Advances in Mathematical
Sciences and Applications
Vol. 26, No. 1 (2017), pp. 119–141

GAKKOTOSHO
TOKYO JAPAN

INITIAL-BOUNDARY VALUE PROBLEMS FOR COMPLEX GINZBURG-LANDAU EQUATIONS IN GENERAL DOMAINS

Takanori Kuroda
Major in Pure and Applied Physics, Graduate School of Advanced Science and Engineering,
Waseda University, 3-4-1 Okubo Shinjuku-ku, Tokyo, 169-8555, JAPAN
(E-mail: 1d_est_quod_est@ruri.waseda.jp)

Mitsuharu Ôtani †
Department of Applied Physics, School of Science and Engineering,
Waseda University, 3-4-1 Okubo Shinjuku-ku, Tokyo, 169-8555, JAPAN
(E-mail: otani@waseda.jp)

and

Shoji Shimizu
NTT DATA Mathematical Systems Inc.
Shinjyuku-ku, Tokyo, 160-0016, JAPAN
(E-mail: shimizu_s@msi.co.jp)

Abstract. In this paper, we are concerned with the global solvability of the initial-boundary value problem for complex Ginzburg-Landau (CGL) equations in general domains. Under a suitable condition on parameters appearing in CGL equations, the global solvability of the initial-boundary value problem for CGL equations is already discussed by several methods. We here introduce a new approach to CGL equations based on the monotone and non-monotone perturbation theory for the parabolic equations governed by subdifferential operators. By using this method together with some approximate procedure and a diagonal argument, the global solvability is shown without assuming any growth conditions on the nonlinear terms.

†Partly supported by the Grant-in-Aid for Scientific Research, # 15K13451, the Ministry of Education, Culture, Sports, Science and Technology, Japan.
Communicated by Toyohiko Aiki; Received December 10, 2016.
2010 *Mathematics Subject Classification.* Primary: 35Q56, 47J35; Secondary: 35K61.
Keywords: Initial boundary value problem, global solvability, complex Ginzburg-Landau equation, unbounded general domain, subdifferential operator.

1 Introduction

In this paper we are concerned with the following complex Ginzburg-Landau equation in a general domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$:

$$\text{(CGL)} \begin{cases} \partial_t u - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u &= f \qquad \text{in } \Omega \times (0,\infty), \\ u &= 0 \qquad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{cases}$$

where $\lambda, \kappa \in \mathbb{R}_+ := (0,\infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $q \geq 2$ are constants; $i = \sqrt{-1}$ is the imaginary unit; $u_0 : \Omega \to \mathbb{C}$ is an initial function; $f : \Omega \times (0,\infty) \to \mathbb{C}$ is an external force; $u : \overline{\Omega} \times [0,\infty) \to \mathbb{C}$ is a complex valued unknown function. In extreme cases, (CGL) gives two well-known equations: nonlinear heat equation (when $\alpha = \beta = 0$) and nonlinear Schrödinger equation (when $\lambda = \kappa = 0$). Thus the general case of (CGL) could be regarded as "intermediate" between nonlinear heat equation and nonlinear Schrödinger equation.

From physical point of view, equation (CGL) describes the finite amplitude evolution of instability waves in a vast variety of dissipative systems, especially near Hopf bifurcations: e.g., Rayleigh-Bénard convection, Taylor-Couette flow (see Cross-Hohenberg [5]); nonlinear traveling waves in a human heart (see Bekki-Harada-Kanai [3]). The equation also arises in the dynamics of self-oscillating fields of the reaction-diffusion type systems: e.g., Belousov-Zhabotinsky reaction (see Kuramoto [9]). As for the overview of various phenomena described by (CGL), we refer to Aranson-Kramer [2], where the relevant solutions are studied to get an insight into non-equilibrium phenomena.

The mathematical study for the existence and the uniqueness of solutions of (CGL) began in the late 80s, and developed in the 90s. In this period, (CGL) had been studied under various boundary conditions and assumptions on the parameters in several function spaces: Temam [16] obtained a unique weak solution by Galerkin method for the case where $N = 1, 2$ and $q = 4$; Yang [17] used the semi-group $\{e^{(\lambda+i\alpha)t\Delta}; t \geq 0\}$ to get mild solutions for the case where $N = 1, 2, 3$ and $q \leq 2 + \frac{4}{N}$; Doering-Gibbon-Levermore [6] established global (in time) weak solutions to (CGL) by an argument similar to that in the proof of Leray's existence theorem for global weak solutions of Navier-Stokes equations; Levermore-Oliver [10] recovered the regularity of mild solutions by the bootstrap argument to get classical solutions in the N-dimensional torus $\mathbb{T}^N$ with $N \leq 5$, $q \leq 2N/(N-2)_+$ and $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{CGL}(c_q^{-1})$ (see (3.1)); Ginibre-Velo [7, 8] treated the case where the initial data and solutions belong to local spaces with no decay conditions at infinity in $\mathbb{R}^N$.

However these works did not make the most use of the parabolicity of (CGL). Okazawa-Yokota [11] developed the complex space version of monotonicity methods in which the theory of maximal monotone operators plays an important role. In their works, the existence of strong solutions is established for either $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{CGL}(c_q^{-1})$ (if $u_0 \in \mathrm{H}_0^1 \cap \mathrm{L}^q$) or $|\beta|/\kappa \leq 2\sqrt{q-1}/(q-2)$ (if $u_0 \in \mathrm{L}^2$). Subsequently, Takeuchi-Asakawa-Yokota [15] improved this result so that for any $u_0 \in \mathrm{L}^2$, there exists a strong solution provided that $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{CGL}(c_q^{-1})$. It should be noted that these works rely essentially on the compactness method requiring the boundedness of Ω.

The solvability of (CGL) in a general domain was first treated in Yokota-Okazawa [18], where the compactness method is replaced by Yosida approximation method. Since this method requires a new type of condition in terms of the Yosida approximation, it is necessary to assume the upper bound condition on q, i.e., $q \leq 2+\frac{4}{N}$. As for these operator theoretical approaches, we refer to Okazawa-Yokota [12], where the well-posedness for (CGL) is discussed for two cases: L^2-initial data case and H^1_0-initial data case.

In this paper, we study the global solvability of (CGL) in a general domain without assuming any upper bound condition on q. Since the embedding $\mathrm{H}^1 \subset \mathrm{L}^2$ is no longer compact in a general domain Ω, we can not directly apply the compactness method. On the other hand, if one tries to apply the contraction mapping principle, one needs to assume some strong upper bound condition on q.

In order to overcome this difficulty, we first introduce suitable approximate problems for (CGL) and establish a priori estimates for solutions of approximate problems. As for the convergence of solutions of approximate problems, on each bounded sub-domains $\Omega' \subset \Omega$, we apply Ascoli's theorem in each restricted function space $\mathrm{C}([0,T];\mathrm{L}^2(\Omega'))$ to derive the locally strong convergence of the nonlinear terms. Then by a diagonal argument, we can construct a sequence of solutions of approximate problems which converges weakly in Ω and strongly on any bounded subsets $\Omega' \subset \Omega$ (see (6.20) and (7.18)).

In carrying out this procedure, we introduce a new approach to (CGL) based on a different perspective of (CGL) than before. In former studies, $-(\lambda + i\alpha)\Delta u$ is always regarded as the leading term and the power-type nonlinear term: $(\kappa + i\beta)|u|^{q-2}u$ as a perturbation. As a matter of fact, this treatment provides some useful properties (e.g., the well-definedness of the semi-group $\{e^{(\lambda+i\alpha)t\Delta}; t \geq 0\}$; monotonicity of $(\kappa + i\beta)|u|^{q-2}u$ if $|\beta|/\kappa \leq c_q^{-1}$). However, when we focus on the parabolic nature of (CGL), this approach might not be suitable one, for instance the leading term $-(\lambda+i\alpha)\Delta u$ is not a self-adjoint operator. From this point of view, we regard the real parts: $-\lambda\Delta u$ and $\kappa|u|^{q-2}u$ as the principal part of (CGL), since both of them can be represented as subdifferential operators: $-\lambda\Delta u = \lambda\partial\varphi(u)$ and $\kappa|u|^{q-2}u = \kappa\partial\psi(u)$ (see (2.7) and (2.8)). In addition, it will be shown that the sum of these can be also represented as a single subdifferential operator (see (2.17)). Therefore (CGL) can be reduced to a parabolic type equation governed by a single subdifferential operator with two perturbation terms: $-i\alpha\Delta u$ (monotone perturbation) and $i\beta|u|^{q-2}u$ (non-monotone perturbation).

Since the subdifferential operators are normally defined in real Hilbert spaces, we work in real product spaces instead of complex spaces, by which some calculations seems to become easier than before, for example (2.1)-(2.4) allow us a systematic manipulation for the cancellation of terms.

This paper consists of seven sections. In §2, we fix some notations and prepare some preliminaries. Two main results are stated in §3 and key inequalities are prepared in §4. In §5, we introduce approximate problems for (CGL) and show their solvability. §6 and §7 are devoted to proofs of main results.

2 Notations and Preliminaries

In this section, in order to formulate (CGL) as an evolution equation in a real product function space, we fix some notations and prepare some related results.

In what follows, we identify $\mathbb{C}$ with $\mathbb{R}^2$ by the correspondence: $u = u_1 + iu_2 \in \mathbb{C} \mapsto$

$U = (u_1, u_2)^{\mathrm{T}} \in \mathbb{R}^2$. Then define the following:

$$(U \cdot V)_{\mathbb{R}^2} := u_1\, v_1 + u_2\, v_2, \qquad U = (u_1, u_2)^{\mathrm{T}},\ V = (v_1, v_2)^{\mathrm{T}} \in \mathbb{R}^2,$$
$$\mathbb{L}^2(\Omega) := \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega),\ (U, V)_{\mathbb{L}^2} := (u_1, v_1)_{\mathrm{L}^2} + (u_2, v_2)_{\mathrm{L}^2},$$
$$U = (u_1, u_2)^{\mathrm{T}}_{,}\ V = (v_1, v_2)^{\mathrm{T}} \in \mathbb{L}^2(\Omega),$$
$$\mathbb{L}^p(\Omega) := \mathrm{L}^p(\Omega) \times \mathrm{L}^p(\Omega), \quad |U|^p_{\mathbb{L}^p} := |u_1|^p_{\mathrm{L}^p} + |u_2|^p_{\mathrm{L}^p}, \quad U \in \mathbb{L}^p(\Omega)\ \ (1 \le p \le \infty),$$
$$\mathbb{H}^1_0(\Omega) := \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega), \quad (U, V)_{\mathbb{H}^1_0} := (u_1, v_1)_{\mathrm{H}^1_0} + (u_2, v_2)_{\mathrm{H}^1_0}, \quad U, V \in \mathbb{H}^1_0(\Omega).$$

We use the differential symbols to act on each component of $\mathbb{H}^1_0(\Omega)$-element:

$$D_i = \frac{\partial}{\partial x_i} : \mathbb{H}^1_0(\Omega) \to \mathbb{L}^2(\Omega),\ D_i U = (D_i u_1, D_i u_2)^{\mathrm{T}} \in \mathbb{L}^2(\Omega)\ (i = 1, \cdots, N),$$
$$\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_N}\right) : \mathbb{H}^1_0(\Omega) \to (\mathrm{L}^2(\Omega))^{2N},\ \nabla U = (\nabla u_1, \nabla u_2)^{\mathrm{T}} \in (\mathrm{L}^2(\Omega))^{2N}.$$

We further define, for $U = (u_1, u_2)^{\mathrm{T}},\ V = (v_1, v_2)^{\mathrm{T}},\ W = (w_1, w_2)^{\mathrm{T}}$,

$$(U(x) \cdot \nabla V(x)) := u_1(x) \nabla v_1(x) + u_2(x) \nabla v_2(x) \in \mathbb{R}^N,$$
$$(U(x) \cdot \nabla V(x)) W(x) := (u_1(x)\, w_1(x) \nabla v_1(x), u_2(x)\, w_2(x) \nabla v_2(x))^{\mathrm{T}} \in \mathbb{R}^{2N},$$
$$(\nabla U(x) \cdot \nabla V(x)) := \nabla u_1(x) \cdot \nabla v_1(x) + \nabla u_2(x) \cdot \nabla v_2(x) \in \mathbb{R}^1,$$
$$|\nabla U(x)| := \left(|\nabla u_1(x)|^2_{\mathbb{R}^N} + |\nabla u_2(x)|^2_{\mathbb{R}^N}\right)^{1/2}.$$

As a counterpart of the multiplication of the imaginary number i in $\mathbb{C}$, we introduce the following matrix I, which is a linear isometry in $\mathbb{R}^2$:

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The realization of I in $\mathbb{L}^p(\Omega)$ is also denoted by I, i.e., $I\,U = (-u_2, u_1)^{\mathrm{T}}$ for all $U = (u_1, u_2)^{\mathrm{T}} \in \mathbb{L}^p(\Omega)$.

