 < t < T,\ s(x)(t) < z < L\}.$$

Throughout this paper, we sometimes omit the parameter x for simplicity as follows : $u = u(x) = u(t,z) = u(x)(t,z)$ and $s = s(x) = s(t) = s(x)(t)$.

For the above problem $\{(1.1) - (1.5)\}$ denoted by $\mathrm{P}(x) := \mathrm{P}_{h,s_0,u_0}(x)$, in [8] we proved the existence of a solution globally in time. Here, we introduce the notation $\tilde{u}(t,y) = u(t,(1-y)s(t) + yL)$ for $y \in [0,1]$ and reformulate $\mathrm{P}(x)$ to the following problem in a cylindrical domain denoted by $\tilde{\mathrm{P}}(x) := \tilde{\mathrm{P}}_{h,s_0,\tilde{u}_0}(x)$:

$$\rho_v \tilde{u}_t - \frac{k}{(L-s(t))^2}\tilde{u}_{yy} = \frac{\rho_v (1-y) s_t(t)}{L - s(t)}\tilde{u}_y \text{ in } Q(T) := (0,T) \times (0,1),$$

$$\tilde{u}(t,1) = h(x,t) \text{ for } 0 \le t \le T,$$

$$\frac{k}{L - s(t)}\tilde{u}_y(t,0) = (\rho_w - \rho_v \tilde{u}(t,0)) s_t(t) \text{ for } 0 \le t \le T,$$

$$s_t(t) = a(\tilde{u}(t,0) - \varphi(s(t))) \text{ for } 0 \le t \le T,$$

$$s(0) = s_0(x) \text{ in } \Omega,$$

$$\tilde{u}(0,y) = u(0,(1-y)s(0) + yL) \text{ on } [0,1].$$

As a important result in [8], we showed that the solution $(s,\tilde{u}) = (s(x),\tilde{u}(x))$ is a continuous in $\mathbb{R} \times L^2(Q(T))$ with respect to $x \in \overline{\Omega}$. From this continuity, we infer that s and $\tilde{u}$ are measurable on $\Omega \times [0,T]$, and on $\Omega \times [0,T] \times (0,1)$, respectively. However, in the result of [8], we impose a strong assumption for h, s_0 and u_0. In this paper, as a sequel of [8], we relax the assumption for h, s_0 and u_0, and consider the existence and uniqueness of a solution of $\tilde{\mathrm{P}}(x)$.

The purpose of this paper is to establish a unique solution $(s,\tilde{u})$ of $\tilde{\mathrm{P}}(x)$ on $[0,T]$ for a.e. $x \in \Omega$ such that $s \in L^2(0,T;L^2(\Omega))$ and $\tilde{u} \in L^\infty(0,T;L^2(\Omega \times (0,1))) \cap$

$L^2(\Omega; L^\infty(0,T;H^1(0,1)))$. By using this property, in near future, we can consider h as the relative humidity in macroscopic domain Ω and consider a two scale problem coupled by a partial differential equation for h in Ω which was studied in [1, 2, 3, 4] and the free boundary problem P(x) in each hole as a mathematical model for moisture transport appearing concrete carbonation process. We refer to [6] for modeling of the two scale problem.

This paper is organized as follows: In section 2, we note the assumptions and the main result concerning about the existence and uniqueness of a solution of $\tilde{\mathrm{P}}(x)$ for a.e. $x \in \Omega$ (Theorem 1). Next, as a property of solutions, we state the regularity and the continuous dependence of the solution thereof (Theorem 2). In section 3, we consider an approximation problem of $\tilde{\mathrm{P}}(x)$, and obtain the uniform estimate for an approximate solution with respect to $x \in \Omega$. By using the result of [8], we prove our main theorem by the limiting process for the solution of the approximation problem of $\tilde{\mathrm{P}}(x)$.

