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ERROR ESTIMATES FOR ELLIPTIC EQUATIONS WITH NOT-EXACTLY PERIODIC COEFFICIENTS

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Abstract. This note is devoted to the derivation of quantitative estimates for linear elliptic equations with coefficients that are not-exactly ε-periodic and the ellipticity constant may degenerate with order $O(\varepsilon^{2\gamma})$. Here, $\varepsilon > 0$ denotes the ratio between the microscopic and the macroscopic length scale, and the coefficients are only periodic with respect to the microscopic scale. It is shown that for $\gamma = 0$ and $\gamma = 1$ the error between the original solution and the effective solution is of order $O(\varepsilon^{1/2})$. Therefore suitable test functions are constructed via the periodic unfolding method and a gradient folding operator making only minimal additional assumptions on the given data and the effective solution with respect to the macroscopic scale.

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1 Introduction

Periodic homogenization is a very powerful tool to derive effective equations for e.g. biological cells [21, 7], binary mixtures [4], modeling of concrete [24, 5], and other processes in porous media [13, 25, 18, 19]. All these articles deal with systems of coupled reaction-diffusion equations which contain linear, elliptic equations of the form

$$-\operatorname{div}\left(\varepsilon^{2\gamma}\mathbb{A}\left(x,\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right)+\mathbb{B}\left(x,\tfrac{x}{\varepsilon}\right)u_\varepsilon=F_\varepsilon(x)\quad\text{ in }\Omega. \tag{1.1}$$

In this text we study equation (1.1) for $\gamma = 0$ and $\gamma = 1$ on a bounded domain $\Omega \subset \mathbb{R}^d$ supplemented with homogeneous Neumann boundary conditions. The coefficients under consideration are not-exactly periodic in the following sense: in addition to the periodicity with respect to the microscopic scale $y = x/\varepsilon$ for $\varepsilon > 0$, the coefficients are allowed to depend arbitrarily on the macroscopic scale $x \in \Omega$. For the sake of quantitative estimates, the x-dependency is assumed to be more regular, namely, Lipschitz- or $\mathrm{H}^1(\Omega)$-regular, cf. assumption (2.7). Within the framework of reaction-diffusion systems $\gamma = 1$ models the slow diffusion of order $O(\varepsilon)$ for some species, whereas for $\gamma = 0$ other species diffuse with order $O(1)$.

The aim is to quantify the error between the original solution u_ε of (1.1) and the effective solution, which is for $\gamma = 0$ a one-scale function $u(x)$ and for $\gamma = 1$ a two-scale function $U(x, y)$ solving the equations (2.8) and (4.2). The weak and strong two-scale convergence of solutions u_ε to (1.1) is studied in [24] and [12] for all $\gamma \in [0,\infty)$. Here, we derive quantitative estimates in the two-scale space $\Omega \times Y$, where $Y = [0, 1)^d$, using the periodic unfolding operator $\mathcal{T}_\varepsilon : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega \times Y)$, cf. [2] or (2.1),

$$\gamma = 0: \qquad \|u_\varepsilon - u\|_{\mathrm{L}^2(\Omega)} + \|\mathcal{T}_\varepsilon(\nabla u_\varepsilon) - [\nabla u+\nabla_y U]\|_{\mathrm{L}^2(\Omega\times Y)} \le \varepsilon^{1/2}C, \tag{1.2}$$

$$\gamma = 1: \qquad \|\mathcal{T}_\varepsilon u_\varepsilon - U\|_{\mathrm{L}^2(\Omega;\mathrm{H}^1(Y))} \le \varepsilon^{1/2}C. \tag{1.3}$$

The constant C does not depend on ε, but on the given data as well as on the norms of the effective solution u and U in the space $\mathrm{H}^2(\Omega)$ and $\mathrm{H}^1(\Omega;\mathrm{H}^1(Y))$, respectively. In the case $\gamma = 0$ the corrector U is given by the well-known unit-cell problem (4.3).

For two-scale functions $\mathbb{F} \in \mathrm{H}^1(\Omega;\mathrm{L}^2(Y))$ which are Y-periodic and in general not continuous, neither with respect to $x \in \Omega$ nor $y \in Y$, the right-hand side of (1.1) is defined via the folding or averaging operator, cf. (2.2),

$$F_\varepsilon(x) = \widehat{F}_\varepsilon(x, x/\varepsilon) \quad\text{ where }\quad \widehat{F}_\varepsilon(x, y) = \fint_{\varepsilon([x/\varepsilon]+Y)} \mathbb{F}(z, y)\,\mathrm{d}z. \tag{1.4}$$

This becomes relevant for systems of parabolic equations which are coupled via Lipschitz-terms $F_\varepsilon(u_\varepsilon, v_\varepsilon)$, where u_ε and v_ε only belong to the space $\mathrm{H}^1(\Omega)$. For such systems the rigorous derivation of effective two-scale equations was proved in [16] using the notion of strong two-scale convergence via periodic unfolding. In the proceedings article [26] the convergence rate of order $O(\varepsilon^{1/4})$ is shown. For elliptic equations with exactly periodic coefficients and $\gamma = 0$ we refer to [8, 9, 23, 10] for unfolding based error estimates of order $O(\varepsilon^{1/2})$ and higher. For coupled reaction-diffusion equations with $\gamma = 0$ only the gradient estimate is of lower order $O(\varepsilon^{1/4})$ in [6]. Estimates of order $O(\varepsilon^{1/2})$ based on asymptotic expansion are derived in [4, 19] containing the cases $\gamma = 0$ and $\gamma = 1$. Therein continuity is assumed for the given data and effective solutions, whereas our approach requires no higher regularity with respect to the microscopic scale.

