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CHEMOTAXIS WITH QUADRATIC DISSIPATION AND LOGISTIC SOURCE

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Abstract. We study systems of chemotaxis with quadratic dissipation and logistic source. Global existence and convergence to the spatially homogeneous steady state are proven in several cases, particularly when the space dimension is two or the dissipation dominates the chemotaxis. The blowup case is also studied for modified nonlinearities.

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1 Introduction

Our purpose is to study chemotaxis systems with source terms. Such models have been proposed in the field of mathematical biology (see, for example [13] concerning the growth case) provided with diffusion, chemotaxis, and self-dissipation or growth. Several transient patterns are observed both numerically and theoretically.

A typical example is

$$\begin{aligned} & u_t = d\Delta u - \chi\nabla\cdot u\nabla v + \alpha f(u), \quad \text{in } \Omega\times(0,T), \\ & d\frac{\partial u}{\partial\nu} - \chi u\frac{\partial v}{\partial\nu}\Big|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0, \end{aligned} \tag{1}$$

with

$$-\Delta v = u - \overline{u} \quad \text{in } \Omega\times(0,T), \quad \overline{v} = 0, \quad \frac{\partial v}{\partial\nu}\Big|_{\partial\Omega} = 0, \tag{2}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outer unit normal vector, $d,\chi,\alpha>0$ are constants, $u_0=u_0(x)$ is a sufficiently regular function on $\overline{\Omega}$, and

$$\overline{w} = \frac{1}{|\Omega|}\int_\Omega w$$

for $w=w(x)$. Here, $u=u(x,t)$ stands for the density of some biological species. The nonlinearity $f=f(u)$ is a smooth function of $u\geq 0$ satisfying $f(0)\geq 0$ which guarantees $u=u(x,t)>0$. The second equation (2) may be replaced by

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0 \tag{3}$$

or

$$\tau v_t - \Delta v = u - \overline{u}, \quad \frac{\partial v}{\partial\nu}\Big|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0(x) \tag{4}$$

where $\tau>0$ is a constant and $v_0=v_0(x)$ is a smooth function.

Known results

In a series of papers, J.I. Tello and M. Winkler [20, 21, 22, 23] studied the global-in-time existence of the solution to such models. For example, if (2) is replaced by

$$-\Delta v + v = u \quad \text{in } \Omega\times(0,T), \quad \frac{\partial v}{\partial\nu}\Big|_{\partial\Omega} = 0, \tag{5}$$

the global-in-time solution exists and is uniformly bounded, provided that

$$\alpha f(u) \leq a - \mu u^2, \quad u\geq 0 \tag{6}$$

with $a>0$, $\mu>0$, and $\mu>(1-\frac{2}{n})\chi$ (see [20]). If $\mu=(1-\frac{2}{n})\chi$, the solution to (1) with (5) for (6) is still global-in-time, but may not be uniformly bounded (see the proof of Theorem 3 of [9], dealing with the case of $\alpha f(u)=ru-\mu u^2$ for $r\geq 0$). More recently, M.A.J. Chaplain and J.I. Tello in [4] studied a model where (2) is replaced by (5) but instead of the linear reaction term v they have a more general term $h(v)$ which is a locally Lipschitz function and its derivative is bounded. They proved the existence of a unique solution and the asymptotic convergence to the steady states. Also recently, X. He and S. Zheng in [7] proved for the parabolic-elliptic model (that is (1) coupled with (5)), under some assumptions, that the positive constant equilibrium is a global attractor. They also proved for the parabolic-parabolic system, again under some assumptions on μ and a, that it admits the non-trivial positive constant equilibrium as a global attractor.

Concerning the system composed of (1) and

$$\tau v_t - \Delta v + v = u, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0(x), \tag{7}$$

furthermore, any $a \geq 0$ admits $\mu_0 > 0$ such that the solution exists global-in-time and is bounded if $\mu \geq \mu_0$ and Ω is convex, where $\tau > 0$ is a constant (see [21]). On the other hand, parabolic-parabolic models with general sources and general sensitivity functions are studied by [16, 17]. Namely, in the case of $n \leq 2$, the global-in-time solution to (1) with (7) exists if $f(0) = 0$ and $f(u) = (-\mu u + \beta)u$, $u \gg 1$, for some constants $\mu, \beta > 0$. Here, the sensitivity function $\chi = \chi(u)$ can be nonlinear, as far as its derivatives up to the third order are bounded. This solution, however, may not be uniformly bounded as $t \uparrow +\infty$. Recently, Lin and Mu in [12] also considered the parabolic-parabolic system with homogeneous Neumann conditions and for the dimensions $N = 2, 3$, the proved the global existence and boundedness of classical solutions, provided that μ and a satisfy some explicit conditions.

It seems that the existence of any uniformly bounded global-in-time solution has not yet been confirmed to (1)-(2) for the above profile of $f(u)$, that is, the Fisher type non-linearity. On the contrary, we have radially symmetric blowup solutions in the case that $f(u) = \lambda u - \mu u^\kappa$ with $\lambda \geq 0$, $\mu > 0$, $1 < \kappa < \frac{3}{2} + \frac{1}{2n-2}$, and $n \geq 5$ (see [22]).

Main results

In this paper, we are mostly concerned with the quadratic dissipation $f(u) = -u^2$ and the logistic source $f(u) = u - u^2$. For these nonlinearities we can show that the solution is global-in-time and uniformly bounded if either $n \leq 2$ or $\alpha > \chi$. Both parabolic-elliptic and parabolic-parabolic chemotaxis systems with the above reaction terms were studied. The main novelty of this work is the proof of the global existence and the study of the asymptotic behavior of the solution which is clarified in several cases.

Quadratic dissipation

First, for the quadratic dissipation, this system takes the form

$$\begin{aligned} &u_t = d\Delta u - \chi\nabla\cdot u\nabla v - \alpha u^2, \quad -\Delta v = u - \overline{u}, \quad \int_\Omega v = 0 \quad \text{in } \Omega\times(0,T), \\ &\left.\frac{\partial}{\partial\nu}(u,v)\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0, \end{aligned} \tag{8}$$

which implies

$$u_t = d\Delta u - \chi \nabla u \cdot \nabla v - \chi \overline{u} u + (\chi - \alpha) u^2, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0. \tag{9}$$

Then the standard comparison theorem guarantees

$$U_-(t) \le u(x,t) \le U_+(t) \tag{10}$$

where $U = U_\pm(t)$ are spatially homogeneous solutions to (9), that is,

$$\frac{dU}{dt} = -\chi \overline{u}(t) U + (\chi - \alpha) U^2, \tag{11}$$

satisfying $U_-|_{t=0} = \min_{\overline{\Omega}} u_0$ and $U_+|_{t=0} = \max_{\overline{\Omega}} u_0$. If $\delta = \alpha - \chi \ge 0$ is the case, therefore, we have $T = +\infty$. This solution satisfies

$$\|u(\cdot,t)\|_\infty \le (\|u_0\|_\infty^{-1} + \delta t)^{-1}. \tag{12}$$

In fact we obtain

$$\frac{dU_+}{dt} \le -\delta U_+^2, \quad U_+|_{t=0} = \|u_0\|_\infty$$

by (11), and then (12) follows from $U_+(t) \le (\|u_0\|_\infty^{-1} + \delta t)^{-1}$.