2 Our main results

In this paper we use the following notations. In general, for a Banach space X we denote by $|\cdot|_X$ its norm. Also, for $D \subset \mathbb{R}^N$ for $N = 1$ and $N = 3$, $H^1(D)$, $H^1_0(D)$ and $H^2(D)$ are the usual Sobolev spaces.

Throughout this paper, we assume the following conditions:

(A1) Ω is a open bounded connected domain of $\mathbb{R}^3$ which has the boundary $\partial\Omega$ in the class of C^2.

(A2) k and a are positive constants.

(A3) $h \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$, $h_t \in L^\infty(\Omega \times (0,T))$ with $0 \le h \le h^* < 1$ a.e. on $\Omega \times (0,T)$, where h^* is a positive constant.

(A4) $\varphi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $\varphi = 0$ on $(-\infty, 0]$, $\varphi \le 1$ on $\mathbb{R}$, $\varphi' > 0$ on $(0, L]$ and $\varphi(L) - h^* > 0$, where h^* is the same constant as in (A3). Also, we denote by $\hat{\varphi}$ the primitive function of φ with $\hat{\varphi}(0) = 0$ and put $C_\varphi = |\varphi'|_{L^\infty(\mathbb{R})}$.

(A5) Two positive constants ρ_w and ρ_v satisfy

$$\rho_w > 2\rho_v, \quad \rho_w \ge \rho_v(C_\varphi + 2), \quad 9aL\rho_v^2 \le k\rho_w.$$

(A6) $s_0 \in L^2(\Omega)$ such that $0 \le s_0 \le L - \delta$ for $\delta > 0$ a.e. on Ω, and the function $x \to |u_0(x)|_{H^1(s_0,L)}$ is bounded a.e. on Ω and $u_0(x, L) = h(x, 0)$ for $x \in \Omega$ and $0 \le u_0 \le 1$ a.e. on $Q_{s_0}(\Omega)$.

Next, for $x \in \Omega$ we state the definition of solutions of P(x) on $[0,T]$.

Definition 1.1 Let $x \in \Omega$, and s and u be functions on $[0,T]$ and $Q_{s(x)}(T)$, respectively, for $T > 0$. We call that a pair $(s,u) = (s(x), u(x))$ is a solution of P(x) on $[0,T]$ if the conditions (S1)-(S6) hold:

(S1) $s(x) \in W^{1,\infty}(0,T)$, $0 \le s(x) < L$ on $[0,T]$, $u(x) \in L^\infty(Q_{s(x)}(T))$, $u_t(x)$, $u_{zz}(x) \in L^2(Q_{s(x)}(T))$ and $|u_z(x)(\cdot)|_{L^2(s(x)(\cdot),L)} \in L^\infty(0,T)$.

(S2) $\rho_v u_t - k u_{zz} = 0$ in $Q_{s(x)}(T)$.

(S3) $u(x)(t, L) = h(x,t)$ for a.e. $t \in [0,T]$.

(S4) $k u_z(t, s(t)) = (\rho_w - \rho_v u(t, s(t)))\, s_t(t)$ for a.e. $t \in [0,T]$.

(S5) $s_t(t) = a\,(u(t, s(t))) - \varphi(s(t))$ for a.e. $t \in [0, T]$.
(S6) $s(x)(0) = s_0(x)$, $u(x)(0, z) = u_0(x, z)$ for $z \in [s_0(x), L]$.

In order to handle the problem $\mathrm{P}(x)$, we can formulate the following problem $\tilde{\mathrm{P}}(x) := \tilde{\mathrm{P}}_{h,s_0,\tilde{u}_0}(x)$ in a cylindrical domain by changes of variables:

$$\tilde{u}(t, y) := u(t, (1-y)s(t) + yL) \text{ for } (t, y) \in [0, T] \times [0, 1], \tag{2.1}$$