Our main result (Theorem 2.1) is novel for $\gamma = 1$ and for $\gamma = 0$ (Theorem 4.1) it is a slight generalization of the estimates obtained in [8]. There arise two difficulties deriving the quantitative estimates (1.2)–(1.3) which we briefly outline for $\gamma = 1$. The periodicity defect of $\mathcal{T}_\varepsilon u_\varepsilon \in \mathrm{L}^2(\Omega; \mathrm{H}^1(Y))$ versus $U \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$ is treated as in [9]. Secondly one has to identify a suitable "folding" or "approximating sequence" for the limit $U(x, y)$ since the naive composition $U(x, x/\varepsilon)$ is only well-defined in $\mathrm{H}^1(\Omega)$ for functions $U \in \mathrm{C}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y))$, see [15] for admissible test functions in two-scale convergence. Also U_ε as in (1.4) is not suitable due to insufficient $\mathrm{H}^1(\Omega)$-regularity. Therefore, we employ the gradient folding operator $\mathcal{G}_\varepsilon : \mathrm{H}^1(\Omega; \mathrm{H}^1_{\mathrm{per}}(Y)) \to \mathrm{H}^1(\Omega)$ as in [12, 16, 26] which is defined by solving the degenerating elliptic problem (3.3). In Theorem 3.4 we control the *folding mismatch* of U_ε and $\mathcal{G}_\varepsilon U$ with respect to the $\mathrm{H}^1(\Omega)$-norm as it is indicated in [26].

This text is structured as follows. In Subsection 2.1 we introduce basic definitions and notations and in Subsection 2.2 our assumptions and Theorem 2.1 for $\gamma = 1$ are stated. The derivation of quantitative estimates is done in Section 3, whereby Subsection 3.1 and 3.2 contain estimates for the approximation errors and the folding mismatch, respectively. The proof of Theorem 2.1 is given in Subsection 3.3. Finally, we show in Section 4 how our approach applies to the case $\gamma = 0$ (Subsection 4.1) as well as to parabolic equations (Subsection 4.3). In Subsection 4.2 we compare our estimates with [8].

2 Notations and definitions

2.1 Periodic unfolding

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Following [2], we denote with $Y := [0, 1)^d$ the unit cube and $\mathcal{Y} := \mathbb{R}^d/_{\mathbb{Z}^d}$ is the torus which we obtain by identifying opposite faces of $\overline{Y}$. Using the mappings $[\,\cdot\,] : \mathbb{R}^d \to \mathbb{Z}^d$ and $\{\,\cdot\,\} : \mathbb{R}^d \to \mathcal{Y}$, any point $x \in \mathbb{R}^d$ admits the additive decomposition $x = [x] + \{x\}$. Here, $[x] := \big(\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor\big)$ is the componentwise Gauss bracket and $\{x\} := x - [x]$ is the remainder. With this we define for $\varepsilon > 0$ the cells $\mathcal{C}_\varepsilon(x) := \varepsilon\big([x/\varepsilon] + Y\big)$ where the macroscopic part $\varepsilon[x/\varepsilon] \in \varepsilon\mathbb{Z}^d$ denotes the node associated to $\mathcal{C}_\varepsilon(x)$ and $y \in \mathcal{Y}$ respective $y \in Y$ is the microscopic part. Since the domain $\Omega \subset \mathbb{R}^d$ is bounded, not all cells $\mathcal{C}_\varepsilon(x)$ are fully contained in Ω. To handle cells intersecting the boundary $\partial\Omega$ we define the following sets

$$\Omega_\varepsilon^- := \mathrm{int}\big(\{x \in \Omega \,|\, \mathcal{C}_\varepsilon(x) \subset \Omega\}\big) \quad \text{and} \quad \Omega_\varepsilon^+ := \mathrm{int}\big(\{x \in \Omega \,|\, \overline{\mathcal{C}_\varepsilon(x)} \cap \overline{\Omega}\}\big)$$

so that $\Omega_\varepsilon^- \subset \Omega \subset \Omega_\varepsilon^+$ as depicted in Figure 2.1. The Lipschitz property of Ω guarantees $\mathrm{vol}(\Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-) \leq \varepsilon 2\sqrt{d}\,\mathrm{meas}(\partial\Omega)$ with finite surface measure $\mathrm{meas}(\partial\Omega)$.

The *periodic unfolding operator* $\mathcal{T}_\varepsilon : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega \times \mathcal{Y})$ is defined as in [2] via

$$(\mathcal{T}_\varepsilon u)(x, y) := u\big(\varepsilon[\tfrac{x}{\varepsilon}] + \varepsilon y\big) \tag{2.1}$$

if $x \in \Omega_\varepsilon^-$ and $(\mathcal{T}_\varepsilon u)(x, y) := 0$ otherwise. Accordingly, we define the *folding operator* (also averaging operator) $\mathcal{F}_\varepsilon : \mathrm{L}^2(\Omega \times \mathcal{Y}) \to \mathrm{L}^2(\Omega)$ via

$$(\mathcal{F}_\varepsilon U)(x) := \fint_{\mathcal{C}_\varepsilon(x)} U\left(z, \{\tfrac{x}{\varepsilon}\}\right)\, \mathrm{d}z \tag{2.2}$$

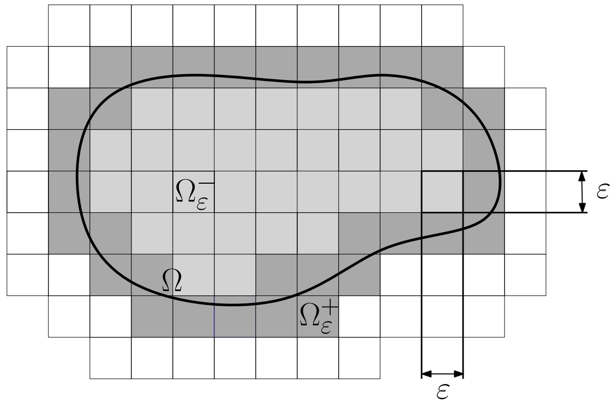

Figure 1: Covering of the domain Ω with microscopic cells.