The first theorem, however, assures global-in-time existence of the solution and its uniform convergence to 0 as $t \uparrow +\infty$ for any parameter region of $d, \chi, \alpha > 0$, in the case of $n \le 2$.

Theorem 1. *If $n \le 2$, it holds that $T = +\infty$ and $\lim_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = 0$ in (8).*

From the proof, we see that the comparison function $U_-(t)$ in (10) must remain bounded for any dimension n (see Remark 2 below). Actually we do not expect blowup of the solution to (8) even if $n \ge 3$.

Theorem 1 is valid to the other models, that is (1) coupled with either (3) or (4) (see Remark 1 and Theorem 5 below). Even in higher space dimensions, the condition $\delta = \alpha - \chi > 0$ is not essential for $T = +\infty$ and $\lim_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = 0$ to hold. The second theorem is an extension of $\alpha > \chi$ for $T = +\infty$ or $\lim_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = 0$ to hold. This theorem also contains Theorem 1 concerning $n \le 2$.

Theorem 2. *The same conclusion as in Theorem 1 holds to (8) even for $n \ge 3$, provided that $\alpha > (1 - \frac{2}{n})\chi$.*

An analogous result to Theorem 2 is known for (1) with (5) (see Theorem 2.5 of [20]). It is also valid to (1) with either (2) or (3) (see Remark 3 below).

Logistic source

If $f(u)$ stands for the logistic source in (1)-(2), we have

$$u_t = d\Delta u - \chi \nabla \cdot u \nabla v + \alpha(u - u^2), \quad -\Delta v = u - \overline{u}, \quad \overline{v} = 0 \quad \text{in } \Omega \times (0,T),$$
$$\left.\frac{\partial}{\partial \nu}(u,v)\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0, \tag{13}$$

which implies

$$u_t = d\Delta u - \chi\nabla u\cdot\nabla v - \chi\overline{u}u + \alpha u + (\chi-\alpha)u^2, \quad \left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0. \tag{14}$$

From the classical comparison theorem, therefore, we have $0 < u(x,t) \leq U(t)$ for $U = U(t)$ satisfying

$$\frac{dU}{dt} = \alpha U + (\chi-\alpha)U^2, \quad U|_{t=0} = \|u_0\|_\infty, \tag{15}$$

and therefore, $\alpha > \chi$ implies

$$T = +\infty, \quad \|u(\cdot,t)\|_\infty \leq \max\{1-\frac{\chi}{\alpha}, \|u_0\|_\infty\}$$

similarly to (12).

The spatially homogeneous part of (13),

$$\frac{du}{dt} = \alpha(u-u^2), \tag{16}$$

is nothing but the logistic ODE. It is well-known that any positive solution to (16) converges to 1 as $t \uparrow +\infty$. The following theorems provide with some criteria for this property to (13). However, we cannot apply any comparison theorems between (16) and (1) with (2) (or (1) with any one of (3), (4), and (5)).

The first theorem on (13), Theorem 3 below, is concerned with the case $\alpha > \chi$. As we have seen above, in this case any solution to (13) is always global-in-time and uniformly bounded. The point in this theorem, therefore, is its uniform convergence to 1. Henceforth, C_i, $i = 1, 2, \cdots, 40$, denote positive constants independent of t.

Theorem 3. *If $\alpha > \chi$, it holds that*

$$\|u(\cdot,t) - 1\|_\infty \leq C_1 e^{-\delta t} \tag{17}$$

in (13), where $\delta > 0$ is a constant.

The proof of Theorem 3 is valid to (1) with (3) or (5) (see Remark 4 below). In particular, the criterion $\alpha > 2\chi$ for $\lim_{t\uparrow+\infty}\|u(\cdot,t)-1\|_\infty = 0$ to occur in (1) with (5) is relaxed as $\alpha > \chi$ (see Theorem 5.1 of [20] for a proof of the former case).

Theorem 4 below, may be compared with Theorem 2 for the quadratic dissipation case.

Theorem 4. *If $\alpha > (1-\frac{2}{n})\chi$, it holds that $T = +\infty$ and $\|u(\cdot,t)\|_\infty \leq C_2$ in (13). If $n \leq 2$, there is $d_0 = d_0(\chi,\alpha) > 0$, such that $d > d_0$ implies (17) with $\delta > 0$.*

Concerning (1) with (4), if $n = 1$ we have the same property. Thus, the solution is global-in-time. If $d \gg 1$, furthermore, then (17) holds (see Theorem 6 below). The above Theorems 3 and 4, however, do not cover all the cases, and there may be blowup of the solution to (13), if either $\chi > \alpha$, or $n \geq 3$, or $0 < d \ll 1$.

This paper is composed of four sections. Section 2 is focused on problem (8) and Theorems 1 and 2 are proven. Section 3 deals with problem (13). Theorems 3 and 4 are proven there. The last section, §4, is devoted to a remark on the blowup or quenching of the solution. In the following we use

$$\|f\|_q = \left(\frac{1}{|\Omega|} \int_\Omega |f|^q \right)^{1/q}, \quad 1 \le q < \infty$$

and

$$(f,g) = \frac{1}{|\Omega|} \int_\Omega fg.$$

2 Quadratic Dissipation Case

This section is devoted to the case $f(u) = -u^2$. First, we show the following proof.

Proof of Theorem 1. We multiply the first equation of (1) with $\log u$, integrate over Ω and use Green's identity to derive:

$$\begin{aligned} &\frac{d}{dt}\frac{1}{|\Omega|}\int_\Omega u(\log u - 1)\ dx + 4d\|\nabla u^{1/2}\|_2^2 + \alpha(u^2, \log u) = \chi(\nabla u, \nabla v) \\ &\quad = \chi(-\Delta v, u) = \chi\left(\|u\|_2^2 - \overline{u}^2\right). \end{aligned} \tag{18}$$

By integrating the first equation of (1) over Ω, on the other hand, we have

$$\frac{d\overline{u}}{dt} = -\alpha\|u\|_2^2 \le -\alpha\overline{u}^2$$

and hence

$$\overline{u}(t) \le (\overline{u_0}^{-1} + \alpha t)^{-1}, \qquad \int_0^T \|u(\cdot,t)\|_2^2 dt \le \alpha^{-1}\overline{u_0}. \tag{19}$$

Noticing

$$\alpha u^2 \log u \ge u(\log u - 1) - C_3, \quad \forall u > 0, \tag{20}$$

we put

$$H(t) = \frac{1}{|\Omega|}\int_\Omega u(\log u - 1) - C_3\ dx.$$

Then it holds that

$$\frac{dH}{dt} + H \le \|u\|_2^2,$$

which implies

$$H(t) \le e^{-t}H(0) + \int_0^t e^{-(t-s)}\|u(\cdot,s)\|_2^2 ds \le e^{-t}H(0) + \int_0^t \|u(\cdot,s)\|_2^2 ds \le C_4.$$

Thus we obtain

$$\int_\Omega u(\log u - 1) + 1\ dx \le C_5. \tag{21}$$

By (21) we can argue similarly to the chemotaxis systems without source terms [2, 5, 15] (see also Chapter 4 of [18]). Below we describe the proof for $n = 2$, because the case $n = 1$ is simpler.

