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# BOUNDS FOR THE NUMERICAL RADIUS OF $3 \times 3$ OPERATOR MATRICES

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**Abstract.** Operator matrices play an important role in operator inequalities. The matrix representation of an operator is always an easier way to look the properties of an operator more closely. Here we obtain bounds for the numerical radius of certain $3 \times 3$ operator matrices not unitarily equivalent to off-diagonal part of $3 \times 3$ operator matrices. Then using those results we establish some upper bounds for the numerical radius of general $3 \times 3$ operator matrices.

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# 1 Introduction.

Let T be a bounded linear operator on a complex Hilbert space H with norm $\|.\|$ and inner product $\langle . \rangle$. The numerical range and numerical radius of T is defined by

$$W(T) = \{< Tx, x >: x \in H, \ ||x|| = 1\}$$

and

$$w(T) = \sup\{| \lambda | \quad : \ \lambda \in \ W(T)\}.$$

The following properties of $w(T)$ are immediate:

1. $w(\alpha T) \ = \ |\alpha| w(T)$ , for any $\alpha \in \mathbb{C}$,
2. $w(U^* T U) = w(T)$ , for any unitary operator U on H.

It is well-known that $w(.)$ defines a norm on B(H) , where B(H) is the space of all bounded linear operators on H. Also this norm is equivalent to the operator norm by the following inequality

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|.$$

The lower bound of this inequality is attained if T is 2-nilpotent operator and the upper bound is attained if T is a normal operator [4]. This classical upper bound of numerical radius was substantially improved by Kittaneh [5] as he proved

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}.$$

Though the inequality of Kittaneh is a landmark in the study of numerical radius inequalities, the term $\|T^2\|^{\frac{1}{2}}$ is very hard to compute even for matrices of order more than 4. For this reason many mathematician over the years tried to find the bounds for numerical radius of an operator using its matrix representation. Some of them are mentioned below:

- Furuta in [3] proved that if a, b, c, d $\in \mathbb{R}^+$ and T $\in M_n(\mathbb{C})$ is of the form

  $$T = \begin{pmatrix} \text{a I} & \text{c A} \\ dA^* & \text{b I} \end{pmatrix},$$

  then

  $$w(T) = \frac{1}{2}(a + b + \sqrt{(a-b)^2 + (c+d)^2\|A\|^2}).$$

- Paul and Bag [6] proved that if a, d $\in \mathbb{R}$ and $T \in M_n(\mathbb{C})$ is of the form

  $$T = \begin{pmatrix} aI_r & \text{B} \\ O_{n-r,r} & dI_{n-r} \end{pmatrix},$$

  then

  $$w(T) = \frac{1}{2}(|a + d| + \sqrt{(a-d)^2 + \|B\|^2}).$$

Also one can go through the papers of Bani-Domi [2, 1] in this context.
Here we give some new numerical radius inequalities for certain $3 \times 3$ operator matrices. A $3 \times 3$ operator matrix T on $H \oplus H \oplus H$ is of the form

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

where each $T_{ij} \in B(H)$. We denote this operator by

$$M(T_{11}, T_{22}, T_{33}, T_{12}, T_{21}, T_{13}, T_{31}, T_{23}, T_{32}).$$

If some of these 9 entries are zero operator on H then we simply omit that position.
As an example

$$\begin{bmatrix} A & O & B \\ O & C & A \\ B & A & O \end{bmatrix}$$

will be denoted by

$$M(A_{11}, C_{22}, B_{13}, B_{31}, A_{23}, A_{32}).$$

As the operator matrices we are dealing with has most of the entries zero this will be a convenient way to write the operator matrices.

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Our desired form of $3 \times 3$ operator matrix is

$$\begin{bmatrix} O & O & A \\ B & O & O \\ O & C & O \end{bmatrix}$$

i.e. $M(A_{13}, B_{21}, C_{32})$ , defined on $H \oplus H \oplus H$, where $A, B, C \in B(H)$. Here we state two basic lemmas :

**Lemma 2.1.** *If $A, B, C \in B(H)$, then*

$$(a) \qquad w(M(A_{11}, B_{22}, C_{33})) = \max(w(A), w(B), w(C))$$

*and*

$$(b) \qquad w(M(A_{11}, A_{22}, A_{33}, B_{12}, C_{21}, C_{13}, B_{31}, B_{23}, C_{32})) =$$
$$\max(w(A + B + C), w(A + \alpha B + \alpha^2 C), w(A + \alpha^2 B + \alpha C)),$$

*where $\alpha$ is cubic root of unity.*

*Proof.* It follows from Lemma 1 of [1]. □

**Lemma 2.2.** *If $A, B, C \in B(H)$, then*

$$
\begin{aligned}
(a) \quad & w(M(A_{13}, B_{21}, C_{32})) = w(M(B_{13}, C_{21}, A_{32})) \\
&= w(M(C_{13}, A_{21}, B_{32})) = w(M(A_{12}, C_{23}, B_{31})) \\
&= w(M(B_{12}, A_{23}, C_{31})) = w(M(C_{12}, B_{23}, A_{31})) \\
&= w(M(\alpha A_{13}, \alpha^2 B_{21}, C_{32})) = w(M(\alpha^2 A_{13}, \alpha B_{21}, C_{32})) \\
&= w(M(\alpha A_{12}, C_{23}, \alpha^2 B_{31})) = w(M(\alpha^2 A_{12}, C_{23}, \alpha B_{31})),
\end{aligned}
$$

$$(b) \quad w(M(A_{13}, A_{21}, A_{32})) = w(A),$$

*where $\alpha$ is a cubic root of unity.*

*Proof.* To prove part (a), let

$$
\begin{aligned}
U_1 &= M(I_{12}, I_{23}, I_{31}), \quad U_2 = M(I_{13}, I_{21}, I_{32}), U_3 = M(I_{11}, I_{23}, I_{32}), \\
U_4 &= M(I_{12}, I_{21}, I_{33}), \quad U_5 = M(I_{13}, I_{22}, I_{31}), U_6 = M(I_{11}, \alpha^2 I_{22}, \alpha^2 I_{33}), \\
U_7 &= M(I_{11}, \alpha I_{22}, \alpha I_{33}), \quad U_8 = M(I_{11}, \alpha^2 I_{23}, \alpha^2 I_{32}), U_9 = M(I_{11}, \alpha I_{23}, \alpha I_{32}).
\end{aligned}
$$