and

$$\begin{cases} \rho_v \tilde{u}_t - \frac{k}{(L-s(t))^2}\tilde{u}_{yy} = \frac{\rho_v(1-y)s_t}{L-s(t)}\tilde{u}_y \text{ a.e. in } Q(T), \\ \tilde{u}(t, 1) = h(x, t) \text{ for a.e. } t \in [0, T], \\ \frac{k}{L-s(t)}\tilde{u}_y(t, 0) = (\rho_w - \rho_v \tilde{u}(t, 0))s_t(t) \text{ for a.e. } t \in [0, T], \\ s_t(t) = a(\tilde{u}(t, 0) - \varphi(s(t))) \text{ for a.e. } t \in [0, T], \\ s(0) = s_0(x) \text{ in } \Omega, \\ \tilde{u}(0, y) = u(0, (1-y)s(0) + yL) =: \tilde{u}_0(y) \text{ for } y \in [0, 1]. \end{cases}$$

For the above problem $\tilde{\mathrm{P}}(x)$, we call that a pair $(s, \tilde{u})$ is a solution of $\tilde{\mathrm{P}}(x)$ on $[0, T]$ if the following (S) and each equation and condition of $\tilde{\mathrm{P}}(x)$ hold:

$$\text{(S)} \begin{cases} s(x) \in W^{1,\infty}(0, T), 0 \le s(x) < L \text{ a.e. on } [0, T], \\ \tilde{u}(x) \in W^{1,2}(0, T; L^2(0, 1)) \cap L^\infty(0, T; H^1(0, 1)) \cap L^\infty(Q(T)) \\ \cap L^2(0, T; H^2(0, 1)). \end{cases}$$

The first result is concerned about the existence and uniqueness of a solution of $\tilde{\mathrm{P}}(x)$ for a.e. $x \in \Omega$.

Theorem 1. *If (A1) $\sim$ (A6) hold, then for any $T > 0$ and a.e. $x \in \Omega$ there exists a unique solution $(s, \tilde{u}) = (s(x), \tilde{u}(x))$ of $\tilde{P}(x)$ on $[0, T]$ such that $0 \le \tilde{u}(x) \le 1$ a.e. on $Q(T)$ and $0 \le s(x) \le s^* < L$ a.e. on $[0, T]$, where s^* is a positive constant which does not depend on x.*

By Theorem 1 and putting $u(x)(t, z) = \tilde{u}(x)\left(t, \frac{z-s(x)}{L-s(x)}\right)$ for $(t, z) \in Q_{s(x)}(T)$ we see that $(s, u) = (s(x), u(x))$ is a unique solution of $\mathrm{P}(x)$ for a.e. $x \in \Omega$. Now, we state our main theorem of this paper.

Theorem 2. *Assume the same assumptions as in Theorem 1.*
(i) Let $(s(x), \tilde{u}(x))$ be a solution of $\tilde{P}(x)$ on $[0, T]$ for a.e. $x \in \Omega$ and $T > 0$. Then, $\tilde{u} \in L^\infty(\Omega; W^{1.2}(0, T; L^2(0, 1))) \cap L^\infty(\Omega; L^\infty(0, T; H^1(0, 1))) \cap L^\infty(\Omega; L^2(0, T; H^2(0, 1))) \cap L^\infty(\Omega; L^\infty(Q(T)))$ and $s \in L^\infty(\Omega; W^{1,\infty}(0, T))$.
(ii) Let $(s_1(x), \tilde{u}_1(x))$ and $(s_2(x), \tilde{u}_2(x))$ be a solution of $\tilde{P}_{h_1,s_0,\tilde{u}_0}(x)$ and $\tilde{P}_{h_2,s_0,\tilde{u}_0}(x)$on $[0, T]$ for a.e. $x \in \Omega$ and $T > 0$, respectively, then it holds that

$$\int_\Omega |\tilde{u}_1(t) - \tilde{u}_2(t)|^2_{L^2(0,1)}dx + \int_\Omega \int_0^t |\tilde{u}_{1y}(t) - \tilde{u}_{2y}(t)|^2_{L^2(0,1)}dxdt$$
$$+|s_1 - s_2|^2_{L^\infty(0,t;L^2(\Omega))} \le C|h_1 - h_2|^2_{W^{1,2}(0,t;L^2(\Omega))} \text{ for } t \in [0, T],$$

where C is a positive constant depending only on k, a, h^*, C_φ, ρ_w, ρ_v and s^*.

