Advances in Mathematical
Sciences and Applications
Vol. 25, No. 1 (2016), pp. 33-42

GAKKOTOSHO
TOKYO JAPAN

INCOMPRESSIBLE FLOWS THROUGH SLENDER OSCILLATING VESSELS PROVIDED WITH DISTRIBUTED VALVES

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Abstract. An incompressible Newtonian fluid flows through a vessel undergoing periodic oscillations. Oscillations would hinder the flow because of the lumen reduction, but if the vessel is equipped with valves forcing the fluid to move in one direction, then they acquire a pumping action which may be advantageous. The underlying biological problem is the so called vasomotion, observed long ago in blood vessels. Here we study the ideal case in which valves are distributed along the vessel and are sufficiently numerous to be considered uniformly distributed, in the sense that backflow can be prevented at any point of the vessel. The problem, characterized by the presence of a free boundary, is solved by means of an upscalig procedure requiring that the aspect ratio radius-to-length is small enough. Conditions are determined ensuring that valves are actually engaged. For a specific example the range of the main parameters is determined so that oscillations enhance the total discharge.

Communicated by Editors; Received September 29, 2015.
AMS Subject Classification 76Z05, 35Q35
Keywords: Vasomotion, upscalig, free boundary

1 Introduction

The present study is inspired by the biological phenomenon of vasomotion, which consists in the synchronous oscillation of small blood vessels. It is particularly important in venules, where it enhances blood flow thanks to the presence of microvalves. The role of such valves is to prevent backflow. Indeed when the driving pressure gradient is weak, as in the case of veins, oscillations can produce flow inversion, but the valves, forcing the fluid to move in one direction exert a pumping action that can be very advantageous. The number of valves is quite variable but in some case they are distributed along the vessel (see [1]–[3]). Here we suppose that there are enough numerous to be considered uniformly distributed, so that the inversion of pressure gradient can be prevented at any point. We will exploit the smallness of the ratio[1]

$$\varepsilon = \frac{R_o^*}{L^*} \ll 1. \tag{1}$$

with R_o^* is the maximum vessel radius and L^* vessel length, to formulate a mathematical model for an incompressible Newtonian flow, characterized by the presence of a free boundary representing the moving location of the engaged valve. We are going to find the conditions ensuring that valves are engaged and the explicit expression of the unknown when $\mathcal{O}(\varepsilon)$ terms are neglected. We will also determine for which range of the parameters the oscillation produces a flow gain with respect to the case of resting vessels.

2 The model with continuous valves distribution

A first rough estimate of the potential contribution of oscillations to flow rate can be found considering the case of a pipe of length L^* whose radius oscillates with period T^*, assuming that there is no applied pressure gradient and that just two valves placed at endpoints ensure one-directional flow. An analysis of the flow when the radius oscillation is modulated in space can be found in [5].

If $\max R^*(t^*) = R_o^*$, and the oscillation amplitude is $2\delta R_o^*$, $\delta \in (0, 1/2)$, the characteristic transversal velocity is

$$v_{2\ ref}^* = \frac{2\delta R_o^*}{T^*}. \tag{2}$$

We take as reference longitudinal velocity $v_{1\ ref}^*$, a value experimentally measured. In particular, [4] reports an average centerline velocity, $v_{1\ ref}^* = 2\ mm/s$, in bat wings venules. We note that, taking $R_o^* = 70\mu m$, $T^* = 6s$, $2\delta = 0.3$, and $L^* = 10\ mm$ (data of [4]), $v_{2\ ref}^*/v_{1\ ref}^* \approx 7\times 10^{-3}$, i.e. $v_{2\ ref}^*/v_{1\ ref}^* = \mathcal{O}(\varepsilon)$.