Then the following properties hold.

1. Skew-symmetric property of the matrix I:

$$(IU \cdot V)_{\mathbb{R}^2} = -(U \cdot IV)_{\mathbb{R}^2}; \quad (IU \cdot U)_{\mathbb{R}^2} = 0 \quad \text{for each } U, V \in \mathbb{R}^2. \tag{2.1}$$

2. Commutative property of the matrix I and the differential operator $D_i = \frac{\partial}{\partial x_i}$:

$$I\, D_i = D_i\, I : \mathbb{H}^1_0 \to \mathbb{L}^2\ (i = 1, \cdots, N). \tag{2.2}$$

3. (In)equalities resulting from the orthogonality of vectors V and IV:

$$(U \cdot V)^2_{\mathbb{R}^2} + (U \cdot IV)^2_{\mathbb{R}^2} = |U|^2_{\mathbb{R}^2} |V|^2_{\mathbb{R}^2} \quad \text{for each } U, V \in \mathbb{R}^2, \tag{2.3}$$
$$(U, V)^2_{\mathbb{L}^2} + (U, IV)^2_{\mathbb{L}^2} \le |U|^2_{\mathbb{L}^2} |V|^2_{\mathbb{L}^2} \quad \text{for each } U, V \in \mathbb{L}^2(\Omega). \tag{2.4}$$

Properties (2.1) and (2.2) are obvious. By virtue of the orthogonality of V and IV, (2.3) is nothing but Pythagorean theorem and (2.4) comes from Bessel's inequality.

Let H be a Hilbert space and denote by $\Phi(\mathrm{H})$ the set of all lower semi-continuous convex function ϕ from H into $(-\infty, +\infty]$ such that the effective domain of ϕ given by $\mathrm{D}(\phi) := \{u \in \mathrm{H};\ \phi(u) < +\infty\}$ is not empty. Then for $\phi \in \Phi(\mathrm{H})$, the subdifferential of ϕ at $u \in \mathrm{D}(\phi)$ is defined by

$$\partial\phi(u) := \{\, f \in \mathrm{H}\,;\ \ (f, v-u)_{\mathrm{H}} \le \phi(v) - \phi(u) \ \text{ for all } v \in \mathrm{H}\,\}.$$

Then $\partial\phi$ becomes a possibly multivalued maximal monotone operator with domain $\mathrm{D}(\partial\phi) = \{\, u \in \mathrm{H}\,;\, \partial\phi(u) \neq \emptyset\,\}$. However for the arguments in what follows, we have only to consider the case where $\partial\phi$ is single valued.

Now we define two functionals $\varphi,\ \psi : \mathbb{L}^2(\Omega) \to [0, +\infty]$ by

$$\varphi(U) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla U(x)|^2 dx & \text{if } U \in \mathbb{H}^1_0(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.5}$$

$$\psi(U) := \begin{cases} \dfrac{1}{q}\displaystyle\int_\Omega |U(x)|^q_{\mathbb{R}^2} dx & \text{if } U \in \mathbb{L}^q(\Omega) \cap \mathbb{L}^2(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

Then it is easy to see that $\varphi, \psi \in \Phi(\mathbb{L}^2(\Omega))$ and their subdifferentials are given by

$$\partial\varphi(U)(\cdot) = -\Delta U(\cdot) \quad \text{with } \mathrm{D}(\partial\varphi) = \{\, U \in \mathbb{H}^1_0(\Omega)\,;\, \Delta U \in \mathbb{L}^2(\Omega)\,\}, \tag{2.7}$$

$$\partial\psi(U)(\cdot) = |U(\cdot)|^{q-2}_{\mathbb{R}^2} U(\cdot) \quad \text{with } \mathrm{D}(\partial\psi) = \mathbb{L}^{2(q-1)}(\Omega) \cap \mathbb{L}^2(\Omega). \tag{2.8}$$

Furthermore for any $\mu > 0$, we can define Yosida approximations $\partial\varphi_\mu,\ \partial\psi_\mu$ of $\partial\varphi,\ \partial\psi$ by

$$\partial\varphi_\mu(U) := \frac{1}{\mu}\,(U - J^{\partial\varphi}_\mu U) = \partial\varphi(J^{\partial\varphi}_\mu U), \quad J^{\partial\varphi}_\mu := (1 + \mu\partial\varphi)^{-1}, \tag{2.9}$$

$$\partial\psi_\mu(U) := \frac{1}{\mu}\,(U - J^{\partial\psi}_\mu U) = \partial\psi(J^{\partial\psi}_\mu U), \quad J^{\partial\psi}_\mu := (1 + \mu\partial\psi)^{-1}. \tag{2.10}$$

Then it is well known that $\partial\varphi_\mu,\ \partial\psi_\mu$ are Lipschitz continuous on $\mathbb{L}^2(\Omega)$ (see [4]).

Here for later use, we prepare some fundamental properties of I in connection with $\partial\varphi,\ \partial\psi,\ \partial\varphi_\mu,\ \partial\psi_\mu$.

Lemma 2.1. *The following orthogonality conditions hold.*

$$(\partial\varphi(U), IU)_{\mathbb{L}^2} = 0 \quad \forall U \in \mathrm{D}(\partial\varphi), \quad (\partial\psi(U), IU)_{\mathbb{L}^2} = 0 \quad \forall U \in \mathrm{D}(\partial\psi), \tag{2.11}$$

$$(\partial\varphi_\mu(U), IU)_{\mathbb{L}^2} = 0, \quad (\partial\psi_\mu(U), IU)_{\mathbb{L}^2} = 0 \quad \forall U \in \mathbb{L}^2(\Omega), \tag{2.12}$$

$$(\partial\varphi(U), I\partial\varphi_\mu(U))_{\mathbb{L}^2} = 0 \quad \forall U \in \mathrm{D}(\partial\varphi), \tag{2.13}$$

$$(\partial\psi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2} = 0 \quad \forall U \in \mathrm{D}(\partial\psi). \tag{2.14}$$

Proof. The first properties (2.11) follows from the integration by parts and a direct calculation. To see (2.12), put $V = J_\mu^{\partial\varphi}U$. Then in view of (2.9), (2.1) and (2.11), we get

$$(\partial\varphi_\mu(U), IU)_{\mathbb{L}^2} = (\partial\varphi(V), IV + \mu I\partial\varphi(V))_{\mathbb{L}^2} = 0.$$

The second orthogonality follows similarly. As for (2.13), by using (2.9), (2.11) and the self-adjointness of $\partial\varphi$, we get

$$\begin{aligned}(\partial\varphi(U), I\partial\varphi_\mu(U))_{\mathbb{L}^2} &= \frac{1}{\mu}(\partial\varphi(U), I(U-V))_{\mathbb{L}^2} = -\frac{1}{\mu}(\partial\varphi(U), IV)_{\mathbb{L}^2}\\ &= -\frac{1}{\mu}(V + \mu\partial\varphi(V), I\partial\varphi(V))_{\mathbb{L}^2} = \frac{1}{\mu}(IV, \partial\varphi(V))_{\mathbb{L}^2} = 0.\end{aligned}$$

To see (2.14), put $W = J_\mu^{\partial\psi}U = (w_1, w_2)^{\mathrm{T}}$. Then by (2.10) and (2.8), we find

$$\begin{aligned}&(\partial\psi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2} = (\partial\psi(W + \mu\partial\psi(W)), I\partial\psi(W))_{\mathbb{L}^2}\\ &= \int_\Omega |W|^{q-2}|W + \mu\partial\psi(W)|^{q-2}\left(-w_2\,(1+\mu|W|^{q-2})\,w_1 + w_1\,(1+\mu|W|^{q-2})\,w_2\right)\,dx = 0.\end{aligned}$$

□

Moreover we can show that the sum of $\partial\varphi$ and $\partial\psi$ is also represented as a single subdifferential operator. To see this, we use the following criterion for the maximal monotonicity of a sum of two maximal monotone operators.

Proposition 2.2 (Brézis, H. [4] Theorem 9.)**.** *Let B be maximal monotone in* H *and $\phi \in \Phi(\mathrm{H})$. Suppose*

$$\phi((1+\mu B)^{-1}u) \le \phi(u), \quad \forall\mu > 0\ \forall u \in \mathrm{D}(\phi). \tag{2.15}$$

Then $\partial\phi + B$ is maximal monotone in H.

Lemma 2.3. *Let $\phi = \varphi$ and $B = \partial\psi$ given by* (2.5) *and* (2.8)*, then the inequality* (2.15) *holds.*

Proof. First we show $(1+\mu\partial\psi)^{-1}\mathrm{D}(\varphi) \subset \mathrm{D}(\varphi)$, where $\mathrm{D}(\varphi) = \mathbb{H}_0^1(\Omega)$. Let $U \in \mathbb{C}_0^1(\Omega) := \mathrm{C}_0^1(\Omega) \times \mathrm{C}_0^1(\Omega)$ and $V := (1+\mu\partial\psi)^{-1}U$, which implies $V(x) + \mu|V(x)|_{\mathbb{R}^2}^{q-2}V(x) = U(x)$ for a.e. $x \in \Omega$. Here define $G : \mathbb{R}^2 \to \mathbb{R}^2$ by $G : V \mapsto G(V) = V + \mu|V|_{\mathbb{R}^2}^{q-2}V$, then we get $G(V(x)) = U(x)$. Note that G is of class C^1 and bijective from $\mathbb{R}^2$ into itself and its Jacobian determinant is given by

$$\det DG(V) = (1+\mu|V|_{\mathbb{R}^2}^{q-2})\{1+\mu(q-2)|V|_{\mathbb{R}^2}^{q-2}\} \ne 0 \quad \text{for each } V \in \mathbb{R}^2.$$

Applying the inverse function theorem, we have $G^{-1} \in \mathrm{C}^1(\mathbb{R}^2;\mathbb{R}^2)$. Hence $V(\cdot) = G^{-1}(U(\cdot)) \in \mathbb{C}_0^1(\Omega)$, which implies $(1+\mu\partial\psi)^{-1}\mathbb{C}_0^1(\Omega) \subset \mathbb{C}_0^1(\Omega)$. Now let $U_n \in \mathbb{C}_0^1(\Omega)$ and $U_n \to U$ in $\mathbb{H}^1(\Omega)$. Then $V_n := (1+\mu\partial\psi)^{-1}U_n \in \mathbb{C}_0^1(\Omega)$ satisfy

$$|V_n - V|_{\mathbb{L}^2} = |(1+\mu\partial\psi)^{-1}U_n - (1+\mu\partial\psi)^{-1}U|_{\mathbb{L}^2} \le |U_n - U|_{\mathbb{L}^2} \to 0 \quad \text{as } n \to \infty,$$

whence it follows that $V_n \to V$ in $\mathbb{L}^2(\Omega)$. Also differentiation of $G(V_n(x)) = U_n(x)$ gives

$$(1+\mu|V_n(x)|_{\mathbb{R}^2}^{q-2})\nabla V_n(x) + \mu(q-2)|V_n(x)|_{\mathbb{R}^2}^{q-4}(V_n(x)\cdot\nabla V_n(x))\,V_n(x) = \nabla U_n(x). \quad (2.16)$$

Multiplying (2.16) by $\nabla V_n(x)$, we easily get $|\nabla V_n(x)|^2 \leq (\nabla U_n(x)\cdot\nabla V_n(x))$. Therefore by the Cauchy-Schwarz inequality, we have $\varphi(V_n) \leq \varphi(U_n) \to \varphi(U)$. Thus the boundedness of $\{|\nabla V_n|\}$ in L^2 assures that $V_n \to V$ weakly in $\mathbb{H}_0^1(\Omega)$, hence we have $(1+\mu\partial\psi)^{-1}\mathrm{D}(\varphi) \subset \mathrm{D}(\varphi)$. Furthermore, from the lower semi-continuity of the norm in the weak topology, we derive $\varphi(V) \leq \varphi(U)$. This is nothing but the desired inequality (2.15). □

Now we see that $\lambda\partial\varphi + \kappa\partial\psi$ is maximal monotone for all $\lambda, \kappa > 0$. Therefore, since the trivial inclusion $\lambda\partial\varphi + \kappa\partial\psi \subset \partial(\lambda\varphi + \kappa\psi)$ holds, we obtain the following relation:

$$\lambda\partial\varphi + \kappa\partial\psi = \partial(\lambda\varphi + \kappa\psi) \quad \text{for all } \lambda, \kappa > 0. \quad (2.17)$$

Thus (CGL) can be reduced to the following evolution equation:

$$\text{(E)} \begin{cases} \dfrac{d}{dt}U(t) + \partial(\lambda\varphi+\kappa\psi)(U(t)) + \alpha I\partial\varphi(U(t)) + \beta I\partial\psi(U(t)) - \gamma U(t) = f(t),\ t \in (0,\infty), \\ U(0) = U_0. \end{cases}$$

3 Main Results

In order to state our main results, we introduce the following so-called CGL-region given by:

$$\mathrm{CGL}(r) := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; xy \geq 0 \text{ or } \frac{|xy|-1}{|x|+|y|} < r \right\}. \quad (3.1)$$

Also, we use the parameter $c_q \in [0,\infty)$ measuring the strength of the nonlinearity:

$$c_q := \frac{q-2}{2\sqrt{q-1}}. \quad (3.2)$$

Then our main results are stated as follows.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a general domain of uniformly C^2-regular class (see, e.g., [1]). Suppose that $f \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ for all $T > 0$ and $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{CGL}(c_q^{-1})$. Then for any $U_0 \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$, there exists a solution $U \in \mathrm{C}([0,\infty);\mathbb{L}^2(\Omega))$ of (E) satisfying*

1. *$U \in \mathrm{W}^{1,2}(0,T;\mathbb{L}^2(\Omega))$ for all $T > 0$,*
2. *$U(t) \in \mathrm{D}(\partial\varphi) \cap \mathrm{D}(\partial\psi)$ for a.e. $t \in (0,\infty)$ and satisfies (E) for a.e. $t \in (0,\infty)$,*
3. *$\partial\varphi(U(\cdot))$, $\partial\psi(U(\cdot)) \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ for all $T > 0$.*

As for the smoothing effect, the following result holds.