Then $U_1, U_2, \cdots, U_9$ are unitary operator matrices, where $I$ is the identity operator in $B(H)$. Consider $M = M(A_{13}, B_{21}, C_{32})$. Now it is easy to prove the following identities:

$$
\begin{aligned}
U_1 M U_1^* &= M(B_{13}, C_{21}, A_{32}), \quad U_2 M U_2^* = M(C_{13}, A_{21}, B_{32}), \\
U_3 M U_3^* &= M(A_{12}, C_{23}, B_{31}), \quad U_4 M U_4^* = M(B_{12}, A_{23}, C_{31}), \\
U_5 M U_5^* &= M(C_{12}, B_{23}, A_{31}), \quad U_6 M U_6^* = M(\alpha A_{13}, \alpha^2 B_{21}, C_{32}), \\
U_7 M U_7^* &= M(\alpha^2 A_{13}, \alpha B_{21}, C_{32}), \quad U_8 M U_8^* = M(\alpha A_{12}, C_{23}, \alpha^2 B_{31}), \\
U_9 M U_9^* &= M(\alpha^2 A_{12}, C_{23}, \alpha B_{31}).
\end{aligned}
$$

Hence from the property of unitarily invariance, we obtain the required results.

Now to prove (b), we take

$$U = \frac{1}{\sqrt{3}} M(I_{11}, \alpha I_{22}, \alpha I_{33}, I_{12}, I_{21}, I_{13}, I_{31}, \alpha^2 I_{23}, \alpha^2 I_{32}).$$

Then $U$ is an unitary operator matrix. Thus

$$U M(A_{13}, A_{21}, A_{32}) U^* = M(A_{11}, \alpha A_{22}, \alpha^2 A_{33})$$

as $1, \alpha, \alpha^2$ are cubic root of unity and $1 + \alpha + \alpha^2 = 0$ . Consequently by Lemma 2.1(a) $w(M(A_{13}, A_{21}, A_{32})) = w(U M(A_{13}, A_{21}, A_{32}) U^*) = w(M(A_{11}, \alpha A_{22}, \alpha^2 A_{33})) = w(A)$. This completes the proof. □

We prove our first result as follows:

**Theorem 2.1.** *Let $A, B, C \in B(H)$ and $M = M(A_{13}, B_{21}, C_{32})$, then for $n = 1, 2, 3, \ldots$ we have*

$$\sqrt[3n]{max(w((ACB)^n), w((BAC)^n), w((CBA)^n))} \leq w(M) \leq \frac{1}{2} \; (\|A\| + \|B\| + \|C\|).$$

*Proof.* To prove the left hand inequality mentioned above, let $M = M(A_{13}, B_{21}, C_{32})$, then

$$M^{3n} = M((ACB)^n_{11}, (BAC)^n_{22}, (CBA)^n_{33}), \text{ for } n = 1, 2, 3, \ldots.$$

So by using by Lemma 2.1(a) we get

$$w(M^{3n}) = \max(w((ACB)^n), w((BAC)^n), w((CBA)^n)).$$

Also $w(M^{3n}) \leq (w(M))^{3n}$, by power inequality. Thus

$$\sqrt[3n]{\max(w((ACB)^n), w((BAC)^n), w((CBA)^n))} \leq w(M) \text{ for } n = 1, 2, 3, \ldots.$$

This completes the proof of the left hand inequality.
We next prove the right hand inequality. Since

$$(M(A_{13}))^2 = (M(B_{21}))^2 = (M(C_{32}))^2 = O,$$

so these three operators are 2-nilpotent and hence

$$w(M(A_{13})) = \frac{1}{2}\|M(A_{13})\|, \; w(M(B_{21})) = \frac{1}{2}\|M(B_{21})\|$$

and

$$w(M(C_{32})) = \frac{1}{2}\|M(C_{32})\|.$$

Therefore

$$\begin{aligned} w(M) &\leq w(M(A_{13})) + w(M(B_{21})) + w(M(C_{32})) \\ &= \frac{1}{2}(\|M(A_{13})\| + \|M(B_{21})\| + \|M(C_{32})\|) \\ &= \frac{1}{2}(\|A\| + \|B\| + \|C\|). \end{aligned}$$

This completes the proof. □

Now we give some bounds for $w(M)$ *i.e.* $w(M(A_{13}, B_{21}, C_{32}))$.