First, we multiply u^p, $p>0$, to (8) and obtain

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1}+\frac{4pd}{(p+1)^2}\|\nabla u^{\frac{p+1}{2}}\|_2^2 &\leq \frac{p\chi}{p+1}\frac{1}{|\Omega|}\int_\Omega \nabla v\cdot\nabla u^{p+1}\\ =\frac{p\chi}{p+1}\frac{1}{|\Omega|}\int_\Omega(-\Delta v)\cdot u^{p+1} &\leq \frac{p\chi}{p+1}\|u\|_{p+2}^{p+2}. \end{aligned}\tag{22}$$

For $p=1$, this inequality means

$$\frac{d}{dt}\|u\|_2^2+2d\|\nabla u\|_2^2\leq \chi\|u\|_3^3. \tag{23}$$

Taking $s>1$, we apply the other form of the Gagliardo-Nirenberg inequality

$$\|z\|_p\leq C_6(p,q,\Omega)\|z\|_q^{1-a}\|z\|_{H^1(\Omega)}^a, \quad 1\leq q\leq p<\infty,\ a=1-\frac{q}{p} \tag{24}$$

for $z=\chi_{u>s}u$ and $q=1$, $p=3$, where $\chi_{u>s}$ denotes the indicator of the set $\{x\in\Omega \mid u(x)>s\}$. Thus, it holds that

$$\begin{aligned}\|u\|_3^3 &= \|\chi_{u>s}u\|_3^3+\|\chi_{u\leq s}u\|_3^3\leq C_7\|\chi_{u>s}u\|_{H^1(\Omega)}^2\|\chi_{u>s}u\|_1+s^3|\Omega|\\ &\leq \frac{C_7}{\log s}\|u\|_{H^1(\Omega)}^2\|u\log u\|_1+s^3|\Omega|.\end{aligned}\tag{25}$$

We also use Poincaré-Wirtinger's inequality

$$\mu_2\|z-\overline{z}\|_2^2\leq\|\nabla z\|_2^2 \tag{26}$$

to deduce

$$\begin{aligned}\|\chi_{u>s}u\|_{H^1(\Omega)}^2 &\leq \|u\|_{H^1(\Omega)}^2=\|\nabla u\|_2^2+\|u\|_2^2\leq\|\nabla u\|_2^2+2(\|u-\overline{u}\|_2^2+\|\overline{u}\|_2^2)\\ &\leq \left(1+\frac{2}{\mu_2}\right)\|\nabla u\|_2^2+\|u\|_1^2.\end{aligned}\tag{27}$$

Here, $\mu_2>0$ denotes the first positive eigenvalue of $-\Delta$ on Ω provided with the Neumann boundary condition.

By (25)-(27), it holds that

$$\|u\|_3^3\leq C_8\left(\|\nabla u\|_2^2+\|u\|_1^2\right)\frac{1}{\log s}\|u\log u\|_1+s^3|\Omega|.$$

Using (19) and (21), therefore, we obtain

$$\frac{d}{dt}\|u\|_2^2+d\|\nabla u\|_2^2\leq C_9$$

with $s\gg 1$, and then it follows that

$$\frac{d}{dt}\|u\|_2^2+\mu_2 d\|u\|_2^2\leq C_{10}$$

from (26) and (19). Thus we obtain

$$\|u\|_2 \leq C_{11} \tag{28}$$

Inequality (28) implies $\|w\|_{4/3} \leq C_{12}$ for $w = u^{3/2}$, and then we can argue similarly, using (22) for $p = 2$. Here, we use $\|u\|_3 \leq \|w\|_3$ in (34), and then it follows that

$$\|u\|_3 \leq C_{13}. \tag{29}$$

Inequality (29) implies $\|v\|_{W^{2,3}(\Omega)} \leq C_{14}$ by the elliptic estimate for the Poisson equation (2), and then

$$\|\nabla v\|_\infty \leq C_{15} \tag{30}$$

by Morrey's inequality, which reduces (22) to

$$\frac{1}{p+1}\frac{d}{dt}\|u^{\frac{p+1}{2}}\|_2^2 + \frac{4pd}{(p+1)^2}\|\nabla u^{\frac{p+1}{2}}\|_2^2 \leq \frac{2p\chi C_{16}}{p+1}\|u^{\frac{p+1}{2}}\|_2\|\nabla u^{\frac{p+1}{2}}\|_2 \tag{31}$$

by $\nabla u^{p+1} = 2u^{\frac{p+1}{2}}\nabla u^{\frac{p+1}{2}}$. From (31) it follows that

$$\frac{d}{dt}\|u^{\frac{p+1}{2}}\|_2^2 + 2d\|\nabla u^{\frac{p+1}{2}}\|_2^2 \leq C_{17}p^2\|u^{\frac{p+1}{2}}\|_2^2, \quad p \geq 1.$$

Then we obtain $\|u(\cdot,t)\|_\infty \leq C_{18}$ with $T = +\infty$ by the iteration scheme (see [1]).

Therefore, the orbit $\mathcal{O} = \{u(\cdot,t)\}_{t\geq 0}$ is compact in $C(\overline{\Omega})$, and hence $\lim_{t\uparrow+\infty}\|u(\cdot,t)\|_\infty = 0$ follows from $\lim_{t\uparrow+\infty}\overline{u}(t) = 0$. □

Remark 1. Even in (1) with (3), inequality (22) is valid by $\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} \leq 0$. Then the same conclusion arises as in Theorem 1 for this system. In the Debye system [3] (or DD (drift-diffusion) model [8]), the second equation (3) is replaced by

$$\Delta v = u, \quad v|_{\partial\Omega} = 0. \tag{32}$$

In this case the above calculation is invalid because of $\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} \geq 0$. Here we use

$$\frac{1}{|\Omega|}\int_\Omega \nabla v \cdot \nabla u^{p+1} = \frac{2}{|\Omega|}\int_\Omega u^{\frac{p+1}{2}}\nabla u^{\frac{p+1}{2}} \cdot \nabla v \leq 2\|\nabla u^{\frac{p+1}{2}}\|_2\|u^{\frac{p+1}{2}}\|_3\|\nabla v\|_6$$

in (22). Regarding $n = 2$, we apply the Gagliardo-Nirenberg inequality in the form of

$$\|\nabla v\|_6 \leq C_{19}\|\nabla v\|_{W^{1,3}(\Omega)}^{1/2}\|\nabla v\|_2^{1/2}, \tag{33}$$

and also the standard L^3 estimate on the Poisson equation (2),

$$\|v\|_{W^{2,3}(\Omega)} \leq C_{20}\|u\|_3. \tag{34}$$

Then we use the elliptic estimate to (2) in the form of

$$\|\nabla v\|_2 \leq C_{21}\,\|u\|_{L\log L} \tag{35}$$

valid to $n = 2$, where $\| \cdot \|_{L \log L}$ denotes the Zygmund norm (see [19]). This inequality implies $\|\nabla v\|_2 \leq C_{22}$ by (21), and hence

$$\frac{d}{dt}\|u\|_2^2 + d\|\nabla u\|_2^2 \leq C_{23}\|u\|_3^3$$

for $p = 1$. Then we can argue similarly, although inequalities (33) and (35) are restricted to $n \leq 2$. Thus, if $n \leq 2$ and $f(u) = -u^2$ any solution to (1) with (32) is global-in-time, and converges to 0 uniformly as $t \uparrow +\infty$.