Theorem 3.2. *Let all assumptions in Theorem 3.1 be satisfied. Then for any $U_0 \in \mathbb{L}^2(\Omega)$, there exists a solution $U \in \mathrm{C}([0,\infty);\mathbb{L}^2(\Omega))$ of (E) satisfying*

1. $U \in \mathrm{W}^{1,2}_{\mathrm{loc}}((0,\infty);\mathbb{L}^2(\Omega))$,

2. $U(t) \in \mathrm{D}(\partial\varphi) \cap \mathrm{D}(\partial\psi)$ *for a.e.* $t \in (0,\infty)$ *and satisfies* (E) *for a.e.* $t \in (0,\infty)$,

3. $\varphi(U(\cdot))$, $\psi(U(\cdot)) \in \mathrm{L}^1(0,T)$ *and* $t\varphi(U(t))$, $t\psi(U(t)) \in \mathrm{L}^\infty(0,T)$ *for all* $T > 0$,

4. $\sqrt{t}\frac{d}{dt}U(t)$, $\sqrt{t}\partial\varphi(U(t))$, $\sqrt{t}\partial\psi(U(t)) \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ *for all* $T > 0$.

4 Key Inequalities

In this section, we prepare some inequalities, which will play an important role in establishing a priori estimates. The same estimates are obtained in [11] and [12] in a framework of complex valued function spaces, whose proof is quite different form ours. Our approach is more direct and seems to be simpler.

Lemma 4.1. *The following inequalities hold for all* $U \in \mathrm{D}(\partial\varphi) \cap \mathrm{D}(\partial\psi)$.

$$|(\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2}| \leq c_q(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}, \tag{4.1}$$

$$|(\partial\varphi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2}| \leq c_q(\partial\varphi(U), \partial\psi_\mu(U))_{\mathbb{L}^2} \leq c_q(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2} \quad \forall\mu > 0, \tag{4.2}$$

where $\partial\psi_\mu(\cdot)$ *is the Yosida approximation of* $\partial\psi(\cdot)$ *given by* (2.10).

Proof. Applying integration by parts to the right-hand side of (4.1), we have

$$(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2} = \int_\Omega \left\{(q-2)|U|^{q-4}_{\mathbb{R}^2}|(U\cdot\nabla U)|^2_{\mathbb{R}^N} + |U|^{q-2}_{\mathbb{R}^2}|\nabla U|^2_{\mathbb{R}^{2N}}\right\} dx. \tag{4.3}$$

Making use of (2.1) and (2.2) with integration by parts to the left-hand side of (4.1), we obtain

$$\begin{aligned}(\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2} &= (q-2)\int_\Omega |U|^{q-4}_{\mathbb{R}^2}\left((U\cdot\nabla U), (IU\cdot\nabla U)\right)_{\mathbb{R}^N} dx \\ &\quad + \int_\Omega |U|^{q-2}_{\mathbb{R}^2}(\nabla U(x)\cdot\nabla IU(x))\, dx \\ &= (q-2)\int_\Omega |U|^{q-4}_{\mathbb{R}^2}\left((U\cdot\nabla U), (IU\cdot\nabla U)\right)_{\mathbb{R}^N} dx. \end{aligned} \tag{4.4}$$

Here by direct calculations, we note

$$|(U\cdot\nabla U)|^2_{\mathbb{R}^N} + |(IU\cdot\nabla U)|^2_{\mathbb{R}^N} = |U|^2_{\mathbb{R}^2}\,|\nabla U|^2_{\mathbb{R}^{2N}}. \tag{4.5}$$

Then by Young's inequality, (4.5) and (4.3), we obtain

$$\begin{aligned}|(\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2}| &\leq (q-2)\int_\Omega |U|^{q-4}_{\mathbb{R}^2}|(U\cdot\nabla U)|_{\mathbb{R}^N}\cdot|(IU\cdot\nabla U)|_{\mathbb{R}^N}\, dx \\ &\leq (q-2)\int_\Omega |U|^{q-4}_{\mathbb{R}^2}\frac{1}{2\sqrt{q-1}}\left\{(q-1)|(U\cdot\nabla U)|^2_{\mathbb{R}^N} + |(IU\cdot\nabla U)|^2_{\mathbb{R}^N}\right\} dx \\ &= c_q\int_\Omega |U|^{q-4}_{\mathbb{R}^2}\left\{(q-2)|(U\cdot\nabla U)|^2_{\mathbb{R}^N} + |U|^2_{\mathbb{R}^2}|\nabla U|^2_{\mathbb{R}^{2N}}\right\} dx \\ &= c_q(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2},\end{aligned}$$

whence follows (4.1).

Let $V := (1+\mu\partial\psi)^{-1}U$, then applying integration by parts, (2.1) and (2.2), we have

$$(\partial\varphi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2} = (\nabla U, \nabla I\partial\psi(V))_{\mathbb{L}^2}$$
$$= (\nabla V + \mu\nabla\partial\psi(V), \nabla I\partial\psi(V))_{\mathbb{L}^2} = (\nabla V, \nabla I\partial\psi(V))_{\mathbb{L}^2}. \tag{4.6}$$

It is clear that (4.1) with U replaced by V is equivalent to

$$|(\nabla V, \nabla I\partial\psi(V))_{\mathbb{L}^2}| \le c_q(\nabla V, \nabla\partial\psi(V))_{\mathbb{L}^2}. \tag{4.7}$$

Hence by (4.6) and (4.7), we obtain

$$|(\partial\varphi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2}| \le c_q(\nabla V, \nabla\partial\psi(V))_{\mathbb{L}^2}$$
$$\le c_q(\nabla V + \mu\nabla\partial\psi(V), \nabla\partial\psi(V))_{\mathbb{L}^2} = c_q(\partial\varphi(U), \partial\psi_\mu(U))_{\mathbb{L}^2},$$

which is the first inequality of (4.2). Finally we show the second inequality of (4.2). We first note, for a.e. $x \in \Omega$,

$$|V(x)|_{\mathbb{R}^2} \le |U(x)|_{\mathbb{R}^2}, \tag{4.8}$$

$$|\nabla V(x)|_{\mathbb{R}^{2N}} \le |\nabla U(x)|_{\mathbb{R}^{2N}}, \tag{4.9}$$

$$\begin{aligned} |(V(x)\cdot\nabla V(x))|_{\mathbb{R}^N} &\le |(V(x)\cdot\nabla U(x))|_{\mathbb{R}^N} \\ &= \frac{|V(x)|_{\mathbb{R}^2}}{|U(x)|_{\mathbb{R}^2}}|(U(x)\cdot\nabla U(x))|_{\mathbb{R}^N} \le |(U(x)\cdot\nabla U(x))|_{\mathbb{R}^N}. \end{aligned} \tag{4.10}$$

Indeed the definition of V gives $V(x)+\mu|V|_{\mathbb{R}^2}^{q-2}V(x) = U(x)$, multiplication of this relation by $V(x)$ immediately gives (4.8). Applying ∇ to this relation, we get

$$(1+\mu|V(x)|_{\mathbb{R}^2}^{q-2})\nabla V(x) + \mu(q-2)|V(x)|_{\mathbb{R}^2}^{q-4}(V(x)\cdot\nabla V(x))V(x) = \nabla U(x). \tag{4.11}$$

Multiplying (4.11) by $\nabla V(x)$, we get

$$(1+\mu|V(x)|_{\mathbb{R}^2}^{q-2})|\nabla V(x)|^2 + \mu(q-2)|V(x)|_{\mathbb{R}^2}^{q-4}|(V(x)\cdot\nabla V(x))|_{\mathbb{R}^N}^2 = (\nabla U(x)\cdot\nabla V(x))$$
$$\le |\nabla U(x)|\,|\nabla V(x)|,$$

whence follows (4.9). Also multiplying (4.11) by $V(x)$ gives

$$\left\{1+\mu(q-1)|V(x)|_{\mathbb{R}^2}^{q-2}\right\}(V(x)\cdot\nabla V(x)) = (V(x)\cdot\nabla U(x)).$$

Hence we have the first inequality of (4.10). Multiplying the definition of V by $\nabla U(x)$, we have

$$(1+\mu|V(x)|_{\mathbb{R}^2}^{q-2})(V(x)\cdot\nabla U(x)) = (U(x)\cdot\nabla U(x)).$$

This yields the second equality of (4.10). Now we use (4.8), (4.9) and (4.10) to get

$$\begin{aligned} (\partial\varphi(U), \partial\psi_\mu(U))_{\mathbb{L}^2} &= \int_\Omega \left\{(q-2)|V|_{\mathbb{R}^2}^{q-4}\big((V\cdot\nabla V),(V\cdot\nabla U)\big)_{\mathbb{R}^N} + |V|_{\mathbb{R}^2}^{q-2}(\nabla V\cdot\nabla U)\right\}dx \\ &\le \int_\Omega \left\{(q-2)|U|_{\mathbb{R}^2}^{q-4}|(U\cdot\nabla U)|_{\mathbb{R}^N}^2 + |U|_{\mathbb{R}^2}^{q-2}|\nabla U|_{\mathbb{R}^{2N}}^2\right\}dx \\ &= (\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}. \end{aligned}$$

Therefore we obtain the desired second inequality of (4.2). □

5 Solvability of Approximate Equation

In this chapter, we introduce the following auxiliary equations:

$$\text{(AE)}\begin{cases} \dfrac{d}{dt}U(t) + \partial(\lambda\varphi + \kappa\psi)(U(t)) + \alpha I\partial\varphi(U(t)) + B(U(t)) = f(t), \quad t \in (0,\infty), \\ U(0) = U_0, \end{cases}$$

where $B : \mathbb{L}^2(\Omega) \to \mathbb{L}^2(\Omega)$ is a Lipschitz continuous operator with Lipschitz constant L_B.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a general domain of uniformly C^2-regular class, $\lambda, \kappa > 0$, $\alpha \in \mathbb{R}$ and $f \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ for all $T > 0$. Then for all $U_0 \in \mathbb{H}^1_0(\Omega) \cap \mathbb{L}^q(\Omega)$, there exists a unique solution $U \in \mathrm{C}([0,\infty);\mathbb{L}^2(\Omega))$ of* (AE) *satisfying*

1. *$U \in \mathrm{W}^{1,2}(0,T;\mathbb{L}^2(\Omega))$ for all $T > 0$,*
2. *$U(t) \in \mathrm{D}(\partial\varphi) \cap \mathrm{D}(\partial\psi)$ for a.e. $t \in (0,\infty)$ and satisfies* (AE) *for a.e. $t \in (0,\infty)$,*
3. *$\partial\varphi(U(\cdot))$, $\partial\psi(U(\cdot)) \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ for all $T > 0$.*

In order to prove Proposition 5.1, we consider the following approximate equation with $\partial\varphi(U)$ replaced by its Yosida approximation $\partial\varphi_\nu(U) = \partial\varphi((1+\nu\partial\varphi)^{-1}U)$.

$$\text{(AE)}_\nu\begin{cases} \dfrac{d}{dt}U(t) + \partial(\lambda\varphi + \kappa\psi)(U(t)) + \alpha I\partial\varphi_\nu(U(t)) + B(U(t)) = f(t), \ \ t \in (0,\infty), \\ U(0) = U_0. \end{cases}$$

Since $\alpha I\partial\varphi_\nu(\cdot) + B(\cdot)$ is Lipschitz in $\mathbb{L}^2(\Omega)$, approximate equation $(\text{AE})_\nu$ has a unique solution $U = U_\nu \in \mathrm{C}([0,\infty);\mathbb{L}^2(\Omega))$ satisfying $U_t, \partial\varphi(U), \partial\psi(U) \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ by the general theory of subdifferential operators (e.g. [4], [13]). We here prepare some a priori estimates for solutions of $(\text{AE})_\nu$.