**Theorem 2.2.** *Let $A, B, C \in B(H)$, then*

$$\begin{aligned} &\frac{1}{3}\max(w(A+B+C), w(A+\alpha B+\alpha^2 C), w(A+\alpha^2 B+\alpha C)) \leq w(M) \\ &\qquad \leq \frac{1}{3}\big(w(A+B+C) + w(A+\alpha B+\alpha^2 C) + w(A+\alpha^2 B+\alpha C)\big). \end{aligned}$$

*Proof.* By Lemma 2.2, we have

$$\begin{aligned} w(A+B+C) &= w(M((A+B+C)_{13}, (A+B+C)_{21}, (A+B+C)_{32})) \\ &= w(M(A_{13}, B_{21}, C_{32}) + M(B_{13}, C_{21}, A_{32}) + M(C_{13}, A_{21}, B_{32})) \\ &\leq w(M(A_{13}, B_{21}, C_{32})) + w(M(B_{13}, C_{21}, A_{32})) \\ &\quad + w(M(C_{13}, A_{21}, B_{32})) \\ &= 3w(M(A_{13}, B_{21}, C_{32})) \\ &= 3w(M). \end{aligned}$$

Therefore,

$$\frac{1}{3}w(A+B+C) \leq w(M).$$

Similarly, by using lemma 2.2(a)

$$\begin{aligned}
& w(A+\alpha B+\alpha^2 C) \\
= \;& w(M((\alpha A+\alpha^2 B+C)_{13},(A+\alpha B+\alpha^2 C)_{21},(\alpha^2 A+B+\alpha C)_{32})). \\
= \;& w(M(\alpha A_{13},\alpha B_{21},\alpha C_{32})+M(\alpha^2 B_{13},\alpha^2 C_{21},\alpha^2 A_{32}) \\
& + M(C_{13},A_{21},B_{32})) \\
\leq \;& w(M(\alpha A_{13},\alpha B_{21},\alpha C_{32}))+w(M(\alpha^2 B_{13},\alpha^2 C_{21},\alpha^2 A_{32})) \\
& + w(M(C_{13},A_{21},B_{32})) \\
= \;& 3w(M(A_{13},B_{21},C_{32})) \\
= \;& 3w(M).
\end{aligned}$$

Therefore

$$\frac{1}{3}w(A+\alpha B+\alpha^2 C) \leq w(M).$$

Similarly we can prove that

$$\frac{1}{3}w(A+\alpha^2 B+\alpha C) \leq w(M).$$

Therefore

$$\frac{1}{3}\max(w(A+B+C),w(A+\alpha B+\alpha^2 C),w(A+\alpha^2 B+\alpha C)) \leq w(M).$$

Now to prove the other side of the inequality suppose

$$U=\frac{1}{\sqrt{3}}M(I_{11},\alpha I_{22},\alpha I_{33},I_{12},I_{21},I_{13},I_{31},\alpha^2 I_{23},\alpha^2 I_{32}) \text{ and } M=M(A_{13},B_{21},C_{32}).$$

Then $U$ is unitary and so

$$\begin{aligned}
& w(M) \\
= \;& w(UMU^*) \\
= \;& \frac{1}{3}w(M((A+B+C)_{11},\alpha(A+B+C)_{22},\alpha^2(A+B+C)_{33},(\alpha A+B+\alpha^2 C)_{12}, \\
& (A+\alpha B+\alpha^2 C)_{21},(\alpha^2 A+B+\alpha C)_{13},(A+\alpha^2 B+\alpha C)_{31}, \\
& (\alpha^2 A+\alpha B+C)_{23},(\alpha A+\alpha^2 B+C)_{32})) \\
= \;& \frac{1}{3}w(M((A+B+C)_{11},\alpha(A+B+C)_{22},\alpha^2(A+B+C)_{33}) \\
& + M((\alpha^2 A+B+\alpha C)_{13},(A+\alpha B+\alpha^2 C)_{21},(\alpha A+\alpha^2 B+C)_{32}) \\
& + M((\alpha A+B+\alpha^2 C)_{12},(\alpha^2 A+\alpha B+C)_{23},(A+\alpha^2 B+\alpha C)_{31})) \\
\leq \;& \frac{1}{3}w(M((A+B+C)_{11},\alpha(A+B+C)_{22},\alpha^2(A+B+C)_{33})) \\
& +\frac{1}{3}w(M((\alpha^2 A+B+\alpha C)_{13},(A+\alpha B+\alpha^2 C)_{21},(\alpha A+\alpha^2 B+C)_{32})) \\
& +\frac{1}{3}w(M((\alpha A+B+\alpha^2 C)_{12},(\alpha^2 A+\alpha B+C)_{23},(A+\alpha^2 B+\alpha C)_{31})).
\end{aligned}$$

Therefore using Lemma 2.1(a), 2.2(a) and 2.2(b) we get

$$w(M) \leq \frac{1}{3}\big(w(A+B+C) + w(A+\alpha B+\alpha^2 C) + w(A+\alpha^2 B+\alpha C)\big).$$

□

**Remark 2.1.** If $A = B = C$, then the inequalities in Theorem 2.2 become equalities.

Before stating the next theorem, we shall go through an observation.
Suppose $X \in B(H)$. Then define

$$R = M(X_{11}, \alpha X_{22}, \alpha^2 X_{33}, X_{12}, \alpha X_{21}, X_{13}, \alpha^2 X_{31}, \alpha X_{23}, \alpha^2 X_{32})$$

and

$$U = \frac{1}{\sqrt{3}} M(I_{11}, \alpha I_{22}, \alpha I_{33}, I_{12}, I_{21}, I_{13}, I_{31}, \alpha^2 I_{23}, \alpha^2 I_{32}).$$