if $(x, y) \in \Omega_\varepsilon^- \times \mathcal{Y}$ and $(\mathcal{F}_\varepsilon U)(x) := 0$ otherwise, where $⨍_A u(z)\,\mathrm{d}z := \mathrm{vol}(A)^{-1}\int_A u(z)\,\mathrm{d}z$ denotes the average. Both operators are linear and bounded, i.e.

$$\|\mathcal{T}_\varepsilon u\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})} \leq \|u\|_{\mathrm{L}^2(\Omega)} \quad \text{and} \quad \|\mathcal{F}_\varepsilon U\|_{\mathrm{L}^2(\Omega)} \leq \|U\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})} \tag{2.3}$$

by Jensen's inequality. We emphasize that the periodic unfolding operator $\mathcal{T}_\varepsilon$ in (2.1) does *not* satisfy the integral identity $\int_{\Omega\times\mathcal{Y}} \mathcal{T}_\varepsilon u\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega u\,\mathrm{d}x$ and we refer to [17] for a unfolding definition which does do so. However, we only use the duality $\mathcal{F}_\varepsilon = \mathcal{T}_\varepsilon{}^*$, cf. [3, Sect. 2.2], meaning for all $u \in \mathrm{L}^2(\Omega)$ and $V \in \mathrm{L}^2(\Omega\times\mathcal{Y})$

$$\int_{\Omega\times\mathcal{Y}} (\mathcal{T}_\varepsilon u)\, V\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega u\,(\mathcal{F}_\varepsilon V)\,.$$

Furthermore, the unfolding operator satisfies $\mathcal{T}_\varepsilon u \in \mathrm{L}^2(\Omega;\mathrm{H}^1(Y))$ for all $u \in \mathrm{H}^1(\Omega)$ and it is $\mathcal{T}_\varepsilon(\varepsilon\nabla u) = \nabla_y(\mathcal{T}_\varepsilon u)$, cf. [3, Eq. (3.1)]. We point out that the spaces $\mathrm{L}^2(\Omega\times\mathcal{Y})$ and $\mathrm{L}^2(\Omega\times Y)$ can be identified, whereas $\mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}))$ is a closed subspace of $\mathrm{L}^2(\Omega;\mathrm{H}^1(Y))$ with $\mathrm{H}^1(\mathcal{Y}) = \mathrm{H}^1_{\mathrm{per}}(Y) \subsetneq \mathrm{H}^1(\mathcal{Y})$.

Using periodic unfolding via $\mathcal{T}_\varepsilon$, *weak (respective strong) two-scale convergence* of bounded sequences $(u_\varepsilon)_\varepsilon \subset \mathrm{L}^2(\Omega)$, as introduced in [22], is equivalent to weak (respective strong) convergence of the unfolded sequence $(\mathcal{T}_\varepsilon u_\varepsilon)_\varepsilon$ in the two-scale space $\mathrm{L}^2(\Omega\times\mathcal{Y})$. It was first proved in [1, Prop. 1.14] that sequences $(u_\varepsilon)_\varepsilon \subset \mathrm{H}^1(\Omega)$ satisfying $\sup_{\varepsilon>0}\{\|u_\varepsilon\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon\nabla u_\varepsilon\|_{\mathrm{L}^2(\Omega)}\} < \infty$ weakly two-scale converge (up to subsequence) to a function $U \in \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}))$, i.e. $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup U$ and $\mathcal{T}_\varepsilon(\varepsilon\nabla u_\varepsilon) \rightharpoonup \nabla_y U$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$. We emphasize that the unfolded function $\mathcal{T}_\varepsilon u_\varepsilon$ is in general not Y-periodic, while its two-scale limit U indeed is, and we call this observation *periodicity defect* as in [8].

2.2 Assumptions and statement of the main result for $\gamma = 1$

We study the following elliptic equation, which is degenerating as $\varepsilon \to 0$,

$$-\operatorname{div}(\varepsilon^2 A_\varepsilon \nabla u_\varepsilon) + B_\varepsilon u_\varepsilon = F_\varepsilon \quad \text{in } \Omega \tag{2.4}$$

supplemented with homogeneous Neumann boundary conditions, i.e. $(\varepsilon^2 A_\varepsilon \nabla u_\varepsilon)\cdot\nu = 0$ on $\partial\Omega$. Here, ν denotes the unit outer normal vector of Ω. The coefficients depend on $x \in \Omega$ and $\varepsilon > 0$ via

$$A_\varepsilon(x) := \mathbb{A}\left(x, \tfrac{x}{\varepsilon}\right), \quad B_\varepsilon(x) := \mathbb{B}\left(x, \tfrac{x}{\varepsilon}\right), \quad \text{and} \quad F_\varepsilon(x) := (\mathcal{F}_\varepsilon \mathbb{F})(x). \tag{2.5}$$

Throughout this text we impose the following assumptions on the given data. Let $\mathbb{A}(x,y) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be a *symmetric* matrix and let $\mathbb{A}$ and $\mathbb{B}$ be *positive definite*, i.e. there exist constants $0 < \alpha \le \beta < \infty$ such that we have uniformly for all $(x, y, \xi) \in \Omega \times \mathcal{Y} \times \mathbb{R}^d$

$$\alpha|\xi|^2 \le \mathbb{A}(x,y)\xi\cdot\xi \le \beta|\xi|^2 \quad \text{and} \quad \alpha \le \mathbb{B}(x,y) \le \beta. \tag{2.6}$$

For the derivation of quantitative estimates we need additional regularity with respect to the macroscopic scale $x \in \Omega$, namely let

$$\mathbb{A} \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{L}^\infty(\mathcal{Y})), \quad \mathbb{B} \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{L}^\infty(\mathcal{Y})), \quad \text{and} \quad \mathbb{F} \in \mathrm{H}^1(\Omega; \mathrm{L}^2(\mathcal{Y})). \tag{2.7}$$