Remark 2. Inequality (21) holds in any space dimension and for any parameters $d, \chi, \alpha > 0$. Therefore, it holds that $U_-(t) \leq C_{24}$ in (10).

Now we show the following proof.

Proof of Theorem 2. We may assume $n \geq 3$, regarding Theorem 1. Taking into account the dissipative term, inequality (22) becomes

$$\begin{aligned} \frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1} + \frac{4pd}{(p+1)^2}\|\nabla u^{\frac{p+1}{2}}\|_2^2 + \alpha\|u\|_{p+2}^{p+2} &= \frac{p\chi}{p+1}\frac{1}{|\Omega|}\int_\Omega \nabla v \cdot \nabla u^{p+1} \\ = \frac{p\chi}{p+1}(-\Delta v, u^{p+1}) \leq \frac{p\chi}{p+1}\|u\|_{p+2}^{p+2} & \end{aligned} \tag{36}$$

by $-\Delta v = u - \overline{u}$. From the assumption, there is $p > \frac{n}{2} - 1$ such that

$$\delta \equiv (p+1)\left(\alpha - \frac{p\chi}{p+1}\right) > 0.$$

Hence it follows that

$$\frac{dX}{dt} + \delta X^{1+\gamma} \leq 0$$

for $X = \|u\|_{p+1}^{p+1}$ and $\gamma = \frac{1}{p+1}$, which implies

$$X^\gamma(t) = \|u(\cdot, t)\|_{p+1} \leq [X^{-\gamma}(0) + \delta t]^{-1} = (\|u_0\|_{p+1}^{-1} + \delta t)^{-1}.$$

Now we use the Gagliardo-Nirenberg inequality in the form of

$$\|w\|_3^3 \leq C_{25}(\|\nabla w\|_2^2 + \|w\|_{n/2}^2)\|w\|_{n/2} \tag{37}$$

and then obtain (28) similarly. Continuing the process as in the proof of the previous theorem, we reach (30) and then the conclusion follows. □

Remark 3. The above proof of Theorem 2 is valid also to (1) with (5), that is, the model studied by [20]. Here, the second equation (5) may be replaced by (2) or (3). In fact, assuming (6), we put $a = \mu = 1$ for simplicity. Then it holds that

$$\frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1} + \frac{4pd}{(p+1)^2}\|\nabla u^{\frac{p+1}{2}}\|_2^2 + \alpha\|u\|_{p+2}^{p+2} \leq \frac{p\chi}{p+1}\|u\|_{p+2}^{p+2} + \alpha\|u\|_p^p \tag{38}$$

for (36). Assuming $\alpha > (1 - \frac{2}{n})\chi$ also, we take $p > \frac{n}{2} - 1$ and $\varepsilon > 0$ such that

$$\delta \equiv (p+1)\left(\alpha - \frac{p\chi}{p+1} - \frac{\alpha p \varepsilon}{p+2}\right) > 0.$$

Then Young's inequality implies

$$\|u\|_p^p \le \|u\|_{p+2}^p \le \frac{p}{p+2} \cdot \varepsilon \|u\|_{p+2}^{p+2} + \frac{2}{p+2} \cdot \varepsilon^{-p(p+2)/2}$$
$$\|u\|_{p+1}^{p+1} \le \|u\|_{p+2}^{p+1} \le \frac{p+1}{p+2}\|u\|_{p+2}^{p+2} + \frac{1}{p+2}$$

and hence

$$\frac{dX}{dt} + \frac{p+2}{p+1}\delta X \le C_{26} \tag{39}$$

for $X = \|u\|_{p+1}^{p+1}$. Inequality (39) implies $\|u\|_{p+1} \le C_{27}$, $p+1 > \frac{n}{2}$. Then we obtain the result similarly to the proof of Theorem 1, using (37).

The proof of Theorem 2, using the quadratic dissipation, provides with an alternative proof of Theorem 1. The original proof of Theorem 1, however, is applicable to the parabolic-parabolic system. We conclude this section with the following theorem.

Theorem 5. *If $n \le 2$, it holds that $T = +\infty$ and $\lim_{t\uparrow+\infty} \|u(\cdot,t)\|_\infty = 0$ in (1) with (4).*

Proof. Similarly to the previous proof, we multiply the first equation of (1) with $\log u$, integrate over Ω and use Green's identity. It holds that

$$\begin{aligned} &\frac{d}{dt}\frac{1}{|\Omega|}\int_\Omega u(\log u - 1) + 4d\|\nabla u^{1/2}\|_2^2 + \alpha(u^2, \log u) = \chi(\nabla u, \nabla v) \\ &\quad = \chi(-\Delta v, u) = \chi(-\Delta v, u - \overline{u}) = \chi(-\Delta v, \tau v_t - \Delta v) = \frac{\tau\chi}{2}\frac{d}{dt}\|\nabla v\|_2^2 + \chi\|\Delta v\|_2^2 \\ &\quad \le \chi\|\Delta v\|_2 \|u\|_2 \le \frac{\chi}{2}\|\Delta v\|_2^2 + \frac{\chi}{2}\|u\|_2^2 \end{aligned} \tag{40}$$

by $(\Delta v, \overline{u}) = \overline{u} \cdot (\Delta v, 1) = 0$. Thus we obtain

$$\tau\frac{d}{dt}\|\nabla v\|_2^2 + \|\Delta v\|_2^2 \le \|u\|_2^2,$$

or

$$(-\Delta v, u) = \frac{\tau}{2}\frac{d}{dt}\|\nabla v\|_2^2 + \|\Delta v\|_2^2 \le \|u\|_2^2 - \frac{\tau}{2}\frac{d}{dt}\|\nabla v\|_2^2.$$

Therefore, (40) becomes,

$$\frac{d}{dt}\frac{1}{|\Omega|}\int_\Omega u(\log u - 1) + \frac{\tau\chi}{2}|\nabla v|^2 \ dx + 4d\|\nabla u^{1/2}\|_2^2 + \alpha(u^2, \log u) \le \chi\|u\|_2^2.$$

By (20) it holds that

$$\frac{dH}{dt} + H \le \|u\|_2^2 + \frac{\tau\chi}{2}\|\nabla v\|_2^2, \tag{41}$$

where

$$H = \frac{1}{|\Omega|}\int_\Omega u(\log u - 1) - C_{28} + \frac{\tau\chi}{2}|\nabla v|^2 \ dx. \tag{42}$$

Equation (4) implies

$$\frac{d\overline{v}}{dt} = 0, \tag{43}$$

while it follows $(\overline{u}, v) = (u, \overline{v})$ and hence

$$\frac{\tau}{2}\frac{d}{dt}\|v\|_2^2 + \|\nabla v\|_2^2 = (u - \overline{u}, v) = (u, v - \overline{v}) \leq \mu_2^{-1}\|\nabla v\|_2\|u\|_2 \leq \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2\mu_2^2}\|u\|_2^2$$

from (26). Since inequality (19) readily holds, we obtain

$$\int_0^T \|u\|_2^2 + \|\nabla v\|_2^2 \, dt \leq C_{29}, \tag{44}$$

which implies

$$H(t) \leq C_{30} \tag{45}$$

by (41).