Clearly $R^2 = O$ and U is an unitary operator. Therefore

$$w(R) = \frac{1}{2}\|R\|.$$

Since operator norm is invariant under unitary transformation, we have

$$\|R\| = \|URU^*\| = \|M((3X)_{31})\| = 3\|X\|.$$

Therefore

$$w(R) = \frac{1}{2}\|R\| = \frac{3}{2}\|X\|.$$

**Theorem 2.3.** *Let $A, B, C \in B(H)$, then for $M = M(A_{13}, B_{21}, C_{32})$ and $P,\ Q \in \{A,\ B,\ C\}$ with $P \neq Q$ we have*

$$w(M) \leq \min\{\delta_1, \delta_2, \delta_3\},$$

*where*

$$\begin{aligned} \delta_1 &= \frac{1}{2}\ \|A+B+C\| \\ &+ \frac{1}{3}\min\{w((\alpha^2+2)P + (\alpha+2)Q) + w((\alpha+2)P + (\alpha^2+2)Q)\}, \\ \delta_2 &= \frac{1}{2}\ \left\|\alpha A + \alpha^2 B + C\right\| \\ &+ \frac{1}{3}\min\{w((\alpha^2+2)\alpha P + (\alpha+2)Q) + w((\alpha+2)\alpha P + (\alpha^2+2)Q)\} \end{aligned}$$

*and*

$$\begin{aligned} \delta_3 &= \frac{1}{2}\ \left\|\alpha^2 A + \alpha B + C\right\| \\ &+ \frac{1}{3}\min\{w((\alpha^2+2)\alpha^2 P + (\alpha+2)Q) + w((\alpha+2)\alpha^2 P + (\alpha^2+2)Q)\}. \end{aligned}$$

*Proof.* First we prove that $w(M) \leq \delta_1$. As we discussed earlier U is an unitary operator on $H \oplus H \oplus H$. and numerical radius is invariant under unitary transformation. Therefore

$$\begin{aligned} w(M) &= w(UMU^*) \\ &= \frac{1}{3}w(M((A+B+C)_{11}, \alpha(A+B+C)_{22}, \alpha^2(A+B+C)_{33}, \\ &\quad (\alpha A+B+\alpha^2 C)_{12}, (A+\alpha B+\alpha^2 C)_{21}, (\alpha^2 A+B+\alpha C)_{13}, \\ &\quad (A+\alpha^2 B+\alpha C)_{31}, (\alpha^2 A+\alpha B+C)_{23}, (\alpha A+\alpha^2 B+C)_{32})). \\ &\quad \text{(Rearranging the elements, we get)} \\ &= \frac{1}{3}w(R+M((-(\alpha^2+2)A-(\alpha+2)C\ )_{12},\ (-(2\alpha+1)A-(\alpha^2+2\alpha)C\ )_{23}, \\ &\quad (-(2\alpha^2+\alpha)A-(1+2\alpha^2)C\ )_{31}\ ) + M((\ -(\alpha+2)A-(\alpha^2+2)C\ )_{13}, \\ &\quad (\ -(2\alpha+\alpha^2)A-(1+2\alpha)C\ )_{21}, (\ -(1+2\alpha^2)A-(\alpha+2\alpha^2)C\ )_{32})), \end{aligned}$$

where R is as in the previous discussion with $X = A+B+C$ . Let us denote

$$\begin{aligned} M_1 &= M((-(\alpha^2+2)A-(\alpha+2)C\ )_{12},\ (-(2\alpha+1)A-(\alpha^2+2\alpha)C\ )_{23}, \\ &\quad (-(2\alpha^2+\alpha)A-(1+2\alpha^2)C\ )_{31}\ ) \end{aligned}$$

and

$$\begin{aligned} M_2 &= M((-(\alpha+2)A-(\alpha^2+2)C\ )_{13},\ (-(2\alpha+\alpha^2)A-(1+2\alpha)C\ )_{21}, \\ &\quad (-(1+2\alpha^2)A-(\alpha+2\alpha^2)C\ )_{32}). \end{aligned}$$

Therefore

$$w(M) = \frac{1}{3}w(R+M_1+M_2) \leq \frac{1}{3}(w(R)+w(M_1)+w(M_2)).$$

Now using the above observation and Lemma 2.2 we get,

$$w(R) = \frac{3}{2}\|A+B+C\|,\ w(M_1) = w((\alpha^2+2)A+(\alpha+2)C)$$

and

$$w(M_2) = w((\alpha+2)A+(\alpha^2+2)C).$$

So

$$\begin{aligned} w(M(A_{13}, B_{21}, C_{32})) &\leq \frac{1}{3}(\frac{3}{2}\|A+B+C\| \ + \ w((\alpha^2+2)A+(\alpha+2)C) \\ &+ \ \ w((\alpha+2)A+(\alpha^2+2)C)). \end{aligned}$$

Similarly

$$\begin{aligned} w(M(B_{13}, C_{21}, A_{32})) &\leq \frac{1}{3}(\frac{3}{2}\|A+B+C\| \ + \ w((\alpha^2+2)A+(\alpha+2)B) \\ &+ \ \ w((\alpha+2)A+(\alpha^2+2)B)) \\ w(M(C_{13}, A_{21}, B_{32})) &\leq \frac{1}{3}(\frac{3}{2}\|A+B+C\| \ + \ w((\alpha^2+2)B+(\alpha+2)C) \\ &+ \ \ w((\alpha+2)B+(\alpha^2+2)C)). \end{aligned}$$

Now from unitarily invariance property of $w$ we get

$$w(M) \leq \delta_1.$$

To prove $w(M) \leq \delta_2$, using similar procedure we get

$$\begin{aligned}
w(M(\alpha A_{13}, \alpha^2 B_{21}, C_{32})) &\leq \frac{1}{3}(\frac{3}{2}\|\alpha A + \alpha^2 B + C\| + w((\alpha^2+2)C + (\alpha+2)\alpha A) \\
&\quad + w((\alpha+2)C + (\alpha^2+2)\alpha A)) \\
w(M(\alpha B_{13}, \alpha^2 C_{21}, A_{32})) &\leq \frac{1}{3}(\frac{3}{2}\|\alpha A + \alpha^2 B + C\| + w((\alpha^2+2)A + (\alpha+2)\alpha B) \\
&\quad + w((\alpha+2)A + (\alpha^2+2)\alpha B)) \\
w(M(\alpha C_{13}, \alpha^2 A_{21}, B_{32})) &\leq \frac{1}{3}(\frac{3}{2}\|\alpha A + \alpha^2 B + C\| + w((\alpha^2+2)B + (\alpha+2)\alpha C) \\
&\quad + w((\alpha+2)B + (\alpha^2+2)\alpha C)).
\end{aligned}$$