The net induced flow rate Q_{vas}^* is the ratio between twice the volume change and the oscillation period T^*, namely

$$Q_{vas}^* = 2\frac{\pi R_o^{*\,2} - \pi R_o^{*\,2}(1-2\delta)^2}{T^*} = 8\pi R_o^{*\,2} L^* \frac{\delta(1-\delta)}{T^*} \tag{3}$$

[1] In case of venules we may have $R_o^* \approx 10\mu m$, $L^* \approx 1.5mm$, so that $\varepsilon \approx 6\times 10^{-3}$, but we will also consider larger and longer vessels.

If we look for the pressure difference generating a similar flow rate in a tube, we find

$$\Delta p_{eq}^* = 8\mu^* \frac{L^*}{\pi {R_o^*}^4} Q_{vas}^* = 64 \frac{\mu^*}{\varepsilon^2} \frac{\delta\,(1-\delta)}{T^*} \approx 0.8\ mmHg\ . \tag{4}$$

It is now necessary to describe if and how oscillations trigger valves engagement. If a pressure gradient is present along the vessel, it can be balanced and eventually overcome by the vessel contraction or expansion if the oscillation is fast enough, thus engaging valves. We disregard the extra pressure needed to keep a valve closed, as well as the valve inertia.

In such a scenario, a closed valve isolates a region in which pressure is constant and equal to inlet (contraction phase) or outlet (expansion phase) pressure. Thus pressure is continuous throughout the vessel.

Now, we define two important ratios

$$\beta = \frac{\Delta p^*}{\Delta p_{eq}^*}\ , \tag{5}$$

where Δp^* is the applied pressure difference, and

$$\gamma = \frac{\Delta p^*}{\Delta p_{ref}^*}\ , \tag{6}$$

having defined $\Delta p_{ref}^* = \dfrac{\mu^* L^*}{{R_o^*}^2} v_{1\ ref}^*$. These two parameters express the relative importance of the applied driving pressure difference with respect to the one induced by oscillations (parameter β), and the amount with which the applied pressure difference actually contributes to the flow (parameter γ). Introducing $\tau = \dfrac{T^*}{L^* /\ v_{1\ ref}^*}$, we have

$$\frac{\beta}{\gamma}\delta\,(1-\delta) = \frac{\tau}{64}. \tag{7}$$

We define the dimensionless pressure

$$p = \frac{p^* - p_{out}^*}{\Delta p^*}, \tag{8}$$

of course assuming $\Delta p^* > 0$, in such a way that $p = 1$ at the inlet and $p = 0$ at the outlet.

Longitudinal and radial velocities are rescaled by the quantities introduced above, and the dimensionless time is $t = t^*/T^*$. Dimensionless space coordinates x, r are defined with the usual double scaling. In particular

$$0 < x < 1, \quad 0 < r < R\,(t)\,, \quad \text{with} \quad R\,(t) = \frac{R^*}{R_o^*}. \tag{9}$$

Concerning the velocity, we adopt the scaling

$$\boldsymbol{v} = \frac{1}{v_{1\ ref}^*}\left[v_1 \boldsymbol{e}_x + \left(\frac{v_{1\ ref}^*}{v_{2\ ref}^*}\right) v_2 \boldsymbol{e}_r\right]. \tag{10}$$

The Reynolds number for this motion turns out to be very small (less than 10^{-3}) and the longitudinal dimensionless component of the Navier-Stokes equation, in which $\mathcal{O}(\varepsilon)$ are neglected, takes the quasi-stationary form

$$\frac{1}{\gamma}\frac{\partial p}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_1}{\partial r}\right). \tag{11}$$

The main information we deduce from the second component is that, at the same order of approximation, $\dfrac{\partial p}{\partial r}=0$. The continuity equation yields

$$\underbrace{\left(\frac{R_o^*}{L^*}\frac{v_{1\ ref}^*}{v_{2\ ref}^*}\right)}_{32(1-\delta)\frac{\beta}{\gamma}}\frac{\partial v_1}{\partial x}+\frac{1}{r}\frac{\partial\left(rv_2\right)}{\partial r}=0. \tag{12}$$