3 Proof of Theorem

At the first of this section, we note a useful lemma. Here, (A3)' and (A6)' are the following conditions:

(A3)' $h \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$ with $0 \le h \le h^* < 1$ on $\Omega \times (0,T)$, where h^* is a positive constant and $h_t \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T;H^2(\Omega))$.

(A6)' $s_0 \in C(\overline{\Omega})$ such that $0 \le s_0(x) < L$ for $x \in \overline{\Omega}$, and $u_0 \in C(\overline{Q_{s_0}(\Omega)})$ such that $u_0(x) \in H^1(s_0(x), L)$ and $u_0(x,L) = h(x,0)$ for $x \in \overline{\Omega}$ and $0 \le u_0 \le 1$ on $\overline{Q_{s_0}(\Omega)}$.

Lemma 1. *If (A1), (A2), (A3)', (A4), (A5), (A6)' hold, then for any $T > 0$ and $x \in \overline{\Omega}$ there exists a unique solution $(s, \tilde{u}) = (s(x), \tilde{u}(x))$ of $\tilde{P}(x)$ on $[0,T]$ such that $\tilde{u} \in C(\overline{\Omega}; L^2(Q(T)))$ and $s \in C(\overline{\Omega}; C([0,T]))$, $0 \le \tilde{u}(x) \le 1$ a.e. on $Q(T)$ and $0 \le s(x) \le s^{**} < L$ a.e. on $[0,T]$, where s^{**} is a positive constant which does not depend on x.*

This lemma is already proved in [8] so that we omit the precise proof. By using lemma 1, we prove Theorems 1 and 2.

Now, we take $\{h_j\} \subset C^\infty(\overline{\Omega \times (0,T)})$ such that $0 \le h_j \le h^*$ on $\Omega \times (0,T)$, $h_j \to h$ in $W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$ as $j \to \infty$ and $\{h_{jt}\}$ is bounded in $L^\infty(\Omega \times (0,T))$. Also, we take $\{s_{0j}\} \subset C^\infty(\overline{\Omega})$ and $\{\tilde{u}_{0j}\} \subset C^\infty(\overline{\Omega \times (0,1)})$ such that $s_{0j} \to s_0$ in $L^2(\Omega)$ as $j \to \infty$, and $0 \le s_{0j} \le L - \frac{\delta}{2}$ on Ω, $\tilde{u}_{0j} \to \tilde{u}_0$ in $L^2(\Omega \times (0,1))$ and for a.e. $x \in \Omega$, $\tilde{u}_{0j}(x) \to \tilde{u}_0(x)$ in $H^1(0,1)$ as $j \to \infty$, and $0 \le \tilde{u}_{0j} \le 1$ on $\overline{\Omega \times (0,1)}$, $\tilde{u}_{0j}(x,1) = h_j(x,0)$ for $x \in \overline{\Omega}$. By using h_j, s_{0j} and $\tilde{u}_{0j}$ we consider the following problem $\tilde{\mathrm{P}}_j(x) := \tilde{\mathrm{P}}_{h_j,s_{0j},\tilde{u}_{0j}}(x)$ for $x \in \overline{\Omega}$:

$$\rho_v \tilde{u}_t - \frac{k}{(L-s(t))^2}\tilde{u}_{yy} = \frac{\rho_v(1-y)s_t}{L-s(t)}\tilde{u}_y \text{ in } Q(T), \tag{3.1}$$

$$\tilde{u}(t,1) = h_j(x,t) \text{ for } t \in (0,T), \tag{3.2}$$

$$\frac{k}{L-s(t)}\tilde{u}_y(t,0) = (\rho_w - \rho_v\tilde{u}(t,0))s_t(t) \text{ for } t \in (0,T), \tag{3.3}$$