Lemma 5.2. *Let U be a solution of $(\text{AE})_\nu$ and fix $T > 0$. Then there exists a positive constant C_1 depending only on $\lambda, \kappa, L_B, T, |B(0)|_{\mathbb{L}^2}$, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |f|^2_{\mathbb{L}^2} dt$ satisfying*

$$\sup_{t\in[0,T]} |U(t)|^2_{\mathbb{L}^2} \le C_1. \tag{5.1}$$

Proof. Noting $|(B(U),U)_{\mathbb{L}^2}| \le (L_B + \frac{1}{2})|U|^2_{\mathbb{L}^2} + \frac{1}{2}|B(0)|^2_{\mathbb{L}^2}$ and multiplying $(\text{AE})_\nu$ by $U(t)$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|U(t)|^2_{\mathbb{L}^2} &+ 2\lambda\varphi(U(t)) + q\kappa\psi(U(t)) + \alpha(I\partial\varphi_\nu(U(t)), U(t))_{\mathbb{L}^2} \\ &= -(B(U(t)), U(t))_{\mathbb{L}^2} + (f(t), U(t))_{\mathbb{L}^2} \\ &\le (L_B + 1)|U(t)|^2_{\mathbb{L}^2} + \frac{1}{2}|B(0)|^2_{\mathbb{L}^2} + \frac{1}{2}|f(t)|^2_{\mathbb{L}^2}. \end{aligned} \tag{5.2}$$

Here we note that $(I\partial\varphi_\nu(U), U)_{\mathbb{L}^2} = 0$ by (2.12). Hence applying Gronwall's inequality to (5.2), we get

$$\begin{aligned} &|U(t)|^2_{\mathbb{L}^2} + 2\int_0^t \{2\lambda\varphi(U(s)) + q\kappa\psi(U(s))\}\, ds \\ &\leq (L_B+2)\, e^{2(L_B+1)t}\left\{T|B(0)|^2_{\mathbb{L}^2} + |U_0|^2_{\mathbb{L}^2} + \int_0^T |f|^2_{\mathbb{L}^2} dt\right\} \end{aligned}$$

for all $t \in [0, T]$, whence follows (5.1). □

Lemma 5.3. *Let U be a solution of* $(\mathrm{AE})_\nu$ *and fix $T > 0$. Then there exists a positive constant C_2 depending only on $\lambda, \kappa, \alpha, L_B, T, \varphi(U_0)$, $\psi(U_0)$, $|B(0)|_{\mathbb{L}^2}$, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |f|^2_{\mathbb{L}^2} dt$ satisfying*

$$\int_0^T \left|\frac{dU(t)}{dt}\right|^2_{\mathbb{L}^2} dt + \int_0^T |\partial\varphi(U(t))|^2_{\mathbb{L}^2} dt + \int_0^T |\partial\psi(U(t))|^2_{\mathbb{L}^2} dt \leq C_2. \tag{5.3}$$

Proof. Multiplying $(\mathrm{AE})_\mu$ by $\partial\varphi(U(t))$, we have, for a.e. $t \in (0, \infty)$,

$$\begin{aligned} &\frac{d}{dt}\varphi(U(t)) + \lambda|\partial\varphi(U(t))|^2_{\mathbb{L}^2} + \kappa(\partial\varphi(U(t)), \partial\psi(U(t)))_{\mathbb{L}^2} + \alpha(I\partial\varphi_\nu(U(t)), \partial\varphi(U(t)))_{\mathbb{L}^2} \\ &\quad = -(B(U(t)), \partial\varphi(U(t)))_{\mathbb{L}^2} + (f(t), \partial\varphi(U(t)))_{\mathbb{L}^2} \\ &\quad \leq \frac{\lambda}{2}|\partial\varphi(U(t))|^2_{\mathbb{L}^2} + \frac{1}{\lambda}\left\{2L_B^2|U(t)|^2_{\mathbb{L}^2} + 2|B(0)|^2_{\mathbb{L}^2} + |f(t)|^2_{\mathbb{L}^2}\right\}. \end{aligned}$$

Note that (4.1) implies $(\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2} \geq 0$ and $(I\partial\varphi_\nu(U), \partial\varphi(U))_{\mathbb{L}^2} = 0$ by (2.13). Hence by Lemma 5.2, we have

$$\frac{d}{dt}\varphi(U(t)) + \frac{\lambda}{2}|\partial\varphi(U(t))|^2_{\mathbb{L}^2} \leq \frac{1}{\lambda}\left\{2L_B^2 C_1 + 2|B(0)|^2_{\mathbb{L}^2} + |f(t)|^2_{\mathbb{L}^2}\right\}. \tag{5.4}$$

Then the integration of (5.4) over $(0, t)$ with respect to $t \in (0, T]$ gives

$$\varphi(U(t)) + \frac{\lambda}{2}\int_0^t |\partial\varphi(U(s))|^2_{\mathbb{L}^2} ds \leq \varphi(U_0) + \frac{1}{\lambda}\left\{2L_B^2 C_1 T + 2T\,|B(0)|^2_{\mathbb{L}^2} + \int_0^T |f|^2_{\mathbb{L}^2} dt\right\}. \tag{5.5}$$

Now multiplying $(\mathrm{AE})_\nu$ by $\partial\psi(U(t))$, we have for a.e. $t \in (0, T)$,

$$\begin{aligned} &\frac{d}{dt}\psi(U(t)) + \lambda(\partial\varphi(U(t)), \partial\psi(U(t)))_{\mathbb{L}^2} + \kappa|\partial\psi(U(t))|^2_{\mathbb{L}^2} \\ &\quad = -\alpha(I\partial\varphi_\nu(U(t)), \partial\psi(U(t)))_{\mathbb{L}^2} - (B(U(t)), \partial\psi(U(t)))_{\mathbb{L}^2} + (f(t), \partial\psi(U(t)))_{\mathbb{L}^2} \\ &\quad \leq \frac{3\kappa}{4}|\partial\psi(U(t))|^2_{\mathbb{L}^2} + \frac{\alpha^2}{\kappa}|\partial\varphi(U(t))|^2_{\mathbb{L}^2} + \frac{1}{\kappa}\left\{2L_B^2 C_1 + 2\,|B(0)|^2_{\mathbb{L}^2} + |f(t)|^2_{\mathbb{L}^2}\right\}. \end{aligned} \tag{5.6}$$

Therefore the integration of (5.6) together with (4.1) yields

$$\begin{aligned} &\psi(U(t)) + \frac{\kappa}{4}\int_0^t |\partial\psi(U(s))|^2_{\mathbb{L}^2} ds \\ &\quad \leq \psi(U_0) + \frac{\alpha^2}{\kappa}\int_0^t |\partial\varphi(U(t))|^2_{\mathbb{L}^2} + \frac{1}{\kappa}\left\{2L_B^2 C_1 T + 2T\,|B(0)|^2_{\mathbb{L}^2} + \int_0^T |f|^2_{\mathbb{L}^2} dt\right\}. \end{aligned} \tag{5.7}$$

Thus from (5.5), (5.7) and $(\mathrm{AE})_\nu$, we derive (5.3). □

Now we are in the position to prove Proposition 5.1.

Proof of Proposition 5.1. Let U_ν be a solution of $(\mathrm{AE})_\nu$ and fix $T > 0$. First we show $\{U_\nu\}_{\nu>0}$ forms a Cauchy in $\mathrm{C}([0,T];\mathbb{L}^2(\Omega))$. To this end, we multiply $(\mathrm{AE})_\nu - (\mathrm{AE})_\mu$ by $U_\nu - U_\mu$ to get

$$\frac{1}{2}\frac{d}{dt}|U_\nu - U_\mu|_{\mathbb{L}^2}^2 + (\partial(\lambda\varphi + \kappa\psi)(U_\nu) - \partial(\lambda\varphi+\kappa\psi)(U_\mu), U_\nu - U_\mu)_{\mathbb{L}^2}$$
$$+ \alpha(I\partial\varphi_\nu(U_\nu) - I\partial\varphi_\mu(U_\mu), U_\nu - U_\mu)_{\mathbb{L}^2} + (B(U_\nu) - B(U_\mu), U_\nu - U_\mu) = 0. \quad (5.8)$$

Since $\partial(\lambda\varphi + \kappa\psi)$ is monotone, the second term of (5.8) is non-negative. Applying Kōmura's trick and using the linearity of $\partial\varphi$, (2.1), (2.9) and (2.11), we obtain

$$\begin{aligned}(I\partial\varphi_\nu(U_\nu) - I\partial\varphi_\mu(U_\mu), U_\nu - U_\mu)_{\mathbb{L}^2} &= (I\partial\varphi_\nu(U_\nu) - I\partial\varphi_\mu(U_\mu), \nu\partial\varphi_\nu(U_\nu) - \mu\partial\varphi_\mu(U_\mu))_{\mathbb{L}^2}\\ &\quad + (I\partial\varphi(J_\nu U_\nu) - I\partial\varphi(J_\mu U_\mu), J_\nu U_\nu - J_\mu U_\mu)_{\mathbb{L}^2}\\ &= (I\partial\varphi_\nu(U_\nu) - I\partial\varphi_\mu(U_\mu), \nu\partial\varphi_\nu(U_\nu) - \mu\partial\varphi_\mu(U_\mu))_{\mathbb{L}^2}\\ &= -\nu(I\partial\varphi_\mu(U_\mu), \partial\varphi_\nu(U_\nu))_{\mathbb{L}^2} - \mu(I\partial\varphi_\nu(U_\nu), \partial\varphi_\mu(U_\mu))_{\mathbb{L}^2},\end{aligned}$$

where $J_\nu := (1 + \nu\partial\varphi)^{-1}$. Hence

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|U_\nu - U_\mu|_{\mathbb{L}^2}^2\\ &\le |\alpha|\left\{\nu|\partial\varphi_\mu(U_\mu)|_{\mathbb{L}^2}|\partial\varphi_\nu(U_\nu)|_{\mathbb{L}^2} + \mu|\partial\varphi_\nu(U_\nu)|_{\mathbb{L}^2}|\partial\varphi_\mu(U_\mu)|_{\mathbb{L}^2}\right\} + L_B|U_\nu - U_\mu|_{\mathbb{L}^2}^2\\ &\le \frac{|\alpha|}{2}(\nu+\mu)\left\{|\partial\varphi(U_\nu)|_{\mathbb{L}^2}^2 + |\partial\varphi(U_\mu)|_{\mathbb{L}^2}^2\right\} + L_B|U_\nu - U_\mu|_{\mathbb{L}^2}^2.\end{aligned}$$