We have also

$$\tau\|v_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 = (u - \overline{u}, v_t) = (u, v_t) \leq \frac{\tau}{2}\|v_t\|_2^2 + \frac{1}{2\tau}\|u\|_2^2$$

because (43) implies $(\overline{u}, v_t) = \overline{u} \cdot (1, v_t) = 0$, and hence

$$\int_0^T \|v_t\|_2^2 \, dt \leq C_{31}. \tag{46}$$

Since $n = 2$, these inequalities (44), (45) with (42), and (46) imply $T = +\infty$ with

$$\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty \leq C_{32} \tag{47}$$

by Gagliardo-Nirenberg's inequality, Moser's iteration scheme, and the semi-group estimate. The proof is similar to the case without the dissipative term (see [15]) and is omitted.

Inequality (47) implies the compactness of the orbit $\{(u(\cdot, t), v(\cdot, t)\}_{t\geq 0}$ in $C(\overline{\Omega})^2$, and then $\lim_{t\uparrow+\infty} \|u(\cdot, t)\|_\infty = 0$ follows similarly to Theorem 1. □

3 Logistic Source Case

This section is devoted to the logistic source, $f(u) = u - u^2$. First, we show the following proof.

Proof of Theorem 3. We have readily shown the existence of uniformly bounded, global-in-time solutions in the case of $\alpha > \chi$. Since

$$\frac{d\overline{u}}{dt} \leq \alpha(\overline{u} - \overline{u}^2),$$

it holds that $\limsup_{t\uparrow+\infty} \overline{u}(t) \leq 1$. Therefore, we may assume $\alpha > \chi \cdot c_0$ for $c_0 = \max\{1, \overline{u_0}\}$, in the case of $\alpha > \chi$.

We have also $\overline{u}(t) \leq \max\{1, \overline{u_0}\} = c_0$, which implies

$$u_t \geq d\Delta u - \chi \nabla u \cdot \nabla v - \chi c_0 u + \alpha u + (\chi - \alpha) u^2, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0$$

by (14). Therefore, it holds that

$$u(x,t) \geq U_-(t)$$

for $U = U_-(t)$ satisfying

$$\frac{dU}{dt} = (\alpha - c_0\chi)U + (\chi - \alpha)U^2, \quad U|_{t=0} = \min_{\overline{\Omega}} u_0. \tag{48}$$

Using the assumption $\alpha > c_0\chi$, therefore, we have $U_-(t) \geq \delta$, and hence

$$T = +\infty, \quad \delta \leq u(x,t) \leq C_{33} \tag{49}$$

for $\delta = \min\{\frac{\alpha - c_0\chi}{\alpha - \chi}, \min_{\overline{\Omega}} u_0\} > 0$ and $C_{33} > 0$ independent of (x,t).

Let $w = u - 1 \geq -1$. Since

$$-\Delta v = w - \overline{w}, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v = 0, \tag{50}$$

it follows that

$$\begin{aligned} w_t &= d\Delta w - \chi \nabla \cdot (w+1)\nabla v + \alpha((w+1) - (w+1)^2) \\ &= d\Delta w - \chi\nabla \cdot w\nabla v + (\chi - \alpha) w - \chi\overline{w} - \alpha w^2, \quad \left.\frac{\partial}{\partial \nu}(w,v)\right|_{\partial\Omega} = 0. \end{aligned} \tag{51}$$

Letting $w_\pm = \max\{w, 0\}$, we multiply $w_\pm^p$, $p > 0$, to (51). Then it follows that

$$\begin{aligned} &\frac{1}{p+1}\frac{d}{dt}\|w_\pm\|_{p+1}^{p+1} + \frac{4pd}{(p+1)^2}\|\nabla w_\pm^{\frac{p+1}{2}}\|_2^2 + \left(\alpha - \frac{\chi p}{p+1}\right)\|w_\pm\|_{p+2}^{p+2} + (\alpha - \chi)\|w_\pm\|_{p+1}^{p+1} \\ &+\overline{w}\left(\frac{\chi p}{p+1}\|w_\pm\|_{p+1}^{p+1} + \|w_\pm\|_p^p\right) = 0 \end{aligned}$$

by

$$-(\nabla \cdot w\nabla v, w_\pm) = (w\nabla v, \nabla w_\pm) = \frac{1}{2}(\nabla v, \nabla w_\pm^2) = \frac{1}{2}(w - \overline{w}, w_\pm^2) = \frac{1}{2}(\|w_\pm\|_3^3 - \overline{w}\|w_\pm\|_2^2).$$

Therefore, it holds that

$$\begin{aligned} &\frac{1}{p+1}\frac{d}{dt}\|w\|_{p+1}^{p+1} + \left(\alpha - \frac{\chi p}{p+1}\right)\|w\|_{p+2}^{p+2} + \left(\alpha - \chi + \frac{\chi p \overline{w}}{p+1}\right)\|w\|_{p+1}^{p+1} \\ &+\overline{w}\|w\|_p^p \leq 0. \end{aligned} \tag{52}$$

Now we show

$$\lim_{t\uparrow+\infty} \overline{w}(t) = 0. \tag{53}$$

In fact, first, we have

$$\frac{d\overline{w}}{dt} + \alpha\overline{w} = -\alpha\|w\|_2^2 \leq 0$$

and hence

$$\limsup_{t\uparrow+\infty} \overline{w}(t) \leq 0. \tag{54}$$

Then we write (51) as

$$w_t = d\Delta w - \chi\nabla v \cdot \nabla w - (w+1)\left((\alpha-\chi)w + \chi\overline{w}\right), \quad \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega} = 0,$$

using (50).

Since $\delta \leq w+1 \leq C_{33}$ and (54), any $\varepsilon > 0$ admits $T > 0$ and $W = W(t)$, $t \geq T$, such that

$$w(\cdot,t) \geq W(t) \geq \delta - 1, \quad t \geq T$$

and

$$\frac{dW}{dt} = -(\alpha-\chi)W(W+1) - \varepsilon.$$

Recalling $\alpha > \chi$, we obtain

$$\liminf_{t\uparrow+\infty} \overline{w}(t) \geq a(\varepsilon)$$

for $0 < \varepsilon \ll 1$, where $W = a(\varepsilon)$ denotes the larger zero of

$$(\alpha-\chi)W(W+1) = -\varepsilon.$$

It follows that

$$\liminf_{t\uparrow+\infty} \overline{w}(t) \geq 0 \tag{55}$$

with $\varepsilon \downarrow 0$, and then inequalities (54) and (55) imply (53).