$$s_t(t) = a(\tilde{u}(t,0) - \varphi(s(t))) \text{ for } t \in (0,T), \tag{3.4}$$

$$s(0) = s_{0j}(x) \text{ in } \Omega, \tag{3.5}$$

$$\tilde{u}(0,y) = \tilde{u}_{0j}(y) \text{ for } y \in [0,1]. \tag{3.6}$$

Obviously, h_j, s_{0j} and $u_{0j}(x,z) := \tilde{u}_{0j}\left(x, \frac{z-s_{0j}(x)}{L-s_{0j}(x)}\right)$ satisfy (A3)' and (A6)'. Therefore, by Lemma 1, for $x \in \overline{\Omega}$ and $j \in \mathbb{N}$ we see that $\tilde{\mathrm{P}}_j(x)$ has a solution $(s_j, \tilde{u}_j) = (s_j(x), \tilde{u}_j(x))$ on $[0,T]$ such that $s_j \in C(\overline{\Omega}; C([0,T]))$ and $\tilde{u}_j \in C(\overline{\Omega}; L^2(Q(T))) \cap C(\overline{\Omega}; L^2(0,T;H^1(0,1)))$, and $0 \le \tilde{u}_j(x) \le 1$ a.e. on $Q(T)$ and $0 \le s_j(x) \le s^{**}_{jx}$ a.e. on $[0,T]$, where s^{**}_{jx} is a positive constant with $s^{**}_{jx} < L$. Here, we show the following lemma.

Lemma 2. *Let $(s_j(x), \tilde{u}_j(x))$ be a solution of $\tilde{P}_j(x)$ on $[0,T]$ for $x \in \overline{\Omega}$ and $j \in \mathbb{N}$. Then, $\{\tilde{u}_j(x); j \in \mathbb{N}\}$ is bounded in $W^{1,2}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^1(0,1))$ and $\{s_j(x); j \in \mathbb{N}\}$ is bounded in $W^{1,\infty}(0,T)$ for $x \in \overline{\Omega}$.*

Proof. For the solution $(s_j, \tilde{u}_j)$, by using the notation $u_j(t,z) = \tilde{u}_j\left(t, \frac{z-s_j(x)}{L-s_j(x)}\right)$, we can obtain the following two inequalities:

$$\begin{aligned}&\frac{\rho_v}{2}\frac{d}{dt}\int_{s_j(t)}^{L}|u_j(t)-h_j(x,t)|^2dz+\frac{k}{2}\int_{s_j(t)}^{L}|u_{jz}(t)|^2dx+\rho_w\frac{d}{dt}\hat{\varphi}(s_j(t))\\ \le&\rho_w(1+h^*)L|h_{jt}(x,t)|+\frac{\rho_w a}{2}\text{ for } x\in\overline{\Omega}\text{ and a.e. } t\in[0,T],\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\frac{\rho_v}{2}\int_0^{t_1}\int_{s_j(t)}^{L}|u_{jt}(t)|^2dzdt+\frac{k}{2}\int_{s_j(t_1)}^{L}|u_{jz}(t_1)|^2dz\\ \le&\frac{k}{2}\int_{s_{0j}}^{L}|\tilde{u}_{0jz}|^2dz+\frac{k}{2}\int_0^{t_1}s_{jt}(t)|u_{jz}(t,s_j(t))|^2dt\\ &+C_1\int_0^{t_1}(|s_{jt}(t)|^2+|h_{jt}(x,t)|^2)dt+C_1\text{ for } x\in\overline{\Omega}\text{ and } t_1\in[0,T],\end{aligned} \tag{3.8}$$

where C_1 is a positive constant. In fact, (3.7) is obtained by testing $\tilde{u}-h$ to (3.1) and testing $\frac{s_t}{a}$ to (3.4). Also, (3.8) is obtained by testing $\frac{\tilde{u}_j(t)-\tilde{u}_j(t-\tau)}{\tau}$ and letting $\tau\to 0$. For the detail derivation, we refer to [5]. Therefore, by the boundedness of $\{h_{jt}\}$ in $L^\infty(\Omega\times(0,T))$ and the fact that $|s_{jt}|\le 2a$ a.e. on $Q(T)$ it is easy to see that there exist $M_1>0$ and $M_2>0$ independent of j such that