Since p is independent of r, equation (11) can be integrated with the boundary conditions

$$v_1\left(x,R\left(t\right),t\right)=0,\quad \text{(no slip)},\qquad \left.\frac{\partial v_1}{\partial r}\right|_{r=0}=0,\quad \text{(symmetry)}. \tag{13}$$

obtaining

$$v_1\left(x,r,t\right)=-\frac{1}{4\gamma}\frac{\partial p}{\partial x}\left(R^2-r^2\right). \tag{14}$$

Using (14) in equation (12) we find

$$\frac{1}{r}\frac{\partial\left(rv_2\right)}{\partial r}-8\left(1-\delta\right)\frac{\beta}{\gamma^2}\frac{\partial^2 p}{\partial x^2}\left(R^2-r^2\right)=0, \tag{15}$$

which is easily integrated with the condition $v_2\left(x,0,t\right)=0$, yielding

$$v_2\left(x,r,t\right)=2\left(1-\delta\right)\frac{\beta}{\gamma^2}\frac{\partial^2 p}{\partial x^2}r\left(2R^2-r^2\right). \tag{16}$$

Recalling that $v_2\left(x,R\left(t\right),t\right)=\dfrac{1}{2\delta}\dot{R}\left(t\right)$, we have

$$\frac{\partial^2 p}{\partial x^2}=\frac{16\gamma}{\tau}\frac{\dot{R}}{R^3}, \tag{17}$$

with τ given by (7). We thus have established that pressure along the venule has a parabolic profile with time dependent curvature.

2.1 Engaging valves

In the biological process of vasomotion the contraction phase is faster than the relaxation phase (see [4]). Therefore it is important to treat them separately. We take the following partition:

- for $0 < t < t_\delta$, $R(t)$ is monotone decreasing (contraction phase) with $R(0) = 1$, and $R(t_\delta) = 1 - 2\delta$;
- for $t_\delta < t < 1$, $R(t)$ is monotone increasing (expansion phase) with $R(1) = 1$.

Typically $t_\delta = 1/4$.

Contraction phase.

Let $x = s(t)$ be the location of the engaged valve. In the region $0 \le x \le s(t)$, the fluid is stationary and p equals the inlet pressure, while in the region $s(t) \le x \le 1$, the blood is flowing. The conditions to be imposed at $x = s(t)$, are

$$p|_{x=s(t)} = 1, \qquad \left.\frac{\partial p}{\partial x}\right|_{x=s(t)} = 0, \tag{18}$$

i.e. continuity of pressure and zero velocity. The unknown function $s(t)$ is then derived by the necessity of matching the residual boundary condition $p|_{x=1} = 0$. Hence, setting[2]

$$A(t) = \frac{16\gamma}{\tau} \frac{\left|\dot{R}\right|}{R^3}, \quad \text{for} \quad 0 < t < t_\delta, \tag{19}$$

and solving (17) with boundary conditions (18) we obtain

$$p(x,t) = \begin{cases} 1, & \text{for} \quad 0 < t < t_\delta, \quad 0 \le x \le s(t), \\ \\ 1 - \dfrac{A(t)}{2}(x - s(t))^2, & \text{for} \quad 0 < t < t_\delta, \quad s(t) \le x \le 1. \end{cases} \tag{20}$$

Imposing then $p|_{x=1} = 0$ we derive

$$s(t) = 1 - \sqrt{\frac{2}{A(t)}}. \tag{21}$$

The obtained value for s is meaningful (i.e. some valve becomes active), provided that the constraint $0 < s(t) < 1$, is satisfied. In particular, $s(t) < 1$, is trivially fulfilled while $s(t) > 0$ requires

$$\sqrt{\frac{2}{A(t)}} < 1 \quad \Leftrightarrow \quad \frac{A(t)}{2} > 1 \quad \Leftrightarrow \quad \frac{8\gamma}{\tau} \frac{\left|\dot{R}(t)\right|}{R^3(t)} > 1. \tag{22}$$

If, during the time interval $(0,\ t_\delta)$, (22) is violated the inward motion of the vessel is not able to overcome the action of the applied pressure gradient and the whole vessel is pervious.