Now from unitarily invariance property of $w$ we get

$$w(M) \leq \delta_2.$$

Lastly to prove $w(M) \leq \delta_3$, using similar procedure we get

$$\begin{aligned}
w(M(\alpha^2 A_{13}, \alpha B_{21}, C_{32})) &\leq \frac{1}{3}(\frac{3}{2}\|\alpha^2 A + \alpha B + C\| + w((\alpha^2+2)C + (\alpha+2)\alpha^2 A) \\
&\quad + w((\alpha+2)C + (\alpha^2+2)\alpha^2 A)) \\
w(M(\alpha^2 B_{13}, \alpha C_{21}, A_{32}) &\leq \frac{1}{3}(\frac{3}{2}\|\alpha^2 A + \alpha B + C\| + w((\alpha^2+2)A + (\alpha+2)\alpha^2 B) \\
&\quad + w((\alpha+2)A + (\alpha^2+2)\alpha^2 B)) \\
w(M(\alpha^2 C_{13}, \alpha A_{21}, B_{32}) &\leq \frac{1}{3}(\frac{3}{2}\|\alpha^2 A + \alpha B + C\| + w((\alpha^2+2)B + (\alpha+2)\alpha^2 C) \\
&\quad + w((\alpha+2)B + (\alpha^2+2)\alpha^2 C)).
\end{aligned}$$

Again from unitarily invariance property of $w$ we get

$$w(M) \leq \delta_3.$$

Therefore

$$w(M) \leq \min\{\delta_1, \delta_2, \delta_3\}.$$

This completes the proof. □

**Remark 2.2.** If $A = B = C$, then the inequalities in Theorem 2.3 become equalities.

# 3 Upper Bounds for the Numerical Radius of a General $3 \times 3$ Operator Matrix

In this section we shall discuss about the upper bounds of a general $3 \times 3$ operator matrix. Before stating our main theorems we prove a lemma which will be useful to get these results.

**Lemma 3.1.** *Suppose $A,\ B,\ C\ \in B(H)$. Then*

$$w(M(A_{11},\alpha A_{22},\alpha^2 A_{33},\alpha B_{12},C_{21},\alpha^2 C_{13},B_{31},\alpha^2 B_{23},\alpha C_{32}))\leq w(A)+w(B)+w(C).$$

*Proof.* We have

$$\begin{aligned}
& w(\ M(A_{11},\alpha A_{22},\alpha^2 A_{33},\alpha B_{12},C_{21},\alpha^2 C_{13},B_{31},\alpha^2 B_{23},\alpha C_{32})\ )\\
\leq\ & w(\ M(A_{11},\alpha A_{22},\alpha^2 A_{33}\ ))\ +\ w(\ M(\alpha B_{12},\ \alpha^2 B_{23},\ B_{31}\ ))\\
& +\ w(\ M(\ \alpha^2 C_{13},\ C_{21},\ \alpha C_{32}\ )\ )\\
\leq\ & w(A)+w(B)+w(C),
\end{aligned}$$

by Lemma 2.1(a), 2.2(a) and 2.2(b). □

Before we prove our next theorem we introduce a notation. By $M(A_{ij*kl},A_{pq*rs})$ we mean a $3\times 3$ operator matrix whose $(k,l)$-th entry is $A_{ij}$, $(r,s)$-th entry is $A_{pq}$ and all other entries are zero.

**Theorem 3.1.** *Let $A_{ij}\in B(H),\ \forall\ i,j=1,2,3$ . Then*

$$\begin{aligned}
w([A_{ij}])\ \leq\ & \max(w(A_{11}),w(A_{22}),w(A_{33}))+\frac{1}{3}\ [w(A_{12}+A_{31}+A_{23})\\
& +w(A_{12}+\alpha A_{31}+\alpha^2 A_{23})+w(A_{12}+\alpha^2 A_{31}+\alpha A_{23})\\
& +w(A_{13}+A_{21}+A_{32})+w(A_{13}+\alpha A_{21}+\alpha^2 A_{32})+w(A_{13}+\alpha^2 A_{21}+\alpha A_{32})].
\end{aligned}$$

*Proof.* Suppose $[A_{ij}]_{3\times 3}\in B(H\oplus H\oplus H)$ where each $A_{ij}\in B(H)$. Let us denote this operator $[A_{ij}]_{3\times 3}$ by $M(A_{11*11},\ A_{22*22},\ A_{33*33},\ A_{12*12},\ A_{21*21},\ A_{13*13},\ A_{31*31},\ A_{23*23},\ A_{32*32})$. Therefore

$$\begin{aligned}
& w([A_{ij}])\\
=\ & w(\ M(A_{11*11},\ A_{22*22},\ A_{33*33},\ A_{12*12},\ A_{21*21},\ A_{13*13},\ A_{31*31},\ A_{23*23},\ A_{32*32})\ )\\
\leq\ & w(\ M(A_{11*11},\ A_{22*22},\ A_{33*33}\ )\ )+w(\ M(\ A_{12*12},\ A_{23*23},\ A_{31*31}\ )\ )\\
& +\ w(\ M(\ A_{13*13},\ A_{21*21},\ A_{32*32}\ )\ )\\
=\ & w(\ M(A_{11*11},\ A_{22*22},\ A_{33*33}\ )\ )+w(\ M(\ A_{12*13},\ A_{23*32},\ A_{31*21}\ )\ )\\
& +\ w(\ M(\ A_{13*13},\ A_{21*21},\ A_{32*32}\ )\ ),
\end{aligned}$$

by Lemma 2.2(a).