$$\int_{s_j(t_1)}^{L}|u_j(x)(t_1,z)|^2dz\le M_1,\quad \int_0^{t_1}\int_{s_j(t)}^{L}|u_{jz}(x)(t,z)|^2dzdt\le M_1, \tag{3.9}$$

and

$$\int_0^{t_1}\int_{s_j(t)}^{L}|u_{jt}(x)|^2dzdt\le M_2,\quad \int_{s_j(t_1)}^{L}|u_{jz}(x)(t_1)|^2dz\le M_2\text{ for } t_1\in[0,T]\text{ and a.e. } x\in\overline{\Omega}. \tag{3.10}$$

Now, by putting

$$s^*:=L-\left(\frac{\varphi(L)-h^*}{2(\sqrt{M_2}+C_\varphi\sqrt{L})}\right)^2,$$

and using the same idea of the proof of [5, 8], we see that $0\le s_j(x)\le s^*<L$ for $t\in[0,T]$ and a.e. $x\in\overline{\Omega}$. By using this estimate for $\{s_j\}$, the notation (2.1) and the proof as in Lemma 2 in [8] we can conclude that Lemma 2 holds. □

Proofs of Theorems 1 and 2. By multiplying $\bar{u}_i-\bar{u}_j$ with $\bar{u}_k=\tilde{u}_k-h_k$ for $k=i,j$ to (3.1) and repeating the argument of the proof as in Lemma 4 of [8], we obtain that for $t_1\in[0,T]$ and a.e. $x\in\overline{\Omega}$ and $i,\, j\in\mathbb{N}$,

$$\begin{aligned}&|\bar{u}_i(x)(t_1)-\bar{u}_j(x)(t_1)|^2_{L^2(0,1)}+|s_i(x)(t_1)-s_j(x)(t_1)|^2+\int_0^{t_1}|\bar{u}_{iy}(x)-\bar{u}_{jy}(x)|^2_{L^2(0,1)}dt\\ \le&C_2\left(\int_0^{t_1}|h_{it}(x,t)-h_{jt}(x,t)|^2dt+\int_0^{t_1}|h_i(x,t)-h_j(x,t)|^2dt+|\bar{u}_i(x)(0)-\bar{u}_j(x)(0)|^2_{L^2(0,1)}\right),\end{aligned} \tag{3.11}$$

where C_2 is a positive constant independent of i and j.

Here, for each $j \in \mathbb{N}$, since $\tilde{u}_j \in C(\overline{\Omega}; C([0,T]; L^2(0,1))) \cap C(\overline{\Omega}; L^2(0,T; H^1(0,1)))$ and $s_j \in C(\overline{\Omega}; C([0,T]))$, we note that $\tilde{u}_j$ and $\tilde{u}_{jy}$ are measurable on $\Omega \times Q(T)$ and s_j is measurable on $\Omega \times (0,T)$. Then, by integrating (3.11) over Ω, we have that for $t_1 \in [0,T]$ and $i, j \in \mathbb{N}$,

$$
\begin{aligned}
&\int_\Omega |\bar{u}_i(x)(t_1) - \bar{u}_j(x)(t_1)|^2_{L^2(0,1)} dx + \int_\Omega \int_0^{t_1} |\bar{u}_{iy}(x) - \bar{u}_{jy}(x)|^2_{L^2(0,1)} dt dx \\
&+ \int_\Omega |s_i(x)(t_1) - s_j(x)(t_1)|^2 dx \\
\le& C_2 \left(\int_\Omega \int_0^{t_1} |h_{it}(x,t) - h_{jt}(x,t)|^2 dt dx + \int_\Omega \int_0^{t_1} |h_i(x,t) - h_j(x,t)|^2 dt dx \right) \\
&+ C_2 \int_\Omega |\bar{u}_i(x)(0) - \bar{u}_j(x)(0)|^2_{L^2(0,1)} dx \\
\le& C_2 \left(\int_0^{t_1} \int_\Omega |h_{it}(x,t) - h_{jt}(x,t)|^2 dx dt + \int_0^{t_1} \int_\Omega |h_i(x,t) - h_j(x,t)|^2 dx dt \right) \\
&+ C_2 \int_\Omega |\bar{u}_i(x)(0) - \bar{u}_j(x)(0)|^2_{L^2(0,1)} dx. \qquad (3.12)
\end{aligned}
$$