[2]Recall that $\dot{R}$ is negative for $t \in (0, t_\delta)$.

In principle a generic contraction phase could take place through a sequence of accelerations and decelerations,causing an alternation between states in which the vessel is pervious and states in which valves are engaged. Therefore, if we want to deal with a general case, we must assume that inequality (22) is fulfilled in N time intervals $I_k \subset (0,\ t_\delta)$, $k = 1,\ ...,\ N$. If $t \in (0,\ t_\delta) \setminus \mathcal{I}$, where $\mathcal{I} = \bigcup_{k=1}^{N} I_k$, no valve is active and the pressure is given by

$$p(x,t) = -\frac{A(t)}{2}x^2 + \left(\frac{A(t)}{2} - 1\right)x + 1, \quad 0 \le x \le 1, \quad t \in (0,\ t_\delta) \setminus \mathcal{I}. \tag{23}$$

Summarizing, we have

$$p(x,t) = \begin{cases} -\frac{A(t)}{2}x^2 + \left(\frac{A(t)}{2} - 1\right)x + 1, \qquad 0 \le x \le 1, & \text{if} \quad t \in (0,\ t_\delta) \setminus \mathcal{I}, \\ \begin{cases} 1, & 0 \le x \le s(t), \\ 1 - \frac{A(t)}{2}(x - s(t))^2, & s(t) \le x \le 1. \end{cases} & \text{if} \quad t \in \mathcal{I}. \end{cases} \tag{24}$$

Having determined the pressure field during the contraction phase, i.e. in the whole time interval $(0, t_\delta)$, we can evaluate the outlet discharge $Q(1,t)$. Exploiting (14) we have

$$Q(1,t) = -\frac{\pi}{8}R^4(t) \left.\frac{\partial p}{\partial x}\right|_{x=1} \underset{(24)\,(21)}{=} \begin{cases} \frac{\pi}{8}R^4(t)\left(\frac{A(t)}{2} + 1\right), & \text{if} \quad t \in (0,\ t_\delta) \setminus \mathcal{I}, \\ \frac{\pi}{8}R^4(t)\sqrt{2A(t)}, & \text{if} \quad t \in \mathcal{I}. \end{cases} \tag{25}$$

Expansion phase.

We now consider the time interval $(t_\delta, 1)$. Still assuming that the active valve is placed at $x = s(t)$, now the fluid is stationary in the region $s(t) \le x \le 1$, where p equals the outlet pressure (that we rescaled to 0). Setting

$$B(t) = \frac{16\gamma}{\tau}\frac{\dot{R}}{R^3}, \quad \text{for} \quad t_\delta < t < 1, \tag{26}$$

the boundary value problem to be solved is

$$\begin{cases} \frac{\partial^2 p}{\partial x^2} = B(t), \quad 0 \le x \le s(t), \quad t_\delta < t < 1, \\ p|_{x=0} = 1, \\ \left.\frac{\partial p}{\partial x}\right|_{x=s(t)} = 0, \end{cases} \tag{27}$$

and then $s(t)$ is obtained imposing $p|_{x=s(t)} = 0$. We thus obtain

$$p(x,t) = \begin{cases} \dfrac{B(t)}{2}x^2 - B(t)\,s(t)\,x + 1, & \text{for} \quad t_\delta < t < 1, \;\; 0 \le x \le s(t), \\ 0, & \text{for} \quad t_\delta < t < 1, \;\; s(t) \le x \le 1, \end{cases} \tag{28}$$

with

$$s(t) = \sqrt{\frac{2}{B(t)}}\,. \tag{29}$$

Again, the value for s is meaningful if $0 < s(t) < 1$. While $0 < s(t)$ is always satisfied, $s(t) < 1$, entails

$$B(t) > 2 \qquad \Leftrightarrow \qquad \frac{8\gamma}{\tau}\frac{\dot{R}(t)}{R^3(t)} > 1. \tag{30}$$