Thus Gronwall's inequality yields

$$|U_\nu(t) - U_\mu(t)|_{\mathbb{L}^2}^2 \le |\alpha|(\nu+\mu)e^{2L_Bt}\int_0^t \left\{|\partial\varphi(U_\nu(s))|_{\mathbb{L}^2}^2 + |\partial\varphi(U_\mu(s))|_{\mathbb{L}^2}^2\right\} ds,$$

for all $t \in [0,\infty)$. Then by Lemma 5.3, we have

$$\sup_{t\in[0,T]} |U_\nu(t) - U_\mu(t)|_{\mathbb{L}^2} \le e^{L_BT}\sqrt{2\,C_2\,|\alpha|(\nu+\mu)},$$

which assures that $\{U_\nu\}_{\nu>0}$ forms a Cauchy in $\mathrm{C}([0,T];\mathbb{L}^2(\Omega))$. Now let $U_\nu \to U$ in $\mathrm{C}([0,T];\mathbb{L}^2(\Omega))$ as $\nu \to 0$. By Lemma 5.3, $\{\frac{d}{dt}U_\nu\}_{\nu>0}$, $\{\partial\varphi(U_\nu)\}_{\nu>0}$ and $\{\partial\psi(U_\nu)\}_{\nu>0}$ are bounded in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$. Hence by the demiclosedness of $\frac{d}{dt}$, $\partial\varphi$ and $\partial\psi$, we have

$$\begin{aligned}&\frac{dU_{\nu_n}}{dt} \rightharpoonup \frac{dU}{dt} \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)),\\ &\partial\varphi(U_{\nu_n}) \rightharpoonup \partial\varphi(U) \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)),\\ &\partial\psi(U_{\nu_n}) \rightharpoonup \partial\psi(U) \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)),\\ &\partial\varphi_{\nu_n}(U_{\nu_n}) \rightharpoonup g \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)),\end{aligned}$$

for some sequence $\{\nu_n\}_{n\in\mathbb{N}}$ such that $\nu_n \to 0$ as $n \to \infty$. Then by the definition of Yosida approximation,

$$
\begin{aligned}
||U_{\nu_n} - J_{\nu_n}U_{\nu_n}||^2_{\mathrm{L}^2(0,T;\mathbb{L}^2)} &= \int_0^T |U_{\nu_n}(s) - J_{\nu_n}U_{\nu_n}(s)|^2_{\mathbb{L}^2} ds \\
&= \nu_n^2 \int_0^T |\partial\varphi_{\nu_n}(U_{\nu_n}(s))|^2_{\mathbb{L}^2} ds \le C_2\, \nu_n^2 \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

This means $J_{\nu_n}U_{\nu_n} \to U$ strongly in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$. Then since $\partial\varphi_\nu(U_\nu) = \partial\varphi(J_\nu U_\nu)$, by the demiclosedness of $\partial\varphi$ we find that $g = \partial\varphi(U)$ and U satisfies

$$
\frac{dU}{dt} + \lambda\partial\varphi(U) + \kappa\partial\psi(U) + \alpha I\partial\varphi(U) + B(U) = f \quad \text{in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)),
$$

i.e., U is the desired solution of (AE). □

6 Proof of Theorem 3.1

To prove Theorem 3.1, we introduce the following approximate equations of (E).

$$
(\mathrm{E})_\mu \begin{cases} \dfrac{d}{dt}U + \partial(\lambda\varphi + \kappa\psi)(U) + \alpha I\partial\varphi(U) + \beta I\partial\psi_\mu(U) - \gamma U = f, \quad t \in (0,\infty) \\ U(0) = U_0, \end{cases}
$$

where $\partial\psi_\mu(U) := \partial\psi((1+\mu\partial\psi)^{-1}U)$ is the Yosida approximation of $\partial\psi(U)$. Since $\partial\psi_\mu(U)$ is Lipschitz continuous, Proposition 5.1 assures that $(\mathrm{E})_\mu$ has a solution $U = U_\mu \in \mathrm{C}([0,\infty);\mathbb{L}^2(\Omega))$ satisfying the same regularities stated in Proposition 5.1. The first step of the proof is to establish some a priori estimates for U.

Lemma 6.1. *Let U be a solution of $(\mathrm{E})_\mu$ and fix $T > 0$. Then there exists a positive constant C_1 depending only on γ, T, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |f|^2_{\mathbb{L}^2} dt$ satisfying*

$$
\sup_{t\in[0,T]} |U(t)|^2_{\mathbb{L}^2} + \int_0^T \varphi(U(s))\, ds + \int_0^T \psi(U(s))\, ds \le C_1. \tag{6.1}
$$

Proof. Multiplying $(\mathrm{E})_\mu$ by $U(t)$, we have, for a.e. $t \in (0,\infty)$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|U(t)|^2_{\mathbb{L}^2} + 2\lambda\varphi(U(t)) + q\kappa\psi(U(t)) \\
&\quad + \alpha(I\partial\varphi(U(t)), U(t))_{\mathbb{L}^2} + \beta(I\partial\psi_\mu(U(t)), U(t))_{\mathbb{L}^2} - \gamma|U(t)|^2_{\mathbb{L}^2} = (f(t), U(t))_{\mathbb{L}^2}.
\end{aligned} \tag{6.2}
$$

Note that (2.11) and (2.12) yield

$$
(I\partial\varphi(U), U)_{\mathbb{L}^2} = 0, \quad (I\partial\psi_\mu(U), U)_{\mathbb{L}^2} = 0.
$$

Hence by (6.2) with Young's inequality, we have

$$
\frac{1}{2}\frac{d}{dt}|U(t)|^2_{\mathbb{L}^2} + 2\lambda\,\varphi(U(t)) + q\,\kappa\,\psi(U(t)) \le (\gamma_+ + \frac{1}{2})|U(t)|^2_{\mathbb{L}^2} + \frac{1}{2}|f(t)|^2_{\mathbb{L}^2}, \tag{6.3}
$$

where $\gamma_+ := \max\{\gamma, 0\}$. Then the Gronwall's inequality yields

$$|U(t)|_{\mathbb{L}^2}^2 + 2\int_0^t \{2\lambda\varphi(U(s)) + q\kappa\psi(U(s))\}\, ds \le (2\gamma_+ + 2)\, e^{(2\gamma_+ + 1)t} \left\{ |U_0|_{\mathbb{L}^2}^2 + \int_0^T |f|_{\mathbb{L}^2}^2 dt \right\}$$

for all $t \in [0, T]$, whence follows (6.1). □

Lemma 6.2. *Let U be a solution of* $(\mathrm{E})_\mu$*, and let* $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{CGL}(c_q^{-1})$*. Then for a fixed $T > 0$, there exists a positive constant C_2 depending only on $\lambda, \kappa, \alpha, \beta, \gamma$, $T, \varphi(U_0), \psi(U_0)$, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |f|_{\mathbb{L}^2}^2 dt$ satisfying*

$$\sup_{t \in [0,T]} \varphi(U(t)) + \int_0^T \left| \frac{dU(t)}{dt} \right|_{\mathbb{L}^2}^2 dt + \int_0^T |\partial\varphi(U(t))|_{\mathbb{L}^2}^2 dt + \int_0^T |\partial\psi(U(t))|_{\mathbb{L}^2}^2 dt \le C_2. \quad (6.4)$$

Proof. Let $V(t) := (1 + \mu\partial\psi)^{-1} U(t)$. Then using $U = V + \mu\partial\psi(V)$, $(\partial\psi(V) \cdot V)_{\mathbb{R}^2} = q\psi(V) \ge 0$ and $\psi(V) + \frac{\mu}{2}|\partial\psi(V)|^2 =: \psi_\mu(U) \le \psi(U)$, we get

$$(\partial\psi(U), \partial\psi_\mu(U))_{\mathbb{L}^2} = \int_\Omega |U|_{\mathbb{R}^2}^{q-2} |V|_{\mathbb{R}^2}^{q-2} (U \cdot V)_{\mathbb{R}^2} \ge \int_\Omega |V|_{\mathbb{R}^2}^{2(q-1)} = |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2,$$

$$(U, \partial\psi_\mu(U))_{\mathbb{L}^2} = q\psi(V) + \mu|\partial\psi(V)|_{\mathbb{L}^2}^2 = q\psi_\mu(U) - (\frac{q}{2} - 1)\mu|\partial\psi(V)|_{\mathbb{L}^2}^2 \le q\psi(U).$$

Hence multiplication of $(\mathrm{E})_\mu$ by $\partial\varphi(U(t))$ and $\partial\psi_\mu(U(t))$ together with (2.1) give

$$\frac{d}{dt}\varphi(U(t)) + \lambda|\partial\varphi(U)|_{\mathbb{L}^2}^2 + \kappa G(t) + \beta B_\mu(t) = 2\gamma\varphi(U(t)) + (f, \partial\varphi(U))_{\mathbb{L}^2}, \quad (6.5)$$

$$\frac{d}{dt}\psi_\mu(U(t)) + \kappa|\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 + \lambda G_\mu(t) - \alpha B_\mu(t) \le q\gamma_+\psi(U(t)) + (f, \partial\psi_\mu(U))_{\mathbb{L}^2}, \quad (6.6)$$

where $\gamma_+ := \max\{\gamma, 0\}$ and

$$G := (\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}, \quad G_\mu := (\partial\varphi(U), \partial\psi_\mu(U))_{\mathbb{L}^2}, \quad B_\mu := (\partial\varphi(U), I\partial\psi_\mu(U))_{\mathbb{L}^2}.$$

We add (6.5)$\times\delta^2$ to (6.6) for some $\delta > 0$ to get

$$\begin{aligned} \frac{d}{dt}\left\{\delta^2\varphi(U) + \psi_\mu(U)\right\} &+ \delta^2\lambda|\partial\varphi(U)|_{\mathbb{L}^2}^2 + \kappa|\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 \\ &+ \delta^2\kappa G + \lambda G_\mu + (\delta^2\beta - \alpha)B_\mu \\ &\le \gamma_+\left\{2\delta^2\varphi(U) + q\psi(U)\right\} + (f, \delta^2\partial\varphi(U) + \partial\psi_\mu(U))_{\mathbb{L}^2}. \quad (6.7) \end{aligned}$$

Let $\epsilon \in (0, \min\{\lambda, \kappa\})$ be a small parameter. By the inequality of arithmetic and geometric means, and the fundamental property (2.4), we have

$$\begin{aligned} &\delta^2\lambda|\partial\varphi(U)|_{\mathbb{L}^2}^2 + \kappa|\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 \\ &\quad = \epsilon\left\{\delta^2|\partial\varphi(U)|_{\mathbb{L}^2}^2 + |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2\right\} + (\lambda - \epsilon)\delta^2|\partial\varphi(U)|_{\mathbb{L}^2}^2 + (\kappa - \epsilon)|\partial\psi_\mu(U)|_{\mathbb{L}^2}^2 \\ &\quad \ge \epsilon\left\{\delta^2|\partial\varphi(U)|_{\mathbb{L}^2}^2 + |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2\right\} + 2\sqrt{(\lambda - \epsilon)(\kappa - \epsilon)\delta^2|\partial\varphi(U)|_{\mathbb{L}^2}^2|\partial\psi_\mu(U)|_{\mathbb{L}^2}^2} \\ &\quad \ge \epsilon\left\{\delta^2|\partial\varphi(U)|_{\mathbb{L}^2}^2 + |\partial\psi_\mu(U)|_{\mathbb{L}^2}^2\right\} + 2\sqrt{(\lambda - \epsilon)(\kappa - \epsilon)\delta^2(G_\mu^2 + B_\mu^2)}. \quad (6.8) \end{aligned}$$

We here recall the key inequality (4.2)

$$G \geq G_\mu \geq c_q^{-1}|B_\mu|. \tag{6.9}$$

Hence (6.7), (6.8) and (6.9) yield

$$\begin{aligned}\frac{d}{dt}\left\{\delta^2\varphi(U)+\psi_\mu(U)\right\}+\epsilon\left\{\delta^2|\partial\varphi(U)|_{\mathbb{L}^2}^2+|\partial\psi_\mu(U)|_{\mathbb{L}^2}^2\right\}+J(\delta,\epsilon)|B_\mu|\\ \leq \gamma_+\left\{2\delta^2\varphi(U)+q\psi(U)\right\}+(f,\delta^2\partial\varphi(U)+\partial\psi_\mu(U))_{\mathbb{L}^2},\end{aligned} \tag{6.10}$$

where

$$J(\delta,\epsilon) := 2\delta\sqrt{(1+c_q^{-2})(\lambda-\epsilon)(\kappa-\epsilon)}+c_q^{-1}(\delta^2\kappa+\lambda)-|\delta^2\beta-\alpha|.$$

Now we are going to show that $(\frac{\alpha}{\lambda},\frac{\beta}{\kappa}) \in \mathrm{CGL}(c_q^{-1})$ assures $J(\delta,\epsilon)\geq 0$ for some δ and ϵ. By the continuity of $J(\delta,\cdot):\epsilon\mapsto J(\delta,\epsilon)$ it suffices to show $J(\delta,0)>0$ for some δ. When $\alpha\beta>0$, it is enough to take $\delta=\sqrt{\alpha/\beta}$. When $\alpha\beta\leq 0$, we have $|\delta^2\beta-\alpha|=\delta^2|\beta|+|\alpha|$. Hence

$$J(\delta,0)=(c_q^{-1}\kappa-|\beta|)\delta^2+2\delta\sqrt{(1+c_q^{-2})\lambda\kappa}+(c_q^{-1}\lambda-|\alpha|).$$

Therefore if $|\beta|/\kappa\leq c_q^{-1}$, we get $J(\delta,0)>0$ for sufficiently large $\delta>0$. If $c_q^{-1}<|\beta|/\kappa$, we find that it is enough to see the discriminant is positive:

$$D/4 := (1+c_q^{-2})\lambda\kappa-(c_q^{-1}\kappa-|\beta|)(c_q^{-1}\lambda-|\alpha|)>0. \tag{6.11}$$

Since

$$D/4>0 \Leftrightarrow \frac{|\alpha|}{\lambda}\frac{|\beta|}{\kappa}-1<c_q^{-1}\left(\frac{|\alpha|}{\lambda}+\frac{|\beta|}{\kappa}\right),$$

the condition $(\frac{\alpha}{\lambda},\frac{\beta}{\kappa})\in\mathrm{CGL}(c_q^{-1})$ yields $D>0$, whence $J(\delta,0)>0$ for some δ.