Thanks to the continuity of $\mathbb{A}$ and $\mathbb{B}$ with respect to $x \in \Omega$, the definitions of A_ε and B_ε in (2.5) are indeed well-defined. Passing to the limit $\varepsilon \to 0$ in (2.4) gives the two-scale convergence of u_ε to a limit function $U \in \mathrm{L}^2(\Omega; \mathrm{H}^1(\mathcal{Y}))$ solving the effective equation (cf. [24, 12, 16])

$$-\operatorname{div}_y(\mathbb{A}\nabla_y U) + \mathbb{B}U = \mathbb{F} \quad \text{in } \Omega \times \mathcal{Y} \tag{2.8}$$

with periodic boundary conditions on the torus $\mathcal{Y}$ and no boundary conditions on Ω. By the Lax–Milgram Theorem and the regularity (2.7) of the given data, the solution U of (2.8) even belongs to the better space $\mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$. This follows by considering difference quotients in $x \in \Omega$, cf. [27, Prop. 2.3.17]. All solutions are uniformly bounded in the sense

$$\sup_{\varepsilon>0}\left\{\|u_\varepsilon\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon\nabla u_\varepsilon\|_{\mathrm{L}^2(\Omega)}\right\} + \|U\|_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))} \le \mathrm{Const.}(\mathbb{A}, \mathbb{B}, \mathbb{F}). \tag{2.9}$$

Our main result is in the case $\gamma = 1$ as follows.

Theorem 2.1. *Let the given data satisfy the assumptions* (2.5)–(2.7) *and let* u_ε *and* U *solve the elliptic equation* (2.4) *and* (2.8)*, respectively. Then, there exists a constant* $C > 0$ *depending on the norm in* (2.9) *such that*

$$\|\mathcal{T}_\varepsilon u_\varepsilon - U\|_{\mathrm{L}^2(\Omega;\mathrm{H}^1(Y))} \le \varepsilon^{1/2} C.$$

The proof (see Subsection 3.3) combines ideas of [26, Thm. 3.2] and [8, Prop. 4.3], and it is shown that $\mathcal{G}_\varepsilon U$ is an approximate solution of the original equation (2.4). Therefore, we proceed in three steps:

1. The choice of an admissible test function for the effective equation (2.8) leads to the periodicity defect.
2. We reformulate by using the duality $\mathcal{F}_\varepsilon = \mathcal{T}_\varepsilon^*$ and control the approximation errors.
3. Choosing a suitable test function for the original equation is equivalent to finding the correct "recovery sequence" (in terms of Γ-convergence) which is $\mathcal{G}_\varepsilon U$ here.

3 Error estimates

3.1 Estimating the periodicity defect and approximation errors

In this subsection we provide preparatory estimates to control the approximation error for the folding operator and the periodicity defect. To handle the error for microscopic cells $\mathcal{C}_\varepsilon(x)$ which are not fully contained in the domain Ω, we exploit for functions $U \in \mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))$ the boundary estimate

$$\|U\|_{\mathrm{L}^2((\Omega\setminus\Omega_\varepsilon^-)\times\mathcal{Y})} \leq \varepsilon^{1/2}C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))}, \tag{3.1}$$

where the constant $C > 0$ only depends on the Lipschitz property of Ω. For one-scale functions, estimate (3.1) can be found in e.g. [8, Eq. (4.6)] or [9, Eq. (2.4)] and for a full proof we refer to [27, Lem. 2.3.5]. The error between $\mathcal{T}_\varepsilon\,\mathcal{F}_\varepsilon$ and $\mathrm{id}_{\mathrm{L}^2(\Omega\times\mathcal{Y})}$ is given via:

Lemma 3.1. *For every $U \in \mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))$ it holds*

$$\|U - \mathcal{T}_\varepsilon\,\mathcal{F}_\varepsilon\,U\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})} \leq (\varepsilon + \varepsilon^{1/2})C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))},$$

where the constant $C > 0$ only depends on the domains Ω and Y.

Proof. Using the paving property $\Omega_\varepsilon^- = \cup_{\lambda_i}\mathcal{C}_\varepsilon(x)$, where $\lambda_i = [x/\varepsilon] \in \mathbb{Z}^d$ for $x \in \Omega_\varepsilon^-$, and applying the Poincaré–Wirtinger inequality yields

$$\begin{aligned}\|U - \mathcal{T}_\varepsilon\,\mathcal{F}_\varepsilon\,U\|^2_{\mathrm{L}^2(\Omega_\varepsilon^-\times\mathcal{Y})} &= \sum_{\lambda_i}\int_{\varepsilon(\lambda_i+Y)}\int_{\mathcal{Y}}\left|U(x,y) - \fint_{\mathcal{C}_\varepsilon(x)} U(z,y)\,\mathrm{d}z\right|^2 \mathrm{d}x\,\mathrm{d}y \\ &\leq \sum_{\lambda_i}\left(\mathrm{diam}(\mathcal{C}_\varepsilon(x)\right)^2\|\nabla_x U\|^2_{\mathrm{L}^2(\varepsilon(\lambda_i+Y)} \leq \varepsilon^2 C\|\nabla_x U\|^2_{\mathrm{L}^2(\Omega\times\mathcal{Y})}.\end{aligned}$$

Here, we used that the Poincaré–Wirtinger constant is bounded by the diameter of the convex set $\mathcal{C}_\varepsilon(x)$, namely, $\varepsilon\mathrm{diam}(Y)$. Since $\|U-\mathcal{T}_\varepsilon\,\mathcal{F}_\varepsilon\,U\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})} = \|U-\mathcal{T}_\varepsilon\,\mathcal{F}_\varepsilon\,U\|_{\mathrm{L}^2(\Omega_\varepsilon^-\times\mathcal{Y})} + \|U\|_{\mathrm{L}^2((\Omega\setminus\Omega_\varepsilon^-)\times\mathcal{Y})}$ the boundary estimate (3.1) gives the desired result. □

Introducing for $\varepsilon > 0$ and $\varphi \in \mathrm{H}^1(\Omega)$ the norm

$$\|\varphi\|_\varepsilon := \|\varphi\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon\nabla\varphi\|_{\mathrm{L}^2(\Omega)} \tag{3.2}$$

we can control the periodicity defect as follows.