$$\begin{aligned}
\leq\ & \max(w(A_{11}),w(A_{22}),w(A_{33}))+\frac{1}{3}\ [w(A_{12}+A_{31}+A_{23})+w(A_{12}+\alpha A_{31}+\alpha^2 A_{23})\\
& +\ w(A_{12}+\alpha^2 A_{31}+\alpha A_{23})+w(A_{13}+A_{21}+A_{32})+w(A_{13}+\alpha A_{21}+\alpha^2 A_{32})\\
& +\ w(A_{13}+\alpha^2 A_{21}+\alpha A_{32})],
\end{aligned}$$

by Lemma 2.1(a) and Theorem 2.2.
This completes the proof. □

**Theorem 3.2.** *Let* $A_{ij} \in B(H), \ \forall \ i,j = 1,2,3$ . *Then*

$$\begin{aligned} w([A_{ij}]) \ \leq \ & \max(w(A_{11}), w(A_{22}), w(A_{33})) \ + \ \min\{\Delta_1, \ \Delta_2, \ \Delta_3\} \\ & + \ \min\{\Phi_1, \ \Phi_2, \ \Phi_3\}, \end{aligned}$$

*where*

$$\begin{aligned} \Delta_1 &= \tfrac{1}{2}\|A_{12}\| \ + \ \tfrac{1}{3}( \ w(A_{31} \ + \ A_{23}) \ + \ w(A_{31} \ + \ \alpha A_{23}) \ + \ w(A_{31} \ + \ \alpha^2 A_{23}) \ ) \\ \Delta_2 &= \tfrac{1}{2}\|A_{23}\| \ + \ \tfrac{1}{3}( \ w(A_{12} \ + \ A_{31}) \ + \ w(A_{12} \ + \ \alpha A_{31}) \ + \ w(A_{12} \ + \ \alpha^2 A_{31}) \ ) \\ \Delta_3 &= \tfrac{1}{2}\|A_{31}\| \ + \ \tfrac{1}{3}( \ w(A_{12} \ + \ A_{23}) \ + \ w(A_{12} \ + \ \alpha A_{23}) \ + \ w(A_{12} \ + \ \alpha^2 A_{23}) \ ) \end{aligned}$$

*and*

$$\begin{aligned} \Phi_1 &= \tfrac{1}{2}\|A_{13}\| \ + \ \tfrac{1}{3}( \ w(A_{21} \ + \ A_{32}) \ + \ w(A_{21} \ + \ \alpha A_{32}) \ + \ w(A_{21} \ + \ \alpha^2 A_{32}) \ ) \\ \Phi_2 &= \tfrac{1}{2}\|A_{21}\| \ + \ \tfrac{1}{3}( \ w(A_{13} \ + \ A_{32}) \ + \ w(A_{13} \ + \ \alpha A_{32}) \ + \ w(A_{13} \ + \ \alpha^2 A_{32}) \ ) \\ \Phi_3 &= \tfrac{1}{2}\|A_{32}\| \ + \ \tfrac{1}{3}( \ w(A_{13} \ + \ A_{21}) \ + \ w(A_{13} \ + \ \alpha A_{21}) \ + \ w(A_{13} \ + \ \alpha^2 A_{21}) \ ). \end{aligned}$$

*Proof.* We have

$$\begin{aligned} & w([A_{ij}]) \\ = \ & w( \ M(A_{11*11}, \ A_{22*22}, \ A_{33*33}, \ A_{12*12}, \ A_{21*21}, \ A_{13*13}, \ A_{31*31}, \ A_{23*23}, \ A_{32*32}) \ ) \\ \leq \ & w( \ M(A_{11*11}, \ A_{22*22}, \ A_{33*33} \ ) \ ) + w( \ M( \ A_{12*12}, \ A_{23*23}, \ A_{31*31} \ ) \ ) \\ & \ + \ w( \ M( \ A_{13*13}, \ A_{21*21}, \ A_{32*32} \ ) \ ). \end{aligned}$$

Now

$$\begin{aligned} & w( \ M( \ A_{12*12}, \ A_{23*23}, \ A_{31*31} \ ) \ ) \\ \leq \ & w( \ M(A_{12*12}) \ ) \ + \ w( \ M(A_{23*23}, \ A_{31*31})) \\ = \ & w( \ M(A_{12*12}) \ ) \ + \ w( \ M(A_{31*21}, \ A_{23*32})), \end{aligned}$$

by Lemma 2.2(a).
Here $M(A_{12*12})$ is a 2-nilpotent operator and from Theorem 2.2 we get

$$\begin{aligned} & w( \ M( \ A_{12*12}, \ A_{23*23}, \ A_{31*31} \ ) \ ) \\ \leq \ & \frac{1}{2}\|A_{12}\| + \frac{1}{3}( \ w(A_{31} \ + \ A_{23}) + w(A_{31} \ + \ \alpha A_{23}) + w(A_{31} \ + \ \alpha^2 A_{23}) \ ) \\ = \ & \Delta_1. \end{aligned}$$