Therefore, by the definition of $\{h_j\}$ and $\{\tilde{u}_{0j}\}$ the above inequality implies that $\{\tilde{u}_j\}$ is a Cauchy sequence in $L^\infty(0,T; L^2(\Omega \times (0,1))) \cap L^2(\Omega; L^2(0,T; H^1(0,1)))$ and $\{s_j\}$ is a Cauchy sequence in $L^\infty(0,T; L^2(\Omega))$. By these results, we see that there exist $\tilde{u} \in L^\infty(0,T; L^2(\Omega \times 0,1))) \cap L^2(\Omega; L^2(0,T; H^1(0,1)))$ and $s \in L^\infty(0,T; L^2(\Omega))$ such that

$$
\tilde{u}_j \to \tilde{u} \text{ in } L^\infty(0,T; L^2(\Omega \times (0,1))) \cap L^2(\Omega; L^2(0,T; H^1(0,1))),
$$

$$
s_j \to s \text{ in } L^\infty(0,T; L^2(\Omega)) \text{ as } j \to \infty.
$$

Namely, $\tilde{u}_j \to \tilde{u}$ in $L^2((0,T) \times \Omega \times (0,1))$ and $s_j \to s$ in $L^2((0,T) \times \Omega)$ as $j \to \infty$. Then, there exists a subsequence $\{j_k\} \subset \{j\}$ and $\Omega_0 \subset \Omega$ with $|\Omega_0| = 0$ such that

$$
\tilde{u}_{j_k}(x) \to \tilde{u}(x) \text{ in } L^2(Q(T)), \tag{3.13}
$$

and

$$
s_{j_k}(x) \to s(x) \text{ in } L^2(0,T) \tag{3.14}
$$

as $k \to \infty$ for $x \in \Omega \setminus \Omega_0$. Moreover, by Lemma 2 and (3.1) $\{\tilde{u}_j(x); j \in \mathbb{N}\}$ is bounded in $L^2(0,T; H^2(0,1))$ for $x \in \Omega \setminus \Omega_0$, and therefore we can take a subsequence $\{j_k(x)\} \subset \{j_k\}$ such that for some $\hat{u}(x)$ and $\hat{s}(x)$, the following convergences hold:

$$
\tilde{u}_{j_k(x)}(x) \to \hat{u}(x) \begin{cases} \text{in } C(\overline{(0,T) \times (0,1)}), \\ \text{weakly in } W^{1,2}(0,T; L^2(0,1)), \\ \text{weakly in } L^2(0,T; H^2(0,1)), \\ \text{weakly-* in } L^\infty(0,T; H^1(0,1)), \end{cases}
$$

$$
\bar{u}_{0j_k(x)}(x) \to \tilde{u}_0(x) \text{ in } C([0,1]),
$$

$$s_{j_k(x)}(x) \to \hat{s}(x) \text{ weakly in } W^{1,2}(0,T).$$

Therefore, by (3.13), (3.14) and the above convergences, we can see that $\hat{u} = \tilde{u}$ in $L^2(Q(T))$ for a.e. on Ω, and $\hat{s} = s$ in $L^2(0,T)$ for a.e. on Ω, and the whole sequences $\{s_j\}$ and $\{\tilde{u}_j\}$ converge s in $L^2(0,T)$ and $\tilde{u}$ in $L^2(Q(T))$ as $j \to \infty$, respectively. Since $(\hat{s}, \hat{u})$ is a solution of $\tilde{\mathrm{P}}(x)$ on $[0,T]$ for a.e. $x \in \Omega$ we can conclude that Theorem 1 holds.