If during the expansion phase (30) is not fulfilled, there is no closed valve and the pressure is

$$p(x,t) = \frac{B(t)}{2}x^2 - \left(\frac{B(t)}{2} + 1\right)x + 1, \qquad 0 \le x \le 1. \tag{31}$$

Assuming that (30) is satisfied in some intervals $J_k \subset (\,t_\delta, 1)$, $k = 1, ..., M$, setting $\mathcal{J} = \bigcup_{k=1}^{M} J_k$, by analogy with (24) we write

$$p(x,t) = \begin{cases} \dfrac{B(t)}{2}x^2 - \left(\dfrac{B(t)}{2} + 1\right)x + 1, \qquad 0 \le x \le 1, & \text{if} \quad t \in (t_\delta, 1) \setminus \mathcal{J}, \\ \begin{cases} \dfrac{B(t)}{2}x^2 - B(t)\,s(t)\,x + 1, & 0 \le x \le s(t), \\ 0, & s(t) \le x \le 1. \end{cases} & \text{if} \quad t \in \mathcal{J}. \end{cases} \tag{32}$$

The outlet discharge $Q(1,t)$ is given by

$$Q(1,t) = -\frac{\pi}{8}R^4(t)\left.\frac{\partial p}{\partial x}\right|_{x=1} \underset{(29)}{\underset{(32)}{=}} \begin{cases} \dfrac{\pi}{8}R^4(t)\left(1 - \dfrac{B(t)}{2}\right), & \text{if} \quad t \in (t_\delta, 1) \setminus \mathcal{J}, \\ 0, & \text{if} \quad t \in \mathcal{J}. \end{cases} \tag{33}$$

When (30) is not fulfilled, i.e. when $t \in (t_\delta, 1) \setminus \mathcal{J}$, the outflow is positive, while when $\left(1 - \dfrac{B(t)}{2}\right)$ is negative the outflow vanishes thus preventing backflow.

Total outlet discharge.

We can compute the total outlet discharge over the period $(0,1)$, $Q_{out} = \int_0^1 Q(1,t)\, dt$. Exploiting (25) and (33), we have

$$\begin{aligned} Q_{out} &= \frac{\pi}{8}\left[\int_{(0,\ t_\delta)\setminus\ \mathcal{I}} R^4(t)\left(\frac{A(t)}{2}+1\right)dt + \int_{\mathcal{I}} R^4(t)\sqrt{2A(t)}dt \right. \\ &\left. + \int_{(t_\delta,1)\setminus\ \mathcal{J}} R^4(t)\left(1-\frac{B(t)}{2}\right)dt\right]. \end{aligned} \tag{34}$$

The latter has to be compared with the discharge computed in absence of vasomotion, i.e. $\frac{\pi}{8}$, and with the total discharge computed in absence of distributed valves, namely

$$Q_{out,\ no\ valves} = \frac{\pi}{8}\int_0^{t_\delta} R^4(t)\left(\frac{A(t)}{2}+1\right)dt + \frac{\beta\pi}{8}\int_{t_\delta}^1 R^4(t)\left(1-\frac{B(t)}{2}\right)dt, \tag{35}$$

in order to quantify the effect of vasomotion.

Example 1 *We approximate the oscillation profile of [4] with the piecewise linear function*

$$R(t) = \begin{cases} -\dfrac{2\delta}{t_\delta}t+1, & \text{if } 0<t<t_\delta\ , \\[2ex] \dfrac{2\delta}{1-t_\delta}(t-1)+1, & \text{if } t_\delta<t<1\ , \end{cases} \tag{36}$$

with $2\delta = 0.3$, and $t_\delta = 1/4$. Accordingly

$$\begin{aligned} A(t) &= 32\frac{\gamma\delta}{\tau}\frac{t_\delta^2}{(t_\delta-2\delta t)^3} = \frac{\gamma^2}{2\beta(1-\delta)}\frac{t_\delta^2}{(t_\delta-2\delta t)^3}, \qquad 0<t<t_\delta\ , \\ B(t) &= 32\frac{\gamma\delta}{\tau}\frac{(1-t_\delta)^2}{(2\delta(t-1)+1-t_\delta)^3} = \frac{\gamma^2}{2\beta(1-\delta)}\frac{(1-t_\delta)^2}{(2\delta(t-1)+1-t_\delta)^3}, \qquad t_\delta<t<1. \end{aligned}$$