Now we take δ and ϵ such that $J(\delta,\epsilon)\geq 0$. Integrating (6.10) and using Young's inequality and Lemma 6.1, we obtain

$$\sup_{t\in[0,T]}\varphi(U(t))+\int_0^T|\partial\varphi(U(s))|_{\mathbb{L}^2}^2ds+\int_0^T|\partial\psi_\mu(U(s))|_{\mathbb{L}^2}^2ds\leq C_2, \tag{6.12}$$

where C_2 depends on the constants stated in Lemma 6.2. We multiply $(\mathrm{E})_\mu$ by $\partial\psi(U)$ to get by (2.14)

$$\begin{aligned}&\frac{d}{dt}\psi(U)+\kappa|\partial\psi(U)|_{\mathbb{L}^2}^2+\lambda(\partial\varphi(U),\partial\psi(U))_{\mathbb{L}^2}\\ &=-\alpha(I\partial\varphi(U),\partial\psi(U))_{\mathbb{L}^2}-\beta(I\partial\psi_\mu(U),\partial\psi(U))_{\mathbb{L}^2}+q\gamma\psi(U)+(f,\partial\psi(U))_{\mathbb{L}^2}\\ &\leq\frac{\kappa}{4}|\partial\psi(U)|_{\mathbb{L}^2}^2+\frac{\alpha^2}{\kappa}|\partial\varphi(U)|_{\mathbb{L}^2}^2+q\gamma_+\psi(U)+\frac{\kappa}{4}|\partial\psi(U)|_{\mathbb{L}^2}^2+\frac{1}{\kappa}|f|_{\mathbb{L}^2}^2.\end{aligned} \tag{6.13}$$

Hence (4.1), (6.12) and the integration of (6.13) yield

$$\int_0^T|\partial\psi(U(s))|_{\mathbb{L}^2}^2ds\leq C_2. \tag{6.14}$$

Thus $(\mathrm{E})_\mu$ together with (6.12) and (6.14) gives the desired estimate (6.4). □

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Let U_μ be a solution of $(\mathrm{E})_\mu$ and fix $T>0$. By Lemma 6.1, Lemma 6.2 and (6.12), there exists a sequence $\mu_n \downarrow 0$ satisfying

$$U_{\mu_n} \rightharpoonup U \qquad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{H}^1_0(\Omega)), \tag{6.15}$$

$$\frac{dU_{\mu_n}}{dt} \rightharpoonup \frac{dU}{dt} \qquad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{6.16}$$

$$\partial\varphi(U_{\mu_n}) \rightharpoonup \partial\varphi(U) \qquad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{6.17}$$

$$\partial\psi(U_{\mu_n}) \rightharpoonup h \qquad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{6.18}$$

$$\partial\psi_{\mu_n}(U_{\mu_n}) \rightharpoonup g \qquad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega)), \tag{6.19}$$

for some function $h, g \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$. Here we used the weak closedness of $\frac{d}{dt}$ and $\partial\varphi$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ in (6.16) and (6.17).

In order to see $h=\partial\psi(U)$, we are going to show that there exists a subsequence $\{\mu'_n\} \subset \{\mu_n\}$ such that

$$U_{\mu'_n}|_{\Omega'} \to U|_{\Omega'} \quad \text{strongly in } \mathrm{C}([0,T];\mathbb{L}^2(\Omega')) \text{ for any bounded subset } \Omega' \text{ of } \Omega. \tag{6.20}$$

To confirm this, we rely on Ascoli's theorem and a diagonal argument. Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be bounded domains in $\mathbb{R}^N$ with smooth boundaries satisfying

(i) $\Omega_k \subset \Omega_{k+1} \subset \Omega$ for each $k \in \mathbb{N}$,

(ii) for all bounded $\Omega' \subset \Omega$ there exists $k \in \mathbb{N}$ such that $\Omega' \subset \Omega_k$.

Fix $k\in\mathbb{N}$, then Lemmas 6.1 and 6.2 assure

$$|U_{\mu_n}(t_2)|_{\Omega_k} - U_{\mu_n}(t_1)|_{\Omega_k}|_{\mathbb{L}^2(\Omega_k)} \le \int_{t_1}^{t_2} \left|\frac{dU_{\mu_n}}{ds}\right|_{\mathbb{L}^2(\Omega)} ds \le \sqrt{C_2}\sqrt{t_2-t_1}, \tag{6.21}$$

$$|U_{\mu_n}(t)|_{\Omega_k}|^2_{\mathbb{H}^1(\Omega_k)} \le |U_{\mu_n}(t)|^2_{\mathbb{L}^2(\Omega)} + |\nabla U_{\mu_n}(t)|^2_{\mathbb{L}^2(\Omega)} \le C_1 + 2\,C_2. \tag{6.22}$$

By (6.21), $\{U_{\mu_n}|_{\Omega_k}\}$ forms an equicontinuous family in $\mathrm{C}([0,T];\mathbb{L}^2(\Omega_k))$. Furthermore by (6.22), $\{U_{\mu_n}(t)|_{\Omega_k}\}$ is relatively compact in $\mathbb{L}^2(\Omega_k)$ for each $t\in(0,T)$. Hence by Ascoli's theorem, there exists a subsequence $\{\mu_n^k\}_{n\in\mathbb{N}}$ of $\{\mu_n\}_{n\in\mathbb{N}}$ such that

$$U_{\mu_n^k}|_{\Omega_k} \to U^k \quad \text{strongly in } \mathrm{C}([0,T];\mathbb{L}^2(\Omega_k)) \text{ as } n\to\infty, \tag{6.23}$$

for some function $U^k \in \mathrm{C}([0,T];\mathbb{L}^2(\Omega_k))$.

Now we can take a sequence of subsequences successively such that

$$\{\mu_n^1\}_{n\in\mathbb{N}} \supset \{\mu_n^2\}_{n\in\mathbb{N}} \supset \cdots \supset \{\mu_n^k\}_{n\in\mathbb{N}} \supset \{\mu_n^{k+1}\}_{n\in\mathbb{N}} \supset \cdots$$

and (6.23) holds for each $k\in\mathbb{N}$. Then the diagonal sequence $\{\mu_n^n\}_{n\in\mathbb{N}} =: \{\mu'_n\}_{n\in\mathbb{N}}$ satisfies

$$U_{\mu'_n}|_{\Omega_k} \to U^k \quad \text{strongly in } \mathrm{C}([0,T];\mathbb{L}^2(\Omega_k)) \text{ as } n\to\infty \quad \text{for each } k\in\mathbb{N}. \tag{6.24}$$

On the other hand, by (6.15), we have

$$U_{\mu_n'}|_{\Omega_k} \rightharpoonup U|_{\Omega_k} \quad \text{weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_k)) \text{ as } n \to \infty \text{ for each } k \in \mathbb{N}. \tag{6.25}$$

Then by the uniqueness of a weak limit, we have $U^k = U|_{\Omega_k}$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_k))$. Finally since $\Omega' \subset \Omega_k$ for some k, we obtain the desired convergence (6.20) from (6.24).

Now we claim that $h = \partial\psi(U) \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$. In fact, by the demiclosedness of the operator: $U \mapsto |U|_{\mathbb{R}^2}^{q-2}U$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))$, we easily find that

$$h(t)|_{\Omega'} = |\,U(t)|_{\Omega'}|_{\mathbb{R}^2}^{q-2}U(t)|_{\Omega'} \quad \text{in } \mathbb{L}^2(\Omega') \text{ for a.e. } t \in (0,T). \tag{6.26}$$

Since (6.26) holds for all $\Omega' \subset \Omega$, we have $|U(t)|_{\mathbb{R}^2}^{q-2}U(t) = h(t) \in \mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ for a.e. $x \in \Omega$, so that $U(t) \in \mathrm{D}(\partial\psi)$ and $h(t) = \partial\psi(U(t))$ for a.e. $t \in (0,T)$.

Finally we show that the function U satisfies equation (E). Note that $J_{\mu_n'}U_{\mu_n'}|_{\Omega'} \to U|_{\Omega'}$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))$, since

$$\begin{aligned}
|J_{\mu_n'}U_{\mu_n'}|_{\Omega'} - &U|_{\Omega'}|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))} \\
&\leq |J_{\mu_n'}U_{\mu_n'} - U_{\mu_n'}|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} + |U_{\mu_n'}|_{\Omega'} - U|_{\Omega'}|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))} \\
&\leq \mu_n'|\partial\psi_{\mu_n'}U|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} + |U_{\mu_n'}|_{\Omega'} - U|_{\Omega'}|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))} \to 0 \quad n \to \infty.
\end{aligned}$$

By virtue of the fact $\partial\psi_\mu(U) = \partial\psi(J_\mu U)$ and the demiclosedness of $\partial\psi$ in $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega'))$, we find that $g = \partial\psi(U) = h$ and hence U satisfies (E).

As for the initial condition, $U(0) = U_0$ can be deduced immediately from (6.20), since $U_{\mu_n'}(0) = U_0$ for each $n \in \mathbb{N}$. The fact that $U \in C([0,T];\mathbb{L}^2(\Omega))$ can be verified by exactly the same arguments in the last part of the proof of Theorem 3.2. □

7 Proof of Theorem 3.2

In this section we give a proof of Theorem 3.2. Let $U_{0n} \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$ such that $U_{0n} \to U_0$ in $\mathbb{L}^2(\Omega)$. By Theorem 3.1, there exists a solution $U_n \in \mathrm{C}([0,T];\mathbb{L}^2(\Omega))$ of (E) with $U_n(0) = U_{0n}$. First we derive some a priori estimates for U_n in terms of $|U_0|_{\mathbb{L}^2(\Omega)}$.

Lemma 7.1. *Let U be a solution of* (E) *satisfying the regularity given in Theorem 3.1 and fix $T > 0$. Then there exists a positive constant C_1 depending only on γ, T, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |f|_{\mathbb{L}^2}^2 dt$ satisfying*

$$\sup_{t\in[0,T]} |U(t)|_{\mathbb{L}^2}^2 + \int_0^T \varphi(U(t))\, dt + \int_0^T \psi(U(t))\, dt \leq C_1. \tag{7.1}$$

Proof. Since proof is almost the same as that of Lemma 6.1, we omit the details. □

Lemma 7.2. *Let U be a solution of* (E) *satisfying the regularity given in Theorem 3.1 and suppose $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{CGL}(c_q^{-1})$. Fix $T > 0$, then there exists a positive constant C_2 depending only on $\lambda, \kappa, \alpha, \beta, \gamma, T$, $|U_0|_{\mathbb{L}^2}$ and $\int_0^T |f|_{\mathbb{L}^2}^2 dt$ satisfying*

$$\begin{aligned}
&\sup_{t\in[0,T]} t\left(\varphi(U(t)) + \psi(U(t))\right) + \int_0^T t\left|\frac{dU}{dt}\right|_{\mathbb{L}^2}^2 dt \\
&\quad + \int_0^T t\,|\partial\varphi(U(t))|_{\mathbb{L}^2}^2\, dt + \int_0^T t\,|\partial\psi(U(t))|_{\mathbb{L}^2}^2\, dt \leq C_2.
\end{aligned} \tag{7.2}$$

Proof. We can derive (7.2) in much the same way as for (6.4) with $\frac{d}{dt}$ replaced by $t\,\frac{d}{dt}$. However we here give another approach.

First we introduce the following four regions in the parameter space.

$$\mathrm{S}_1(r) := \left\{(x,y) \in \mathbb{R}^2;\ |x| \le r\right\}, \quad \mathrm{S}_2(r) := \left\{(x,y) \in \mathbb{R}^2;\ |y| \le r\right\},$$
$$\mathrm{S}_3(r) := \left\{(x,y) \in \mathbb{R}^2;\ xy > 0\right\}, \qquad \mathrm{S}_4(r) := \left\{(x,y) \in \mathbb{R}^2;\ |1+xy| < r|x-y|\right\}.$$

Noting that $\mathrm{CGL}(c_q^{-1}) = \mathrm{S}_1(c_q^{-1}) \cup \mathrm{S}_2(c_q^{-1}) \cup\ \mathrm{S}_3(c_q^{-1}) \cup \mathrm{S}_4(c_q^{-1})$, we are going to establish a priori estimates in four regions, $\mathrm{S}_1(c_q^{-1}),\ \mathrm{S}_2(c_q^{-1}),\ \mathrm{S}_3(c_q^{-1})$ and $\mathrm{S}_4(c_q^{-1})$.