Theorem 3.2 ([9, Thm. 2.2])**.** *For every $\varphi \in \mathrm{H}^1(\Omega)$, there exists a $\mathcal{Y}$-periodic function $\Phi_\varepsilon \in \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}))$ such that*

$$\|\Phi_\varepsilon\|_{\mathrm{H}^1(Y;\mathrm{L}^2(\Omega))} \leq C\|\varphi\|_\varepsilon \quad \textit{and} \quad \|\mathcal{T}_\varepsilon\,\varphi - \Phi_\varepsilon\|_{\mathrm{H}^1(Y;\mathrm{H}^1(\Omega)^*)} \leq (\varepsilon + \varepsilon^{1/2})C\|\varphi\|_\varepsilon,$$

where the constant $C > 0$ only depends in the domains Ω and Y.

3.2 Gradient folding operator and folding mismatch

By the Sobolev extension Theorem, there exists a linear and continuous operator, cf. e.g. [20, Thm. 3.9], $\mathcal{E} : \mathrm{H}^1(\Omega; \mathrm{L}^2(\mathcal{Y})) \to \mathrm{H}^1(\mathbb{R}^d; \mathrm{L}^2(\mathcal{Y}))$ such that $(\mathcal{E}U)|_{\Omega\times\mathcal{Y}} = U$ and $\|\mathcal{E}U\|_{\mathrm{H}^1(\mathbb{R}^d;\mathrm{L}^2(\mathcal{Y}))} \leq C\|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))}$. Moreover, let $\mathcal{E}$ be such that its restriction to one-scale functions $u \in \mathrm{H}^1(\Omega)$ satisfies $(\mathcal{E}u)|_\Omega = u$ and $\|\mathcal{E}u\|_{\mathrm{H}^1(\mathbb{R}^d)} \leq C\|u\|_{\mathrm{H}^1(\Omega)}$. Taking care of cells $\mathcal{C}_\varepsilon(x)$ intersecting the boundary of Ω, we define the extended folding operator $\mathcal{F}_\varepsilon^+ : \mathrm{H}^1(\Omega; \mathrm{L}^2(\mathcal{Y})) \to \mathrm{L}^2(\mathbb{R}^d)$ via

$$(\mathcal{F}_\varepsilon^+ U)(x,y) := \fint_{\mathcal{C}_\varepsilon(x)} (\mathcal{E}U)\left(z, \{\tfrac{x}{\varepsilon}\}\right) \mathrm{d}z.$$

The *gradient folding operator* $\mathcal{G}_\varepsilon : \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y})) \to \mathrm{H}^1(\Omega_\varepsilon^+)$ is defined as in [16] following [17, 12]: for $U \in \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$ given, $\mathcal{G}_\varepsilon U := \widehat{u}_\varepsilon \in \mathrm{H}^1(\Omega_\varepsilon^+)$ is the unique solution of the elliptic problem

$$\int_{\Omega_\varepsilon^+} (\widehat{u}_\varepsilon - \mathcal{F}_\varepsilon^+ U)\varphi + (\varepsilon\nabla\widehat{u}_\varepsilon - \mathcal{F}_\varepsilon^+(\nabla_y U)) \cdot \varepsilon\nabla\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathrm{H}^1(\Omega_\varepsilon^+). \tag{3.3}$$

The Lax–Milgram Theorem implies that $\mathcal{G}_\varepsilon U$ is well-defined and bounded in the sense $\|\mathcal{G}_\varepsilon U\|_\varepsilon \leq \|\mathcal{E}U\|_{\mathrm{H}^1(\mathbb{R}^d;\mathrm{H}^1(\mathcal{Y}))}$. Throughout the following calculations the operator $\mathcal{E}$ is omitted in notation, however, all generic constants C will also depend on its norm.

Having defined two different folding operators $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$, we want to control their difference which we call *folding mismatch*. Therefore we introduce the so-called scale-splitting operator $\mathcal{Q}_\varepsilon : \mathrm{H}^1(\Omega) \to \mathrm{W}^{1,\infty}(\mathbb{R}^d)$ following [3, Def. 4.1]: for $x \in \mathcal{C}_\varepsilon(x)$ and every $\kappa = (\kappa_1, \ldots, \kappa_d) \in \{0,1\}^d$, we set

$$\bar{x}_l^{(\kappa_l)} := \begin{cases} \frac{x_l - \varepsilon[x/\varepsilon]_l}{\varepsilon} & \text{if } \kappa_l = 1 \\ 1 - \frac{x_l - \varepsilon[x/\varepsilon]_l}{\varepsilon} & \text{if } \kappa_l = 0 \end{cases}$$

and

$$(\mathcal{Q}_\varepsilon w)(x) := \sum_{\kappa\in\{0,1\}^d} (\mathcal{F}_\varepsilon^+ w)\left(\varepsilon[\tfrac{x}{\varepsilon}] + \varepsilon\kappa\right) \cdot \bar{x}_1^{(\kappa_1)} \cdots \bar{x}_d^{(\kappa_d)}. \tag{3.4}$$

The function $\mathcal{Q}_\varepsilon w$ interpolates the values of $\mathcal{F}_\varepsilon^+ w$ at the nodes $\varepsilon[x/\varepsilon]$ via $\mathcal{Q}_1$-Lagrange elements as customary in the finite elements methods. Indeed, it is $\mathcal{F}_\varepsilon^+ w \in \mathrm{L}^\infty(\Omega_\varepsilon^+)$ such that for $z \in \mathrm{L}^2(\mathcal{Y})$ the products

$$x \mapsto \left(\mathcal{F}_\varepsilon^+ w\right)(x) z\left(\tfrac{x}{\varepsilon}\right) \quad \text{and} \quad x \mapsto \left(\mathcal{Q}_\varepsilon w\right)(x) z\left(\tfrac{x}{\varepsilon}\right) \tag{3.5}$$

belong to the space $\mathrm{L}^2(\Omega_\varepsilon^+)$, see e.g. [15, Thm.4]. According to [3, Prop. 4.5], there exists a constant $C > 0$ only depending on Ω and Y such that

$$\|\mathcal{Q}_\varepsilon w\|_{\mathrm{H}^1(\Omega)} \leq C\|w\|_{\mathrm{H}^1(\Omega)} \quad \text{for all } w \in \mathrm{H}^1(\Omega). \tag{3.6}$$

The following auxiliary estimate is proved in [27, Lem. 3.6] or [26, Lem. 2.3.9] based on ideas from [8, Prop. 3.2].