Here we put $X = \|w\|_{p+1}^{p+1}$, $\gamma = \frac{1}{p+1}$, and $\delta = \frac{\alpha-\chi}{2} > 0$, noting $\|w\|_p^p \leq \|w\|_{p+1}^p$. By (52)-(53), any $\varepsilon > 0$ admits $T > 0$ such that

$$\gamma\frac{dX}{dt} + \delta X \leq \varepsilon X^{1-\gamma}, \quad t \geq T \tag{56}$$

for any $p \geq 1$. Inequality (56) means

$$\frac{dY}{dt} + \varepsilon Y^2 \geq \delta Y, \quad t \geq T$$

or

$$\frac{dY^{-1}}{dt} + \delta Y^{-1} \leq \varepsilon, \quad t \geq T$$

for $Y = X^{-\gamma} = \|w\|_{p+1}^{-1}$, which implies

$$Y^{-1}(t) \leq e^{-\delta(t-T)}Y^{-1}(T) + \frac{\varepsilon}{\delta}\left(1 - e^{-\delta(t-T)}\right).$$

Thus we end up with

$$\|w(\cdot,t)\|_{p+1} \leq e^{-\delta(t-T)}\|w(\cdot,T)\|_{p+1} + \frac{\varepsilon}{\delta}\left(1-e^{-\delta(t-T)}\right), \quad t \geq T.$$

Sending $p \uparrow +\infty$, $t \uparrow +\infty$, and $\varepsilon \downarrow 0$, we obtain $\limsup_{t\uparrow+\infty} \|w(\cdot,t)\|_\infty \leq 0$, and hence

$$\lim_{t\uparrow+\infty} \|u(\cdot,t)-1\|_\infty = 0. \tag{57}$$

Once (57) is established, the linearization theory assures the exponential decay. Here we rewrite (51) as

$$w_t = d\Delta w - (\alpha-\chi)w - \chi\overline{w} - F(w), \quad \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega} = 0 \tag{58}$$

for $F(w) = \chi\nabla\cdot w\nabla(-\Delta)^{-1}w + \alpha w^2$, where $v = (-\Delta)^{-1}w$ denotes that v is the solution to (50). Its linear part takes the form

$$w_t = d\Delta w - (\alpha-\chi)w - \chi\overline{w}, \quad \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega} = 0, \quad w|_{t=0} = w_0 \tag{59}$$

and the solution to (59) is written as $w(\cdot,t) = e^{tL}w_0$, where L is a self-adjoint operator in $L^2(\Omega)$ with compact resolvent. Since $|\overline{w}(t)| \leq |\overline{w_0}|e^{-\alpha t}$ holds in (59) by

$$\frac{d\overline{w}}{dt} = -\alpha\overline{w}, \quad \overline{w}|_{t=0} = \overline{w_0}$$

the operator L is negative definite.

From the elliptic and parabolic regularity [6, 11], the convergence (57) implies that of $w(\cdot,t) \to 0$ in C^∞ topology. Then we have

$$F(w) = \nabla(-\Delta)^{-1}w\cdot\nabla w + (\alpha-\chi)w^2$$

which implies

$$w_t = L(t)w, \quad \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega} = 0$$

for $L(t) = L + a(x,t)\cdot\nabla + (\alpha-\chi)w(\cdot,t)$, where $a(\cdot,t) = \nabla(-\Delta)^{-1}w(\cdot,t)$ and $w(\cdot,t)$ converge to zero in C^∞ topology as $t \uparrow +\infty$. Hence there is $\delta > 0$ such that $\|w(\cdot,t)\|_\infty \leq C_{34}e^{-\delta t}$, or (17), from the perturbation theory of spectrum of operators and the theory of evolution equations (see [10, 24]). □

Remark 4. The above proof is valid even to (1) with either (3) or (5). Hence we obtain (17), provided that $\alpha > \chi$.

If $\chi > \alpha$, conversely, blowup of the solution $U = U(t)$ occurs in (48), if

$$U_0 > \frac{c_0\chi - \alpha}{\chi - \alpha}. \tag{60}$$

Any $u_0 = u_0(x) > 0$, however, does not satisfy (60) for $U_0 = \min_{\overline{\Omega}} u_0$ and $c_0 = \max\{1, \overline{u_0}\}$. In fact we have the following proof.

Proof of Theorem 4. In (13), equality (38) is replaced by

$$\frac{1}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1}+\frac{4pd}{(p+1)^2}\|\nabla u^{\frac{p+1}{2}}\|_2^2+\alpha\|u\|_{p+2}^{p+2}\leq\frac{p\chi}{p+1}\|u\|_{p+2}^{p+2}+\alpha\|u\|_{p+1}^{p+1}. \tag{61}$$

We use

$$\|u\|_{p+1}^{p+1}\leq\|u\|_{p+2}^{p+1}\frac{p+1}{p+2}\cdot\varepsilon\cdot\|u\|_{p+2}^{p+2}+\frac{\varepsilon^{-(p+1)(p+2)}}{p+2}$$

for $p>\frac{n}{2}-1$, $\varepsilon>0$ satisfying

$$\alpha-\frac{p\chi}{p+1}-\frac{p+1}{p+2}\alpha\varepsilon>0$$

to obtain $\|u(\cdot,t)\|_{p+1}\leq C_{35}$. Then $T=+\infty$ with $\|u(\cdot,t)\|_\infty\leq C_{36}$ follows as in Remark 3.

We turn to the asymptotic behavior of the solution. We assure $d_0=d_0(\chi,\alpha)>0$ independent of the initial value such that $0<d<d_0$ implies (17), assuming $n\leq 2$.

Since

$$\frac{d\overline{u}}{dt}=\alpha(\overline{u}-\|u\|_2^2)=\alpha(\overline{u}-\overline{u}^2-\|u-\overline{u}\|_2^2)\leq\alpha(\overline{u}-\overline{u}^2) \tag{62}$$

two cases arise, that is, $\overline{u}(t)\geq 1$ for $t\in[0,T)$ and $\overline{u}(t)<1$ for $t\in(t_1,T)$ with some $t_1\in[0,T)$. In the first case, it holds that

$$\lim_{t\uparrow+\infty}\overline{u}(t)=1 \tag{63}$$

and then, we obtain (57) by the compactness of the orbit. This property implies (17) similarly to the previous theorem.

In the second case of $\overline{u}(t)<1$, $t\in(t_1,T)$, we use $w=u-\overline{u}$ to get

$$-\Delta v=w,\quad \left.\frac{\partial}{\partial\nu}(v,w)\right|_{\partial\Omega}=0$$

and

$$\frac{d\overline{u}}{dt}=\alpha(\overline{u}-\overline{u}^2-\|w\|_2^2)$$

by (62). Then it follows that

$$\begin{aligned} w_t &= u_t-\frac{d\overline{u}}{dt}=d\Delta u-\chi\nabla\cdot u\nabla v+\alpha(u-u^2)-\alpha(\overline{u}-\overline{u}^2)+\alpha\|w\|_2^2 \\ &= d\Delta w-\chi\nabla\cdot(w+\overline{u})\nabla v+\alpha[(w+\overline{u})-(w+\overline{u})^2]-\alpha(\overline{u}-\overline{u}^2)+\alpha\|w\|_2^2 \\ &= d\Delta w-\chi\nabla\cdot w\nabla v+\overline{u}(\chi-2\alpha)w+\alpha(w-w^2)+\alpha\|w\|_2^2. \end{aligned} \tag{64}$$