First let $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{S}_1(c_q^{-1})$. Multiply (E) by $\partial\psi(U(t))$, then we get

$$\begin{aligned}\frac{d}{dt}\psi(U) + \kappa|\partial\psi(U)|_{\mathbb{L}^2}^2 + \lambda G - \alpha B &= q\,\gamma\,\psi(U) + (f, \partial\psi(U))_{\mathbb{L}^2} \\ &\le q\,\gamma_+\,\psi(U) + \frac{\kappa}{2}\,|\partial\psi(U)|_{\mathbb{L}^2}^2 + \frac{1}{2\kappa}\,|f|_{\mathbb{L}^2}^2,\end{aligned} \tag{7.3}$$

where $G := (\partial\varphi(U), \partial\psi(U))_{\mathbb{L}^2}$ and $B := (\partial\varphi(U), I\partial\psi(U))_{\mathbb{L}^2}$. By (4.1) and $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{S}_1(c_q^{-1})$, we easily see that $\lambda G - \alpha B \ge (c_q^{-1}\lambda - |\alpha|)|B| \ge 0$. Hence multiplication of (7.3) by $t \in (0,T)$ yields

$$\frac{d}{dt}\left\{t\,\psi(U)\right\} + \frac{\kappa}{2}\,t\,|\partial\psi(U)|_{\mathbb{L}^2}^2 \le (1 + q\,\gamma_+\,T)\,\psi(U) + \frac{T}{2\kappa}\,|f|_{\mathbb{L}^2}^2. \tag{7.4}$$

So integrating (7.4) over $(0,t)$ with respect to $t \in (0,T]$, we obtain by Lemma 7.1

$$\sup_{0<t<T} t\,\psi(U(t)) + \int_0^T t\,|\partial\psi(U(t))|_{\mathbb{L}^2}^2\,dt \le C_2, \tag{7.5}$$

where C_2 is the constant appearing in Lemma 7.2. Now multiplying (E) by $\partial\varphi(U)$, we have

$$\begin{aligned}\frac{d}{dt}\varphi(U) + \lambda|\partial\varphi(U)|_{\mathbb{L}^2}^2 &= 2\,\gamma\,\varphi(U) - \kappa\,G - \beta B + (f, \partial\varphi(U))_{\mathbb{L}^2} \\ &\le 2\,\gamma_+\,\varphi(U) + \frac{3\lambda}{4}\,|\partial\varphi(U)|_{\mathbb{L}^2}^2 + \frac{\kappa^2+\beta^2}{\lambda}\,|\partial\psi(U)|_{\mathbb{L}^2}^2 + \frac{1}{\lambda}\,|f|_{\mathbb{L}^2}^2.\end{aligned} \tag{7.6}$$

Therefore, in parallel with (7.5), multiplying (7.6) by $t \in (0,T)$ and integrating over $(0,t)$ with respect to $t \in (0,T]$, we obtain

$$\sup_{0<t<T} t\,\varphi(U(t)) + \int_0^T t\,|\partial\varphi(U(t))|_{\mathbb{L}^2}^2\,dt \le C_2. \tag{7.7}$$

Thus by using (E), we obtain the desired estimate (7.2).

As for the case where $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{S}_2(c_q^{-1})$, we can derive (7.2) in a way similar to that for the previous case: $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{S}_1(c_q^{-1})$. More precisely, the difference is just the order of multiplication by $\partial\varphi(U)$ and $\partial\psi(U)$. That is to say, in this case, we first multiply (E) by $\partial\varphi(U)$ to get the estimate (7.7) and next multiply (E) by $\partial\psi(U)$ to get the estimate (7.5).

Next we consider the case where $\big(\frac{\alpha}{\lambda},\frac{\beta}{\kappa}\big)\in \mathrm{S}_3(c_q^{-1})$. Multiply (E) by $\big(|\alpha|\,\partial\varphi(U)+|\beta|\,\partial\psi(U)\big)$. Then, since $|\alpha|\beta=\alpha|\beta|$, we get by (2.1)

$$(\alpha I\partial\varphi(U)+\beta I\partial\psi(U),|\alpha|\,\partial\varphi(U)+|\beta|\,\partial\psi(U))_{\mathbb{L}^2}=(-\alpha\,|\beta|+|\alpha|\,\beta)\,(\partial\varphi(U),I\partial\psi(U))=0.$$

Hence we get

$$\begin{aligned}&\frac{d}{dt}\{\,|\alpha|\,\varphi(U)+|\beta|\,\psi(U)\,\}+\lambda\,|\alpha|\,|\partial\varphi(U)|^2_{\mathbb{L}^2}+\kappa\,|\beta|\,|\partial\psi(U)|^2_{\mathbb{L}^2}+(\,\kappa\,|\alpha|+\lambda\,|\beta|\,)\,G\\&=2\,|\alpha|\,\gamma\,\varphi(U)+q\,|\beta|\,\gamma\,\psi(U)+(f,|\alpha|\,\partial\varphi(U)+|\beta|\,\partial\psi(U))_{\mathbb{L}^2}\\&\le 2|\alpha|\gamma\varphi(U)+q|\beta|\,\gamma\psi(U)+\frac{\lambda|\alpha|}{2}|\partial\varphi(U)|^2_{\mathbb{L}^2}+\frac{\kappa|\beta|}{2}|\partial\psi(U)|^2_{\mathbb{L}^2}+\frac{1}{2}(\frac{|\alpha|}{\lambda}+\frac{|\beta|}{\kappa})|f|^2_{\mathbb{L}^2}.\end{aligned}\tag{7.8}$$

Then multiplying (7.8) by $t\in(0,T)$ and integrating over $(0,t)$ with respect to $t\in(0,T]$, we can obtain (7.2).

Finally let $\big(\frac{\alpha}{\lambda},\frac{\beta}{\kappa}\big)\in \mathrm{S}_4(c_q^{-1})$. Multiplying (E) by (i) $\frac{dU}{dt}$; (ii) $I\partial\varphi(U)$; (iii) $I\partial\psi(U)$, we get

$$\begin{aligned}\left|\frac{dU}{dt}\right|^2_{\mathbb{L}^2}+\frac{d}{dt}\{\lambda\varphi(U)+\kappa\psi(U)\}+\alpha(I\partial\varphi(U),\frac{dU}{dt})_{\mathbb{L}^2}+\beta(I\partial\psi(U),\frac{dU}{dt})_{\mathbb{L}^2}\\=(f,\frac{dU}{dt})_{\mathbb{L}^2}+\frac{\gamma}{2}\frac{d}{dt}|U|^2_{\mathbb{L}^2},\end{aligned}\tag{7.9}$$

$$(I\partial\varphi(U),\frac{dU}{dt})_{\mathbb{L}^2}=\kappa\,B-\alpha\,|I\partial\varphi(U)|^2_{\mathbb{L}^2}-\beta\,G+(f,I\partial\varphi(U))_{\mathbb{L}^2},\tag{7.10}$$

$$(I\partial\psi(U),\frac{dU}{dt})_{\mathbb{L}^2}=-\lambda\,B-\alpha\,G-\beta\,|I\partial\psi(U)|^2_{\mathbb{L}^2}+(f,I\partial\psi(U))_{\mathbb{L}^2}.\tag{7.11}$$

Substituting (7.10) and (7.11) into (7.9), we obtain

$$\begin{aligned}\left|\frac{dU}{dt}\right|^2_{\mathbb{L}^2}+\frac{d}{dt}\{\lambda\varphi(U)+\kappa\psi(U)\}-\alpha^2|\partial\varphi(U)|^2_{\mathbb{L}^2}-\beta^2|\partial\psi(U)|^2_{\mathbb{L}^2}\\-2\alpha\beta G+(\kappa\alpha-\lambda\beta)B=(f,\frac{dU}{dt}+\alpha I\partial\varphi(U)+\beta I\partial\psi(U))_{\mathbb{L}^2}+\frac{\gamma}{2}\frac{d}{dt}|U|^2_{\mathbb{L}^2}.\end{aligned}\tag{7.12}$$

Here by virtue of (7.3) and (7.6), $J_\epsilon:=\frac{\alpha^2+\epsilon}{\lambda}\times(\mathrm{E})\times\partial\varphi(U)+\frac{\beta^2+\epsilon}{\kappa}\times(\mathrm{E})\times\partial\psi(U)$ with $\epsilon>0$ is given by

$$\begin{aligned}J_\epsilon=&\frac{\alpha^2+\epsilon}{\lambda}\frac{d}{dt}\varphi(U)+\frac{\beta^2+\epsilon}{\kappa}\frac{d}{dt}\psi(U)+(\alpha^2+\epsilon)|\partial\varphi(U)|^2_{\mathbb{L}^2}+(\beta^2+\epsilon)|\partial\psi(U)|^2_{\mathbb{L}^2}\\&+\left\{\frac{\kappa}{\lambda}(\alpha^2+\epsilon)+\frac{\lambda}{\kappa}(\beta^2+\epsilon)\right\}G+\left\{\frac{\beta}{\lambda}(\alpha^2+\epsilon)-\frac{\alpha}{\kappa}(\beta^2+\epsilon)\right\}B\\&=(f,\frac{\alpha^2+\epsilon}{\lambda}\,\partial\varphi(U)+\frac{\beta^2+\epsilon}{\kappa}\,\partial\psi(U))_{\mathbb{L}^2}+\frac{2}{\lambda}(\alpha^2+\epsilon)\,\gamma\,\varphi(U)+\frac{q}{\kappa}\,(\beta^2+\epsilon)\,\gamma\,\psi(U)\end{aligned}$$

Adding J_ϵ to (7.12), we get

$$\left|\frac{dU}{dt}\right|_{\mathbb{L}^2}^2 + \left(\lambda + \frac{\alpha^2+\epsilon}{\lambda}\right)\frac{d}{dt}\varphi(U) + \left(\kappa + \frac{\beta^2+\epsilon}{\kappa}\right)\frac{d}{dt}\psi(U) + \epsilon|\partial\varphi(U)|_{\mathbb{L}^2}^2 + \epsilon|\partial\psi(U)|_{\mathbb{L}^2}^2$$
$$+\left\{\frac{\kappa}{\lambda}(\alpha^2+\epsilon) + \frac{\lambda}{\kappa}(\beta^2+\epsilon) - 2\alpha\beta\right\}G + \left\{(\kappa\alpha - \lambda\beta) + \frac{\beta}{\lambda}(\alpha^2+\epsilon) - \frac{\alpha}{\kappa}(\beta^2+\epsilon)\right\}B$$
$$= (f, \frac{dU}{dt} + \alpha I\partial\varphi(U) + \beta I\partial\psi(U) + \frac{\alpha^2+\epsilon}{\lambda}\,\partial\varphi(U) + \frac{\beta^2+\epsilon}{\kappa}\,\partial\psi(U))_{\mathbb{L}^2}$$
$$+ \frac{2}{\lambda}\,(\alpha^2+\epsilon)\,\gamma\,\varphi(U) + \frac{q}{\kappa}\,(\beta^2+\epsilon)\,\gamma\,\psi(U) + \frac{\gamma}{2}\frac{d}{dt}|U|_{\mathbb{L}^2}^2. \tag{7.13}$$

Here the coefficients of G and B are

$$I_1(\epsilon) := \left\{\frac{\kappa}{\lambda}(\alpha^2+\epsilon) + \frac{\lambda}{\kappa}(\beta^2+\epsilon) - 2\alpha\beta\right\},$$
$$I_2(\epsilon) := \left\{(\kappa\alpha - \lambda\beta) + \frac{\beta}{\lambda}(\alpha^2+\epsilon) - \frac{\alpha}{\kappa}(\beta^2+\epsilon)\right\}.$$

Now we claim that $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{S}_4(c_q^{-1})$ assures $I_1(\epsilon)\,G + I_2(\epsilon)\,B \geq 0$ for some $\epsilon > 0$. In order to show this fact, it suffices to show $I_1(0)\,G + I_2(0)\,B > 0$ by the continuity of $\epsilon \mapsto I_1(\epsilon)\,G + I_2(\epsilon)\,B$. By (4.1), we have

$$I_1(0)\,G + I_2(0)\,B = \lambda\kappa\left(\frac{\alpha}{\lambda} - \frac{\beta}{\kappa}\right)^2 G + \lambda\kappa\left(\frac{\alpha}{\lambda} - \frac{\beta}{\kappa}\right)\left(1 + \frac{\alpha}{\lambda}\frac{\beta}{\kappa}\right)B$$
$$\geq \lambda\kappa\left|\frac{\alpha}{\lambda} - \frac{\beta}{\kappa}\right|\left(c_q^{-1}\left|\frac{\alpha}{\lambda} - \frac{\beta}{\kappa}\right| - \left|1 + \frac{\alpha}{\lambda}\frac{\beta}{\kappa}\right|\right)|B| > 0$$

for all $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{S}_4(c_q^{-1})$ with $|B| > 0$. For the case $|B| = 0$, the claim is obvious, since $I_1(\epsilon) \geq 2\epsilon$ and $G \geq 0$. Therefore, in (7.13), we can neglect terms containing G and B. Furthermore, applying Young's inequality for terms in the right-hand side of (7.13) containing $\partial\varphi(U)$ and $\partial\psi(U)$, we can show that these terms can be canceled by the good terms $\epsilon\,|\partial\varphi(U)|_{\mathbb{L}^2}^2$ and $\epsilon\,|\partial\psi(U)|_{\mathbb{L}^2}^2$ in the left-hand side.