Lemma 3.3. *For $w \in \mathrm{H}^1(\Omega)$ and $z \in \mathrm{L}^2(\mathcal{Y})$, it holds*

$$\|(\mathcal{F}_\varepsilon^+ w - \mathcal{Q}_\varepsilon w) z(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} \leq \varepsilon C \|w\|_{\mathrm{H}^1(\Omega)} \|z\|_{\mathrm{L}^2(\mathcal{Y})},$$

where the constant $C > 0$ only depends on the dimension d.

Finally, we have collected all ingredients to control the folding mismatch.

Theorem 3.4. *For every $U \in \mathrm{H}^1(\Omega; \mathrm{H}^1(\mathcal{Y}))$, the folding mismatch is*

$$\|\mathcal{G}_\varepsilon U - \mathcal{F}_\varepsilon U\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon \nabla(\mathcal{G}_\varepsilon U) - \mathcal{F}_\varepsilon(\nabla_y U)\|_{\mathrm{L}^2(\Omega)} \leq \varepsilon^{1/2} C \|U\|_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))}, \quad (3.7)$$

$$\|\mathcal{G}_\varepsilon U - \mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} + \|\varepsilon \nabla(\mathcal{G}_\varepsilon U) - \mathcal{F}_\varepsilon^+(\nabla_y U)\|_{\mathrm{L}^2(\Omega_\varepsilon^+)} \leq \varepsilon C \|U\|_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))}, \quad (3.8)$$

where the constant $C > 0$ only depends on the domains Ω and Y.

Proof. For brevity we set $\mathrm{L}^2_\varepsilon := \mathrm{L}^2(\Omega_\varepsilon^+)$ troughout this proof.

Step 1: Treatment of the boundary. The triangle inequality, the identity $(\mathcal{F}_\varepsilon U)|_{\Omega_\varepsilon^-} = (\mathcal{F}_\varepsilon^+ U)|_{\Omega_\varepsilon^-}$, Jensen's inequality, and the boundary estimate (3.1) imply

$$\begin{aligned}\|\mathcal{G}_\varepsilon U - \mathcal{F}_\varepsilon U\|_{\mathrm{L}^2(\Omega)} &\leq \|\mathcal{G}_\varepsilon U - \mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2(\Omega)} + \|\mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2(\Omega \setminus \Omega_\varepsilon^-)} \\ &\leq \|\mathcal{G}_\varepsilon U - \mathcal{F}_\varepsilon^+ U\|_{\mathrm{L}^2_\varepsilon} + (\varepsilon + \varepsilon^{1/2}) C \|U\|_{\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))}.\end{aligned}$$

An analog estimate holds for the gradient term so that it remains to prove (3.8) and (3.7) follows immediately.

Step 2: Proof of estimate (3.8) *for products $U(x,y) = w(x)z(y)$.* Let U satisfy the decomposition $U(x,y) = w(x)z(y)$ with $w \in \mathrm{H}^1(\Omega)$ and $z \in \mathrm{H}^1(\mathcal{Y})$. We introduce the function $\vartheta_\varepsilon(x) := (\mathcal{Q}_\varepsilon w)(x) z(x/\varepsilon)$ which belongs to the space $\mathrm{H}^1(\Omega_\varepsilon^+)$ by construction. The definition of $\mathcal{G}_\varepsilon U$ in (3.3) is equivalent to

$$\begin{aligned}&\int_{\Omega_\varepsilon^+} (\mathcal{G}_\varepsilon U - \vartheta_\varepsilon)\varphi + \varepsilon \nabla(\mathcal{G}_\varepsilon U - \vartheta_\varepsilon) \cdot \varepsilon \nabla \varphi \,\mathrm{d}x \\ &\qquad = \int_{\Omega_\varepsilon^+} (\mathcal{F}_\varepsilon^+ U - \vartheta_\varepsilon)\varphi + (\mathcal{F}_\varepsilon^+(\nabla_y U) - \varepsilon \nabla \vartheta_\varepsilon) \cdot \varepsilon \nabla \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in \mathrm{H}^1(\Omega_\varepsilon^+).\end{aligned}$$

Choosing $\varphi = \mathcal{G}_\varepsilon U - \vartheta_\varepsilon$ as well as applying Hölder's inequality, Lemma 3.3, and the boundedness of $\mathcal{Q}_\varepsilon$ in (3.6) gives

$$\begin{aligned}&\|\mathcal{G}_\varepsilon U - \vartheta_\varepsilon\|_\varepsilon \leq \|\mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_{\mathrm{L}^2_\varepsilon} + \|\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon \nabla \vartheta_\varepsilon\|_{\mathrm{L}^2_\varepsilon} \\ &= \|(\mathcal{F}_\varepsilon^+ w - \mathcal{Q}_\varepsilon w) z(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^2_\varepsilon} + \|(\mathcal{F}_\varepsilon^+ w - \mathcal{Q}_\varepsilon w) \nabla_y z(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^2_\varepsilon} + \|\varepsilon \nabla_x (\mathcal{Q}_\varepsilon w) z(\tfrac{\cdot}{\varepsilon})\|_{\mathrm{L}^2_\varepsilon} \\ &\leq \varepsilon C \|w\|_{\mathrm{H}^1(\Omega)} \|z\|_{\mathrm{H}^1(\mathcal{Y})}.\end{aligned}$$