If $\beta/\gamma^2 \le 0.2$, conditions (22) and (30) are fulfilled in the whole interval. Hence $\mathcal{I} \equiv (0,t_\delta)$ and $\mathcal{J} \equiv (t_\delta,1)$. According to (25) and (33) we have

$$Q(1,t) = \begin{cases} \dfrac{\pi}{8t_\delta^3}\sqrt{\dfrac{\gamma^2}{(1-\delta)\beta}}(t_\delta-2\delta t)^{5/2}, & 0<t<t_\delta, \\[2ex] 0, & t_\delta<t<1, \end{cases} \tag{37}$$

and

$$Q_{out} = \frac{\pi\ t_\delta^{5/2}}{8t_\delta^2}\sqrt{\frac{\gamma^2}{(1-\delta)\beta}}\int_0^{t_\delta}\left(1-2\delta\frac{t}{t_\delta}\right)^{5/2}\frac{dt}{t_\delta} = \frac{\sqrt{t_\delta}\pi}{56\delta}\sqrt{\frac{\gamma^2}{(1-\delta)\beta}}\left(1-(1-2\delta)^{7/2}\right) \tag{38}$$

that we compare in Fig. 1, with $\frac{\pi}{8}$, i.e. the discharge in absence of vasomotion. In particular, Q_{out} is larger than the virtual Poiseuille-like discharge for $\beta/\gamma^2 \le 0.14$. Under that condition, such a positive difference between Q_{out} and $\frac{\pi}{8}$, expresses the advantageous contribution.

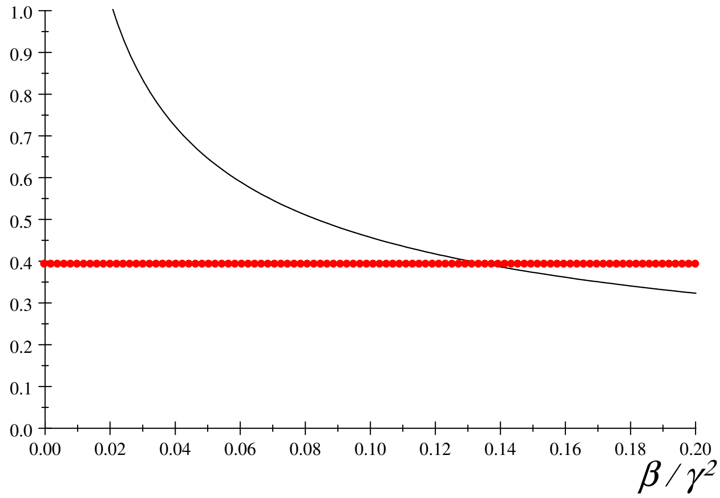

Figure 1: *Continuous line* Q_{out} *given by (38). Dotted line* $Q_{out} = \frac{\pi}{8}$.

3 Conclusions

We have analyzed a Newtonian flow in a vessel whose walls oscillate synchronously. The vessel is equipped with a series of valves which we assume to be arranged continuously along the entire vessel. The purpose of valves is to prevent retrograde flow. The vessel contraction causes the closing of the valves near the inlet, while, when the vessel relaxes, the engaging valves are located in the outlet region. The location of the shutting valve varies in time and is calculated explicitly, along with the pressure field and the longitudinal velocity.

Such a periodic constriction-relaxation of the vessel plays a key role. Indeed, when the mechanical action exerted by the vessel motion overcomes the weak hydraulic pressure gradient, the flow due to vasoconstriction exceeds the one due to the applied pressure. Conditions are found on the applied pressure field and on the oscillation parameters ensuring that valves play an advantageous role.

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