Multiplying by $w_\pm$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_\pm\|_2^2+d\|\nabla w_\pm\|_2^2 &= \frac{\chi}{2}\|w_\pm\|_3^3+\overline{u}(\chi-2\alpha)\|w_\pm\|_2^2+\alpha(\|w_\pm\|_2^2-\|w_\pm\|_3^3)+\alpha\|w_\pm\|_2^2\|w_\pm\|_1 \\ &\leq \frac{\chi}{2}\|w_\pm\|_3^3+\overline{u}(\chi-2\alpha)\|w_\pm\|_2^2+2\alpha\|w_\pm\|_2^2 \end{aligned}$$

by

$$-(\nabla \cdot w\nabla v, w_\pm) = \frac{1}{2}(\nabla v, \nabla w_\pm^2) = \frac{1}{2}(-\Delta v, w_\pm^2) = \frac{1}{2}\|w_\pm\|_3^3 \tag{65}$$

and $\|w_\pm\|_1 \leq 2\overline{u} \leq 2$. Therefore, it holds that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|_2^2 + d\|\nabla w\|_2^2 &\leq \frac{\chi}{2}\|w\|_3^3 + (\overline{u}(\chi - 2\alpha) + 2\alpha)\|w\|_2^2 \\ &\leq K\chi\|\nabla w\|_2^2 + (\overline{u}(\chi - 2\alpha) + 2\alpha)\|w\|_2^2 \end{aligned} \tag{66}$$

recalling $n \leq 2$, where $K > 0$ is the constant arising in the Gagliardo-Nirenberg inequality (24), that is, $K = C_6(p, q, \Omega)^3$ for $p = 3$, $q = 1$.

If $(d - K\chi)\mu_2 > (\chi - 2\alpha)_+ + 2\alpha$, therefore, we obtain

$$\frac{d}{dt}\|w\|_2^2 + \delta\|w\|_2^2 \leq 0, \quad t > t_1$$

with $\delta > 0$ by $\overline{w} = 0$, recalling that $\mu_2 > 0$ denotes the first positive eigenvalue of $-\Delta$ provided with the Neumann boundary condition. Hence it follows that

$$\|w(\cdot, t)\|_2^2 \leq C_{37}e^{-\delta t}, \qquad \int_0^T \|w\|_2^2 \, dt \leq C_{38}.$$

By (62), therefore, any $\varepsilon > 0$ admits $T > 0$ such that

$$\frac{d\overline{u}}{dt} \geq \alpha(\overline{u} - \overline{u}^2) - \varepsilon, \quad t \geq T$$

which implies either

$$\lim_{t\uparrow+\infty} \overline{u}(t) = 0 \tag{67}$$

or (63). In the latter case we obtain (57) from the compactness of the orbit.

We shall show that (67) does not arise for $\chi \geq \alpha$. In fact, in the reverse case of $\alpha > \chi$ Theorem 3 implies (17) concluding to a contradiction. Specifically, at first (13) implies

$$u_t = d\Delta u - \chi\nabla v \cdot \nabla u + \chi u(u - \overline{u}) + \alpha(u - u^2), \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0,$$

and therefore, it holds that $u(\cdot, t) \geq U(t)$ for $U = U(t)$, satisfying

$$\frac{dU}{dt} = \chi U(U - \overline{u}) + \alpha(U - U^2) = U(\alpha - \chi\overline{u} + (\chi - \alpha)U) \tag{68}$$

with $U|_{t=0} = \min u_0 > 0$. The right-hand side on (68), however, is estimated below by $\alpha U/2$ in $t \gg 1$ because of (63) and $\chi \geq \alpha$. Then we obtain $\lim_{t\uparrow+\infty} U(t) = +\infty$, and hence $\lim_{t\uparrow+\infty} \overline{u}(t) = +\infty$ by $\overline{u} \geq U$, a contradiction. □

Remark 5. The first part of the above proof shows that if $n = 2$, the global-in-time classical solution exists without any restrictions on d, χ, and α. For this part, we may apply the method used for the proof of Theorem 1. In particular, inequality (21) holds for any n, and therefore, any spatially homogeneous sub-solution of $u = u(x, t)$ is bounded.

We conclude this section with the following theorem concerning the system (1) with (4), that is,

$$\begin{aligned} & u_t = \Delta u - \chi \nabla \cdot u \nabla v + \alpha(u - u^2), \quad \tau v - \Delta v = u - \overline{u} \quad \text{in } \Omega \times (0,T) \\ & \frac{\partial}{\partial \nu}(u,v)\bigg|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0(x) > 0, \ v\big|_{t=0} = 0. \end{aligned} \tag{69}$$

Theorem 6. *If $n = 1$, any solution to (69) is global-in-time and uniformly bounded. Furthermore, there is $d_0 = d_0(\chi, \alpha) > 0$ such that $d > d_0$ implies (17) with $\delta > 0$.*

Proof. We modify the proof of Theroem 5, using

$$\tau \frac{d}{dt}\|\nabla v\|_2^2 + \|\Delta v\|_2^2 \leq \|u - \overline{u}\|_2^2, \quad (-\Delta v, u - \overline{u}) \leq \|u - \overline{u}\|_2^2 - \frac{\tau}{2}\frac{d}{dt}\|\nabla v\|_2^2$$

derived from

$$(-\Delta v, u - \overline{u}) = \frac{\tau}{2}\frac{d}{dt}\|\nabla v\|_2^2 + \|\Delta v\|_2^2 \leq \frac{1}{2}\|\Delta v\|_2^2 + \frac{1}{2}\|u - \overline{u}\|_2^2.$$

Then inequality (41) takes the form

$$\frac{dH}{dt} + H \leq \|u - \overline{u}\|_2^2 + \frac{\tau\chi}{2}\|\nabla v\|_2^2$$

for $H = H(t)$ defined by (42).

We have

$$\begin{aligned} & \frac{\tau}{2}\frac{d}{dt}\|v\|_2^2 + \|\nabla v\|_2^2 = (u - \overline{u}, v) \leq \|u - \overline{u}\|_2 \|v\|_2 \leq \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2\mu_2^2}\|u - \overline{u}\|_2^2 \\ & \tau\|v_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 = (u - \overline{u}, v_t) \leq \frac{\tau}{2}\|v_t\|_2^2 + \frac{1}{2\tau}\|u - \overline{u}\|_2^2, \end{aligned}$$

which implies

$$\begin{aligned} & \int_0^T \|\nabla v\|_2^2 \, dt \leq \mu_2^{-2} \int_0^T \|u - \overline{u}\|_2^2 \, dt + \tau\|v_0\|_2^2 \\ & \int_0^T \|v_t\|_2^2 \, dt \leq \tau^{-2} \int_0^T \|u - \overline{u}\|_2^2 \, dt + \tau^{-1}\|\nabla v_0\|_2^2. \end{aligned} \tag{70}$$

If

$$\int_0^T \|u - \overline{u}\|_2^2 \, dt \leq C_{39}, \tag{71}$$

we can argue similarly, and it holds that $T = +\infty$ with $\|u(\cdot, t)\|_\infty \leq C_{40}$. If

$$\lim_{t \uparrow +\infty} \overline{u}(t) = 1, \tag{72}$$

furthermore, then (17) follows.