Finally, the triangle inequality yields

$$\begin{aligned}&\|\mathcal{G}_\varepsilon U - \mathcal{F}_\varepsilon U\|_{\mathrm{L}^2_\varepsilon} + \|\varepsilon \nabla(\mathcal{G}_\varepsilon U) - \mathcal{F}_\varepsilon(\nabla_y U)\|_{\mathrm{L}^2_\varepsilon} \\ &\leq \|\mathcal{G}_\varepsilon U - \vartheta_\varepsilon\|_\varepsilon + \|\mathcal{F}_\varepsilon U - \vartheta_\varepsilon\|_{\mathrm{L}^2_\varepsilon} + \|\mathcal{F}_\varepsilon(\nabla_y U) - \varepsilon \nabla \vartheta_\varepsilon\|_{\mathrm{L}^2_\varepsilon} \\ &\leq 2\varepsilon C \|w\|_{\mathrm{H}^1(\Omega)} \|z\|_{\mathrm{H}^1(\mathcal{Y})}\end{aligned} \quad (3.9)$$

and the desired estimate (3.8) follows for functions U of product form.

Step 3: Proof of estimate (3.8) *for general functions* $U(x,y)$. Let $\{\Phi_i\}_{i=1}^{\infty}$ be an orthonormal basis in $\mathrm{H}^1(\mathcal{Y})$ which is also orthogonal in $\mathrm{L}^2(\mathcal{Y})$. Then we can express $U \in \mathrm{H}^1(\Omega_\varepsilon^+; \mathrm{H}^1(\mathcal{Y}))$ (extended by $\mathcal{E}$) via the linear combination

$$U(x,y) = \sum_{i=1}^{\infty} u_i(x)\Phi_i(y) \quad \text{where} \quad u_i(x) := \int_{\mathcal{Y}} U(x,y)\Phi_i(y)\,\mathrm{d}y. \tag{3.10}$$

By construction it is $u_i \in \mathrm{H}^1(\Omega_\varepsilon^+)$ and we set $U_i(x,y) := u_i(x)\Phi_i(y)$.

The assumptions on Φ_i imply $\Phi_i(\cdot/\varepsilon) \perp \Phi_j(\cdot/\varepsilon)$ and $\nabla_y\Phi_i(\cdot/\varepsilon) \perp \nabla_y\Phi_j(\cdot/\varepsilon)$ in L^2_ε for all $i \neq j$, i.e. $\{\Phi_i(\cdot/\varepsilon)\}_i$ is an orthogonal system in $\mathrm{H}^1(\Omega_\varepsilon^+)$ for each $\varepsilon > 0$. (Here, $u \perp v$ if and only if $(u,v)_{\mathrm{L}^2_\varepsilon} = 0$.) Indeed, using the paving property $\Omega_\varepsilon^+ = \cup_{\lambda_i}\mathcal{C}_\varepsilon(x)$, where $\lambda_i = [x/\varepsilon] \in \mathbb{Z}^d$ for $x \in \Omega_\varepsilon^+$, a substitution of variables yields

$$\begin{aligned}\int_{\Omega_\varepsilon^+} \Phi_i(\tfrac{x}{\varepsilon})\Phi_j(\tfrac{x}{\varepsilon})\,\mathrm{d}x &= \sum_{\lambda_i}\int_{\mathcal{C}_\varepsilon(x)} \Phi_i(\tfrac{x}{\varepsilon})\Phi_j(\tfrac{x}{\varepsilon})\,\mathrm{d}x \\ &= \sum_{\lambda_i}\frac{1}{\varepsilon^d}\int_{[\frac{x}{\varepsilon}]+Y} \Phi_i(y)\Phi_j(y)\,\mathrm{d}y = 0\end{aligned} \tag{3.11}$$

and analogously for $\nabla_y\Phi_i$. Applying the folding operator $\mathcal{F}_\varepsilon^+$ to U_i gives

$$(\mathcal{F}_\varepsilon^+ U_i)(x) = (\mathcal{F}_\varepsilon^+ u_i)(x)\Phi_i(\tfrac{x}{\varepsilon}) \quad \text{and} \quad (\mathcal{F}_\varepsilon^+[\nabla_y U_i])(x) = (\mathcal{F}_\varepsilon^+ u_i)(x)\nabla_y\Phi_i(\tfrac{x}{\varepsilon}).$$

Since $(\mathcal{F}_\varepsilon^+ u_i)(x) = \fint_{\mathcal{C}_\varepsilon(x)} u_i(z)\,\mathrm{d}z$ is constant on each cell $\mathcal{C}_\varepsilon(x)$, we have as well $\mathcal{F}_\varepsilon^+ U_i \perp \mathcal{F}_\varepsilon^+ U_j$ and $\mathcal{F}_\varepsilon^+(\nabla_y U_i) \perp \mathcal{F}_\varepsilon^+(\nabla_y U_j)$ in L^2_ε for all $i \neq j$. Therefore, it suffices to consider the basis functions Φ_i. By the definition of $\mathcal{G}_\varepsilon$ we have for $v_i^\varepsilon := \mathcal{G}_\varepsilon\,\Phi_i$

$$\int_{\Omega_\varepsilon^+} \big(v_i^\varepsilon - \Phi_i(\tfrac{\cdot}{\varepsilon})\big)\,\varphi + \big(\varepsilon\nabla v_i^\varepsilon - \nabla_y\Phi_i(\tfrac{\cdot}{\varepsilon})\big)\cdot\varepsilon\nabla\varphi\,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathrm{H}^1(\Omega_\varepsilon^+). \tag{3.12}$$

Inserting the test function $\varphi(x) = \Phi_j(x/\varepsilon)$ in (3.12) yields with (3.11) for all $i \neq j$

$$\int_{\Omega_\varepsilon^+} v_i^\varepsilon\Phi_j(\tfrac{\cdot}{\varepsilon}) + \varepsilon\nabla v_i^\varepsilon\cdot\nabla_y\Phi_j(\tfrac{\cdot}{\varepsilon})\,\mathrm{d}x = 0.$$