Similarly it is easy to see that

$$w( \ M( \ A_{12*12}, \ A_{23*23}, \ A_{31*31} \ ) \ ) \leq w( \ M(A_{23*23}) \ ) \ + \ w( \ M(A_{12*12}, \ A_{31*31}))$$

$$\leq \frac{1}{2}\|A_{23}\| + \frac{1}{3}( \ w(A_{12} \ + \ A_{31}) + w(A_{12} \ + \ \alpha A_{31}) + w(A_{12} \ + \ \alpha^2 A_{31}) \ ) = \Delta_2$$

and

$$w( \ M( \ A_{12*12}, \ A_{23*23}, \ A_{31*31} \ ) \ ) \leq w( \ M(A_{31*31}) \ ) \ + \ w( \ M(A_{12*12}, \ A_{23*23}))$$

$$\leq \frac{1}{2}\|A_{31}\| + \frac{1}{3}(\ w(A_{12}\ +\ A_{23}) + w(A_{12}\ +\ \alpha A_{23}) + w(A_{12}\ +\ \alpha^2 A_{23})\ ) = \Delta_3.$$

Therefore

$$w(\ M(\ A_{12*12},\ A_{23*23},\ A_{31*31}\ )\ ) \leq \min\{\ \Delta_1,\ \Delta_2,\ \Delta_3\}.$$

From this deduction, it is also evident that

$$w(\ M(\ A_{13*13},\ A_{21*21},\ A_{32*32}\ )\ ) \leq \min\{\ \Phi_1,\ \Phi_2,\ \Phi_3\}.$$

This completes the proof. □

**Theorem 3.3.** *Let* $A_{ij} \in B(H),\ \forall\ i,j = 1,2,3$ *. Then for* $0 \leq p \leq 1$

$$\begin{aligned} w([A_{ij}]) \ \leq \ & \max(w(A_{11}), w(A_{22}), w(A_{33})) \ + \ \min\{\ \Delta_1',\ \Delta_2',\ \Delta_3'\} \\ & + \ \min\{\ \Phi_1',\ \Phi_2',\ \Phi_3'\}, \end{aligned}$$

*where*

$$\begin{aligned}
\Delta'_1 \ &= \ \frac{1}{3}(w(pA_{12} + A_{23}) \ + \ w(pA_{12} + \alpha A_{23}) \ + \ w(pA_{12} + \alpha^2 A_{23})) \\
&+ \frac{1}{3}(w((1-p)A_{12} + A_{31}) \ + \ w((1-p)A_{12} + \alpha A_{31}) \ + \ w((1-p)A_{12} + \alpha^2 A_{31})), \\
\Delta'_2 \ &= \ \frac{1}{3}(w(pA_{23} + A_{12}) \ + \ w(pA_{23} + \alpha A_{12}) \ + \ w(pA_{23} + \alpha^2 A_{12})) \\
&+ \frac{1}{3}(w((1-p)A_{23} + A_{31}) \ + \ w((1-p)A_{23} + \alpha A_{31}) \ + \ w((1-p)A_{23} + \alpha^2 A_{31})), \\
\Delta'_3 \ &= \ \frac{1}{3}(w(pA_{31} + A_{12}) \ + \ w(pA_{31} + \alpha A_{12}) \ + \ w(pA_{31} + \alpha^2 A_{12})) \\
&+ \frac{1}{3}(w((1-p)A_{31} + A_{23}) \ + \ w((1-p)A_{31} + \alpha A_{23}) \ + \ w((1-p)A_{31} + \alpha^2 A_{23}))
\end{aligned}$$

*and*

$$\begin{aligned}
\Phi'_1 \ &= \ \frac{1}{3}(w(pA_{13} + A_{21}) \ + \ w(pA_{13} + \alpha A_{21}) \ + \ w(pA_{13} + \alpha^2 A_{21})) \\
&+ \frac{1}{3}(w((1-p)A_{13} + A_{32}) \ + \ w((1-p)A_{13} + \alpha A_{32}) \ + \ w((1-p)A_{13} + \alpha^2 A_{32})), \\
\Phi'_2 \ &= \ \frac{1}{3}(w(pA_{21} + A_{13}) \ + \ w(pA_{21} + \alpha A_{13}) \ + \ w(pA_{21} + \alpha^2 A_{13})) \\
&+ \frac{1}{3}(w((1-p)A_{21} + A_{32}) \ + \ w((1-p)A_{21} + \alpha A_{32}) \ + \ w((1-p)A_{21} + \alpha^2 A_{32})), \\
\Phi'_3 \ &= \ \frac{1}{3}(w(pA_{32} + A_{13}) \ + \ w(pA_{32} + \alpha A_{13}) \ + \ w(pA_{32} + \alpha^2 A_{13})) \\
&+ \frac{1}{3}(w((1-p)A_{32} + A_{21}) \ + \ w((1-p)A_{32} + \alpha A_{21}) \ + \ w((1-p)A_{32} + \alpha^2 A_{21})).
\end{aligned}$$

*Proof.* We have

$$\begin{aligned}
& w([A_{ij}]) \\
= \ & w(\ M(A_{11*11},\ A_{22*22},\ A_{33*33},\ A_{12*12},\ A_{21*21},\ A_{13*13},\ A_{31*31},\ A_{23*23},\ A_{32*32})\ ) \\
\leq \ & w(\ M(A_{11*11},\ A_{22*22},\ A_{33*33}\ )\ ) + w(\ M(\ A_{12*12},\ A_{23*23},\ A_{31*31}\ )\ ) \\
& + w(\ M(\ A_{13*13},\ A_{21*21},\ A_{32*32}\ )\ )
\end{aligned}$$