Thus multiplying (7.13) by $t \in (0, T)$ and integrating over $(0, t)$ with $t \in (0, T]$, we can repeat the same arguments as above to get (7.2). □

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let U_n be a solution of (E) with $U_n(0) = U_{0n} \in \mathbb{H}_0^1(\Omega) \cap \mathbb{L}^q(\Omega)$ such that $U_{0n} \to U_0$ in $\mathbb{L}^2(\Omega)$. By Lemmas 7.1 and 7.2, there exists a subsequence $\{m_n\}_{n\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ satisfying

$$U_{m_n} \rightharpoonup U \qquad \text{weakly in } \mathrm{L}^2_{\mathrm{loc}}((0,\infty); \mathbb{H}_0^1(\Omega)), \tag{7.14}$$
$$\sqrt{t}\,\frac{dU_{m_n}}{dt} \rightharpoonup \sqrt{t}\,\frac{dU}{dt} \qquad \text{weakly in } \mathrm{L}^2(0,T; \mathbb{L}^2(\Omega)), \tag{7.15}$$
$$\sqrt{t}\,\partial\varphi(U_{m_n}) \rightharpoonup \sqrt{t}\,\partial\varphi(U) \qquad \text{weakly in } \mathrm{L}^2(0,T; \mathbb{L}^2(\Omega)), \tag{7.16}$$
$$\sqrt{t}\,\partial\psi(U_{m_n}) \rightharpoonup \sqrt{t}\,h \qquad \text{weakly in } \mathrm{L}^2(0,T; \mathbb{L}^2(\Omega)), \tag{7.17}$$

for some function g, h. Here we used the weak closedness of $\frac{d}{dt}$ and $\partial\varphi$ in $\mathrm{L}^2(\delta, T; \mathbb{L}^2(\Omega))$ for any $\delta \in (0, T)$.

Furthermore by the same argument as those in the proof of Theorem 3.1, we note

$$U_{m_n'} \to U \quad \text{strongly in } \mathrm{C}(\delta, T; \mathbb{L}^2(\Omega')) \text{ for any bounded } \Omega' \subset \Omega,\ \forall \delta \in (0, T) \tag{7.18}$$

for some subsequence $\{m_n'\} \subset \{m_n\}$. Hence this assures $h = \partial\psi(U) \in \mathrm{L}^2(\delta, T; \mathbb{L}^2(\Omega))$ for any $\delta \in (0, T)$, i.e., $h = \partial\psi(U)$ for a.e. $t \in (0, T)$. Thus we find that U satisfies equation (E). Then in order to complete the proof, it suffices to check

$$U(t) \to U_0 \quad \text{in } \mathbb{L}^2(\Omega) \text{ as } t \downarrow 0. \tag{7.19}$$

First we show $U(t) \rightharpoonup U_0$ weakly in $\mathbb{L}^2(\Omega)$. Multiplying the approximate equation by $W \in \mathbb{C}_0^\infty(\Omega)$, we have

$$\begin{aligned} \frac{d}{dt}(U_n(t), W)_{\mathbb{L}^2} &= \gamma\,(U_n(t), W)_{\mathbb{L}^2} + (f(t), W)_{\mathbb{L}^2} \\ &\quad - ((\lambda + \alpha I)\,\partial\varphi(U_n(t)), W)_{\mathbb{L}^2} - ((\kappa + \beta I)\,\partial\psi(U_n(t)), W)_{\mathbb{L}^2}. \end{aligned} \tag{7.20}$$

Integrating (7.20) over $(0, t)$ and taking the absolute value, we get

$$\begin{aligned} |(U_n(t) - U_{0n}, W)_{\mathbb{L}^2}| &\leq |\gamma||W|_{\mathbb{L}^2} \int_0^t |U_n(s)|_{\mathbb{L}^2} ds + |W|_{\mathbb{L}^2} \int_0^t |f(s)|_{\mathbb{L}^2} ds \\ &\quad + (\lambda + |\alpha|)|\nabla W|_{\mathbb{L}^2} \int_0^t |\nabla U_n(s)|_{\mathbb{L}^2} ds \\ &\quad + (\kappa + |\beta|) \int_0^t \int_\Omega |U_n(s)|_{\mathbb{R}^2}^{q-1} |W|_{\mathbb{R}^2} dx ds. \end{aligned}$$

Then using Hölder's inequality and Lemma 7.1, we obtain

$$\begin{aligned} |(U_n(t) - U_{0n}, W)_{\mathbb{L}^2}| &\leq |\gamma| \sqrt{C_1}\, |W|_{\mathbb{L}^2}\, t + |f(s)|_{L^2(0,t;\mathbb{L}^2(\Omega))}\, |W|_{\mathbb{L}^2}\, t^{\frac{1}{2}} \\ &\quad + (\lambda + |\alpha|)\, \sqrt{2C_1}\, |\nabla W|_{\mathbb{L}^2}\, t^{\frac{1}{2}} + (\kappa + |\beta|)\, (q\, C_1)^{\frac{q-1}{q}}\, |W|_{\mathrm{L}^q}\, t^{\frac{1}{q}}. \end{aligned} \tag{7.21}$$

Letting $n = m_n' \to \infty$, we obtain $|(U(t) - U_0, W)_{\mathbb{L}^2}| \leq Ct^{\frac{1}{q}}$ for sufficiently small $t > 0$, which implies that $U(t) \to U_0$ in $\mathcal{D}'(\Omega)$. Since $\mathbb{C}^\infty(\Omega) \subset \mathbb{L}^2(\Omega)$ is dense, we find that $U(t) \rightharpoonup U_0$ weakly in $\mathbb{L}^2(\Omega)$.

Then, in order to derive (7.19), it suffices to show that $|U(t)|_{\mathbb{L}^2}^2 \to |U_0|_{\mathbb{L}^2}^2$. Let $U_k := U|_{\Omega_k}$ with $\{\Omega_k\}_{k \in \mathbb{N}}$ given in section 6. By the same argument as for (6.3), we have

$$|U_n(t)|_{\Omega_k}|_{\mathbb{L}^2(\Omega_k)}^2 \leq |U_n(t)|_{\mathbb{L}^2}^2 \leq e^{(2\gamma_+ + 1)t} \left\{ |U_{0n}|_{\mathbb{L}^2}^2 + \int_0^t |f(s)|_{\mathbb{L}^2}^2 ds \right\} \quad \forall t \in [0, T]. \tag{7.22}$$

Then by virtue of (6.20), we let $n \to \infty$ to obtain

$$|U_k(t)|_{\mathbb{L}^2(\Omega_k)}^2 = |\tilde{U}_k(t)|_{\mathbb{L}^2}^2 \leq e^{(2\gamma_+ + 1)t} \left\{ |U_0|_{\mathbb{L}^2}^2 + \int_0^t |f(s)|_{\mathbb{L}^2}^2 ds \right\} \quad \forall t \in [0, T], \tag{7.23}$$

where $\tilde{U}_k$ denotes the zero extension of U_k. By the definition of U_k, it is clear that $\{|\tilde{U}_k|\}_{k\in\mathbb{N}}$ forms a pointwise monotonically increasing sequence. Hence (7.23) and Beppo Levi's theorem yields that $\tilde{U}_k$ converges to U in $\mathbb{L}^2(\Omega)$ for all $t \in [0,T]$ and that U satisfies

$$|U(t)|_{\mathbb{L}^2}^2 \leq e^{(2\gamma_+ +1)t} \left\{ |U_0|_{\mathbb{L}^2}^2 + \int_0^t |f(s)|_{\mathbb{L}^2}^2 ds \right\} \quad \forall t \in [0,T].$$

Here letting $t \downarrow 0$, we have $\overline{\lim}_{t\downarrow 0}|U(t)|_{\mathbb{L}^2}^2 \leq |U_0|_{\mathbb{L}^2}^2$. On the other hand, by virtue of the lower semicontinuity of the norm with respect to the weak convergence $U(t) \rightharpoonup U_0$, we get $|U_0|_{\mathbb{L}^2}^2 \leq \underline{\lim}_{t\downarrow 0}|U(t)|_{\mathbb{L}^2}^2$. Thus we can conclude that $|U(t)|_{\mathbb{L}^2}^2 \to |U_0|_{\mathbb{L}^2}^2$. □

References

[1] Amann H., Existence and regularity for semilinear parabolic evolution equation, Ann. Scuola Norm. Sup. Pisa. Cl. Sci. 11 no. 4 (1984), 593-676.

[2] Aranson I. S.; Kramer L., The World of the Complex Ginzburg-Landau Equation, Rev. Mod. Phys. 74, 99, Published 4 February 2002.

[3] Bekki N.; Harada Y.; Kanai H., Bekki-Nozaki Hole in Traveling Excited Waves on Human Cardiac Interventricular Septum. Journal of the Physical Society of Japan 81 2012.

[4] Brézis, H., Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, Contributions to Nonlinear Functional Analysis, E. Zarantonello ed., Academic Press, New York, 1971, 101-156.

[5] Cross M. C.; Hohenberg P. C., Pattern formation outside of equilibrium, Rev. Mod. Phys. 65, 851, Published 1 July 1993.

[6] Doering, C. R.; Gibbon, J. D.; Levermore, C. D., Weak and strong solutions of the complex Ginzburg-Landau equation. Phys. D 71 (1994), no. 3, 285-318.

[7] Ginibre, J.; Velo, G., The Cauchy problem in local spaces for the complex Ginzburg-Laudau equation. I. Compactness methods. Phys. D 95 (1996), no. 3-4, 191-228.

[8] Ginibre, J.; Velo, G., The Cauchy problem in local spaces for the complex Ginzburg-Landau equation. II. Contraction methods. Comm. Math. Phys. 187 (1997), no. 1, 45-79.

[9] Kuramoto, Y.; Chemical Oscillations, Waves, and Turbulence, Springer Series in Synergetics, Volume 19, 1984.

[10] Levermore, C. D.; Oliver, M., The complex Ginzburg-Landau equation as a model problem. Dynamical systems and probabilistic methods in partial differential equations (Berkeley, CA, 1994), 141-190, Lectures in Appl. Math., 31, Amer. Math. Soc., Providence, RI,

[11] Okazawa, N.; Yokota, T., Global existence and smoothing effect for the complex Ginzburg-Landau equation with p-Laplacian. J. Differential Equations 182 (2002), no. 2, 541-576.

[12] Okazawa, N.; Yokota, T., Subdifferential operator approach to strong wellposedness of the complex Ginzburg-Landau equation, Discrete Contin. Dyn. Syst. 28 (2010), no. 1, 311-341.

[13] Showalter, R. E., Monotone operators in Banach space and nonlinear partial differential equations. Mathematical Surveys and Monographs, 49. American Mathematical Society, Providence, RI, 1997.

[14] Simon, J., Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl. (4) 146 (1987), 65-96.

[15] Takeuchi, S.; Asakawa, H.; Yokota, T., Complex Ginzburg-Landau type equations with nonlinear Laplacian. Nonlinear partial differential equations and their applications, 315-332, GAKUTO Internat. Ser. Math. Sci. Appl., 20, Gakkotosho, Tokyo, 2004.

[16] Temam, R., Infinite-dimensional dynamical systems in mechanics and physics. Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1988.

[17] Yang, Y. S., On the Ginzburg-Landau wave equation. Bull. London Math. Soc. 22 (1990), no. 2, 167-170.

[18] Yokota, T.; Okazawa, N., Smoothing effect for the complex Ginzburg-Landau equation (general case). Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 13B (2006), suppl., 305-316.