To establish (71)-(72), we have only to take the case $\overline{u}(t) < 1$, $t \in (t_1, T)$ for some $t_1 \in (0, T)$, and to derive (71). Then we use (64) with $w = u - \overline{u}$. Equality (65) then becomes

$$(-\Delta v, w_\pm^2) = (w - \tau v_t, w_\pm^2) \leq \|w_\pm\|_3^3 + \tau \|v_t\|_2 \|w_\pm\|_4^2. \tag{73}$$

Thus we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + d\|\nabla w\|_2^2 \leq \frac{\chi}{2}\|w\|_3^3 + ((\chi - 2\alpha)_+ + 2\alpha)\,\|w\|_2^2 + \|v_t\|_2^2 + \frac{\tau^2}{4}\|w\|_4^4$$

for (66). Then Gagliardo-Nirenberg inequality valid to $n = 1$,

$$\|w\|_q^q \leq K\|\nabla w\|_2^2 \|w\|_1^{q-2}, \quad 2 \leq q \leq 4,$$

implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + d\|\nabla w\|_2^2 &\leq \frac{\chi}{2}K\|\nabla w\|_2^2\|w\|_1 + ((\chi - 2\alpha)_+ + 2\alpha)\,\|w\|_2^2 \\ &+ \|v_t\|_2^2 + \frac{\tau^2}{4}K\|\nabla w\|_2^2\|w\|_1^2.\end{aligned}$$

Since $\|w\|_1 \leq 2\|u\|_1 = 2\overline{u} < 2$, it holds that

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + (d - \chi K - \tau^2 K)\|\nabla w\|_2^2 \leq ((\chi - 2\alpha)_+ + 2\alpha)\,\|w\|_2^2 + \|v_t\|_2^2$$

and hence

$$\left\{\left(d - \tau^2 K - \chi K\right)\mu_2 - (\chi - 2\alpha)_+ - \alpha\right\}\int_0^T \|u - \overline{u}\|_2^2 \, dt \leq \int_0^T \|v_t\|_2^2 \, dt$$

for $d > (\tau^2 + \chi)K$, recalling (26). Then (71) follows from (70), provided that d is sufficiently large as

$$\left(d - (\tau^2 + \chi)K\right)\mu_2 > (\chi - 2\alpha)_+ + 2\alpha + \tau^{-2}.$$

The proof is complete. □

4 A Remark

Here we show the following theorem.

Theorem 7. *If $f = f(s)$, $s \geq 0$, is concave, $f(0) < -\frac{4d}{\alpha}n(n-1)$, and $f'(0) > 0$, then there is a solution which is not global-in-time to (1)-(2).*

We use the following lemma.

Lemma 1. *Let $n \geq 2$, $\Omega = B \equiv B(0,1)$, and $u_0 = u_0(r)$, $r = |x|$, or $n = 1$, $\Omega = (-1,1)$, $n = 1$, and $u_0(-x) = u_0(x)$, in (1)-(2). Assume, furthermore, that $f = f(u)$ is concave. Then it holds that*

$$\frac{dX}{dt} \leq 4dn(n-1)\overline{u} + \chi\overline{u}X - \chi\overline{u}^2 + \alpha f(X) \tag{74}$$

in (1)-(2), where

$$X = \frac{\int_\Omega r^n u}{\int_\Omega r^n}. \tag{75}$$

Proof. First, we have

$$-r^{n-1}v_r(r,t) = \int_0^r s^{n-1}(u(s,t) - \overline{u}(t))ds = \int_0^r s^{n-1}u(s,t)ds - \frac{r^n}{n}\overline{u}(t)$$

by (2), which implies

$$\begin{aligned} \int_0^1 r^{2(n-1)}uv_r dr &= \frac{\overline{u}}{n}\int_0^1 r^{2n-1}udr - \int_0^1 r^{n-1}u(r,t)dr \cdot \int_0^r s^{n-1}u(s,t)ds \\ &= \frac{\overline{u}}{n}\int_0^1 r^{2n-1}u\ dr - \frac{1}{2}\left(\int_0^1 r^{n-1}u(r,t)\ dr\right)^2. \end{aligned}$$

Then it follows that

$$\begin{aligned} \frac{d}{dt}\int_\Omega r^n u &= \int_\Omega |x|^n\left(\nabla\cdot(d\nabla u - \chi u\nabla v) + \alpha f(u)\right) = \int_\Omega -\nabla|x|^n\cdot(d\nabla u - \chi u\nabla v) + \alpha r^n f(u)\ dx \\ &= \int_\Omega -dnr^{n-2}x\cdot\nabla u + \chi nr^{n-1}uv_r + \alpha r^n f(u)\ dx = \int_{\partial\Omega} -dnr^{n-2}(x\cdot\nu)u \\ &+ \int_\Omega dnu\nabla\cdot r^{n-2}x + \alpha r^n f(u)\ dx + \chi n\omega_{n-1}\left\{\frac{\overline{u}}{n}\int_0^1 r^{2n-1}u\ dr - \frac{1}{2}\left(\int_0^1 r^{n-1}u\ dr\right)^2\right\}, \end{aligned}$$

where ω_{n-1} denotes the $(n-1)$-dimensional volume of $\partial\Omega$. By $\nabla\cdot r^{n-2}x = 2(n-1)r^{n-2}$ and $|\Omega| = \frac{\omega_{n-1}}{n}$, we obtain

$$\begin{aligned} \frac{d}{dt}\frac{1}{|\Omega|}\int_\Omega r^n u &= -\frac{dn^2}{|\partial\Omega|}\int_{\partial\Omega} u + \frac{1}{|\Omega|}\int_\Omega dn(n-1)r^{n-2}u + \alpha r^n f(u)\ dx \\ &+ \frac{\chi\overline{u}}{|\Omega|}\int_\Omega r^n u - \frac{\chi\overline{u}^2}{2}. \end{aligned} \tag{76}$$

Here we multiply both sides by 2, regarding $\int_\Omega r^n = \frac{\omega_{n-1}}{2n}$. Then (74) follows from the concavity of $f = f(u)$. □

Now we give the following proof.

Proof of Theorem 7. By (74) we have

$$\begin{aligned}\frac{1}{\alpha}\frac{dX}{dt} &\leq \frac{\chi}{\alpha}\left\{-\left(\overline{u}-\frac{X}{2}\right)^2+\frac{X^2}{4}+\frac{4d}{\chi}n(n-1)\right\}+f(X)\\ &\leq \frac{\chi}{\alpha}\left\{\frac{X^2}{4}+\frac{4d}{\chi}n(n-1)\right\}+f(X)\equiv g(X).\end{aligned}$$

From the assumption, we have $g(0)<0<g'(0)$. Hence $T<+\infty$ arises if $X(0)\ll 1$. □

Remark 6. Since the symmetry and the decreasing property of $u_0=u_0(x)$ is kept for $u=u(\cdot,t)$, there may arise two cases if $\lim_{t\uparrow T}X(t)=0$, that is, either $\lim_{t\uparrow T}\|u(\cdot,t)\|_\infty=0$ or $\lim_{t\uparrow T}\|u(\cdot,t)\|_\infty=+\infty$. In the latter case the blowup set is composed of the origin, and a sufficient condition for this property is $\limsup_{t\uparrow T}\overline{u}(t)>0$.

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