Now for $0 \le p \le 1$, using Lemma 2.2(a) and Theorem 2.2 we get

$$\begin{aligned}
& w(\ M(\ A_{12*12},\ A_{23*23},\ A_{31*31}\ )\ ) \\
= \ & w(\ M(\ (p+(1-p))A_{12*12},\ A_{23*23},\ A_{31*31}\ )\ ) \\
= \ & w(\ M(pA_{12*12},\ A_{23*23}\ ) + \ M((1-p)A_{12*12},\ A_{31*31}\ )\ ) \\
\le \ & w(\ M(pA_{12*12},\ A_{23*23}\ )\ ) + w(\ M((1-p)A_{12*12},\ A_{31*31}\ )\ ) \\
= \ & w(\ M(pA_{12*13},\ A_{23*32}\ )\ ) + w(\ M((1-p)A_{12*13},\ A_{31*21}\ )\ ) \\
\le \ & \frac{1}{3}(\ w(pA_{12}+A_{23}) + w(pA_{12}+\alpha A_{23}) + w(pA_{12}+\alpha^2 A_{23})\ ) \\
& +\frac{1}{3}(\ w((1-p)A_{12}+A_{31}) + w((1-p)A_{12}+\alpha A_{31}) + w((1-p)A_{12}+\alpha^2 A_{31})\ ) \\
= \ & \Delta_1^{'}.
\end{aligned}$$

In a similar way we can prove the following inequalities

$$\begin{aligned}
& w(\ M(\ A_{12*12},\ A_{23*23},\ A_{31*31}\ )\ ) \\
= \ & w(\ M(\ A_{12*12},\ (p+(1-p))A_{23*23},\ A_{31*31}\ )\ ) \\
\le \ & \frac{1}{3}(\ w(pA_{23}+A_{12}) + w(pA_{23}+\alpha A_{12}) + w(pA_{23}+\alpha^2 A_{12})\ ) \\
& +\frac{1}{3}(\ w((1-p)A_{23}+A_{31}) + w((1-p)A_{23}+\alpha A_{31}) + w((1-p)A_{23}+\alpha^2 A_{31})\ ) \\
= \ & \Delta_2^{'}
\end{aligned}$$

and

$$\begin{aligned}
& w(\ M(\ A_{12*12},\ A_{23*23},\ A_{31*31}\ )\ ) \\
= \ & w(\ M(\ A_{12*12},\ A_{23*23},\ (p+(1-p))A_{31*31}\ )\ ) \\
\le \ & \frac{1}{3}(\ w(pA_{31}+A_{12}) + w(pA_{31}+\alpha A_{12}) + w(pA_{31}+\alpha^2 A_{12})\ ) \\
& +\frac{1}{3}(\ w((1-p)A_{31}+A_{23}) + w((1-p)A_{31}+\alpha A_{23}) + w((1-p)A_{31}+\alpha^2 A_{23})\ ) \\
= \ & \Delta_3^{'}.
\end{aligned}$$

Again if we consider $M(\ A_{13*13},\ A_{21*21},\ A_{32*32})$, then for $0 \le p \le 1$

$$\begin{aligned}
& w(\ M(\ A_{13*13},\ A_{21*21},\ A_{32*32}\ )\ ) \\
= \ & w(\ M(\ (p+(1-p))A_{13*13},\ A_{21*21},\ A_{32*32}\ )\ ) \\
= \ & w(\ M(pA_{13*13},\ A_{21*21}\ ) + \ M((1-p)A_{13*13},\ A_{32*32}\ )\ ) \\
\le \ & w(\ M(pA_{13*13},\ A_{21*21}\ )\ ) + w(\ M((1-p)A_{13*13},\ A_{32*32}\ )\ ) \\
\le \ & \frac{1}{3}(\ w(pA_{13}+A_{21}) + w(pA_{13}+\alpha A_{21}) + w(pA_{13}+\alpha^2 A_{21})\ ) \\
& +\frac{1}{3}(\ w((1-p)A_{13}+A_{32}) + w((1-p)A_{13}+\alpha A_{32}) + w((1-p)A_{13}+\alpha^2 A_{32})\ ) \\
= \ & \Phi_1^{'}.
\end{aligned}$$

Similarly

$$
\begin{aligned}
& w(\ M(\ A_{13*13},\ A_{21*21},\ A_{32*32}\ )\ ) \\
= \ & w(\ M(\ A_{13*13},\ (p+(1-p))A_{21*21},\ A_{32*32}\ )\ ) \\
\leq \ & \frac{1}{3}(\ w(pA_{21}+A_{13})+w(pA_{21}+\alpha A_{13})+w(pA_{21}+\alpha^2 A_{13})\ ) \\
& +\frac{1}{3}(\ w((1-p)A_{21}+A_{32})+w((1-p)A_{21}+\alpha A_{32})+w((1-p)A_{21}+\alpha^2 A_{32})\ ) \\
= \ & \Phi_2^{'}
\end{aligned}
$$

and

$$
\begin{aligned}
& w(\ M(\ A_{13*13},\ A_{21*21},\ A_{32*32}\ )\ ) \\
= \ & w(\ M(\ A_{13*13},\ A_{21*21},\ (p+(1-p))A_{32*32}\ )\ ) \\
\leq \ & \frac{1}{3}(\ w(pA_{32}+A_{13})+w(pA_{32}+\alpha A_{13})+w(pA_{32}+\alpha^2 A_{13})\ ) \\
& +\frac{1}{3}(\ w((1-p)A_{32}+A_{21})+w((1-p)A_{32}+\alpha A_{21})+w((1-p)A_{32}+\alpha^2 A_{21})\ ) \\
= \ & \Phi_3^{'}.
\end{aligned}
$$

Therefore

$$
w([A_{ij}]) \leq \max(w(A_{11}),w(A_{22}),w(A_{33}))\ +\ \min\{\Delta_1^{'},\ \Delta_2^{'},\ \Delta_3^{'}\}\ +\ \min\{\Phi_1^{'},\ \Phi_2^{'},\ \Phi_3^{'}\}.
$$

This completes the proof. □

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