Next, by $s \in L^\infty(0,T;L^2(\Omega))$ and $\tilde{u} \in L^\infty(0,T;L^2(\Omega\times(0,1)))\cap L^2(\Omega;L^2(0,T;H^1(0,1)))$ in the proof of Theorem 1, it is easy to see that Theorem 2 (i) holds. Also, let $(s_1, \tilde{u}_1)$ and $(s_2, \tilde{u}_2)$ be a solution of $\tilde{\mathrm{P}}_1(x)$ and $\tilde{\mathrm{P}}_2(x)$ on $[0,T]$ for a.e. $x \in \Omega$ and $\bar{u}_i = u_i - h$ for $i = 1, 2$, then we note that (3.9) and (3.10) replaced s_j and u_j by s_i and u_i hold. Therefore, by the same derivation of (3.12) we have

$$\begin{aligned}
&\int_\Omega |\bar{u}_1(x)(t) - \bar{u}_2(x)(t)|^2_{L^2(0,1)} dx + \int_\Omega \int_0^t |\bar{u}_{1y}(x) - \bar{u}_{2y}(x)|^2_{L^2(0,1)} dt dx \\
&+ \int_\Omega |s_1(x)(t) - s_2(x)(t)|^2 dx \\
\le& C_2 \left(\int_0^t \int_\Omega |h_{1t}(x,\tau) - h_{2t}(x,\tau)|^2 dx d\tau + \int_0^t \int_\Omega |h_1(x,\tau) - h_2(x,\tau)|^2 dx d\tau \right) \text{ for } t \in [0,T].
\end{aligned}$$

This yields that Theorem 2 (ii) holds. Thus, Theorem 2 is also proved.

References

[1] T. Aiki and K. Kumazaki, Mathematical model for hysteresis phenomenon in moisture transport in concrete carbonation process, *Phys. B,* **407** (2012), 1424–1426.

[2] T. Aiki and K. Kumazaki, Mathematical modelling of concrete carbonation process with hysteresis effect, *RIMS, Kyoto Univ., sūrikaisekikenkyūsho, kōkyūroku,* **1792** (2012), 99–107.

[3] T. Aiki and K. Kumazaki, Uniqueness of a solutions to a mathematical model describing moisture transport in concrete materials, *Netw., Heterogneous media,* **9** (2014), no. 4,

[4] T. Aiki and K. Kumazaki, Well-posedness of a mathematical model for moisture transport appearing in concrete carbonation process, *Adv. Math. Sci. Appl.,* **21** (2011), 361–381.

[5] T. Aiki and Y. Murase, On a large time behavior of a solution to a one-dimensional free boundary problem for adsorption phenomena, to appear in *J. Math. Anal. Appl.*

[6] T. Aiki, Y. Murase, N. Sato and K. Kumazaki, A two scale model for concrete carbonation process in a three dimensional domain, *RIMS, Kyoto Univ., sūrikaisekikenkyūsho, kōkyūroku*, **1997** (2016), 133-139.

[7] T. Aiki, Y. Murase, N. sato and K. Shirakawa, A mathematical model for a hysteresis appearing in adsorption phenomena, *RIMS, Kyoto Univ., sūrikaisekikenkyūsho, kōkyūroku,* **1856** (2013), 1–11.

[8] K. Kumazaki, Continuous dependence of a solution of a free boundary problem describing adsorption phenomenon for a given data, *Adv. Math. Sci. Appl.*,, **25** (2016), no. 1, 283-299.

[9] N. Sato, T. Aiki, Y. Murase and K. Shirakawa, A one dimensional free boundary problem for adsorption phenomena, Netw., Heterogeneous media., **9** (2014), no. 4, 655–668.