Furthermore, choosing $\varphi(x) = v_j^\varepsilon(x)$ in (3.12) yields with the latter equality

$$\int_{\Omega_\varepsilon^+} v_i^\varepsilon v_j^\varepsilon + \varepsilon\nabla v_i^\varepsilon\cdot\varepsilon\nabla v_j^\varepsilon\,\mathrm{d}x = 0.$$

Hence, it is $v_i^\varepsilon \perp v_j^\varepsilon$ in $\mathrm{H}^1(\Omega_\varepsilon^+)$ for all $i \neq j$. We continue to estimate the folding mismatch of $\mathcal{F}_\varepsilon^+$ and $\mathcal{G}_\varepsilon$ and set $u_i^\varepsilon := \mathcal{G}_\varepsilon\,U_i$. Replacing $\mathcal{F}_\varepsilon^+\,\Phi_i$ and $\mathcal{F}_\varepsilon^+(\nabla_y\Phi_i)$ with $\mathcal{F}_\varepsilon^+\,U_i$ and $\mathcal{F}_\varepsilon^+(\nabla_y U_i)$ in (3.12), respectively, and exploiting that $\mathcal{F}_\varepsilon^+\,u_i$ is constant on each cell $\mathcal{C}_\varepsilon(x)$ implies $u_i^\varepsilon \perp u_j^\varepsilon$ in $\mathrm{H}^1(\Omega_\varepsilon^+)$ for all $i \neq j$.

Finally, we apply the result of Step 2 with $w = u_i$ and $z = \Phi_i$. Since $\mathcal{F}_\varepsilon^+$ and $\mathcal{G}_\varepsilon$ are linear and continuous operators, we have in particular $u^\varepsilon = \sum_{i=1}^{\infty} u_i^\varepsilon$. Therefore, we square estimate (3.9) so that the mixed product terms vanish for $i \neq j$, namely,

$$\begin{aligned}
&\|u^\varepsilon - \mathcal{F}_\varepsilon^+ U\|^2_{\mathrm{L}^2_\varepsilon} + \|\varepsilon\nabla u^\varepsilon - \mathcal{F}_\varepsilon^+(\nabla_y U)\|^2_{\mathrm{L}^2_\varepsilon} \\
&= \|\textstyle\sum_{i=1}^\infty (u_i^\varepsilon - \mathcal{F}_\varepsilon^+ U_i)\|^2_{\mathrm{L}^2_\varepsilon} + \|\textstyle\sum_{i=1}^\infty (\varepsilon\nabla u_i^\varepsilon - \mathcal{F}_\varepsilon^+(\nabla_y U_i))\|^2_{\mathrm{L}^2_\varepsilon} \\
&= \textstyle\sum_{i=1}^\infty \|u_i^\varepsilon - \mathcal{F}_\varepsilon^+ U_i\|^2_{\mathrm{L}^2_\varepsilon} + \textstyle\sum_{i=1}^\infty \|\varepsilon\nabla u_i^\varepsilon - \mathcal{F}_\varepsilon(\nabla_y U_i)\|^2_H \\
&\leq \textstyle\sum_{i=1}^\infty \varepsilon^2 C\|u_i\|^2_{\mathrm{H}^1(\Omega)}\|z\|^2_{\mathrm{H}^1(\mathcal{Y})} = \varepsilon^2 C\|U\|^2_{\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))}.
\end{aligned}$$

The last equality follows by Parseval's identity $\|U\|^2_{\mathrm{H}^1(\Omega_\varepsilon^+;\mathrm{H}^1(\mathcal{Y}))} = \sum_{i=1}^\infty \|u_i\|^2_{\mathrm{H}^1(\Omega_\varepsilon^+)}$. □

3.3 Proof of Theorem 2.1

Proof of Theorem 2.1. Step 1: Periodicity defect. The weak formulation of the effective equation (2.8) reads

$$\int_{\Omega\times\mathcal{Y}} \mathbb{A}\nabla_y U\cdot\nabla_y\Phi + \mathbb{B}U\Phi - \mathbb{F}\Phi \,\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{for all } \Phi\in\mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y})). \tag{3.13}$$

Thanks to Theorem 3.2 we can choose $\Phi_\varepsilon \in \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}))$ to estimate the periodicity defect of $\mathcal{T}_\varepsilon\varphi \in \mathrm{L}^2(\Omega;\mathrm{H}^1(Y))$. Indeed, exploiting the higher x-regularity (2.7) of the given data and the limit $U\in\mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))$ yields

$$\begin{aligned}
&\left|\int_{\Omega\times\mathcal{Y}} \mathbb{A}\nabla_y U\cdot[\nabla_y(\mathcal{T}_\varepsilon\varphi) - \nabla_y\Phi_\varepsilon] + \big(\mathbb{B}U - \mathbb{F}\big)[\mathcal{T}_\varepsilon\varphi - \Phi_\varepsilon]\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\leq \|\mathbb{A}\nabla_y U\|_{\mathrm{L}^2(\mathcal{Y};\mathrm{H}^1(\Omega))}\|\nabla_y(\mathcal{T}_\varepsilon\varphi) - \nabla_y\Phi_\varepsilon\|_{\mathrm{L}^2(\mathcal{Y};\mathrm{H}^1(\Omega)^*)} \\
&\quad + \big(\||\mathbb{B}U| + |\mathbb{F}|\|_{\mathrm{L}^2(\mathcal{Y};\mathrm{H}^1(\Omega))}\big)\|\mathcal{T}_\varepsilon\varphi - \Phi_\varepsilon\|_{\mathrm{L}^2(\mathcal{Y};\mathrm{H}^1(\Omega)^*)} \\
&\leq \mathrm{Const.}(\mathbb{A},\mathbb{B},\mathbb{F},U)(\varepsilon + \varepsilon^{1/2})C\|\varphi\|_\varepsilon.
\end{aligned}$$

Note that the identification of the spaces $\big(\mathrm{L}^2(\mathcal{Y};\mathrm{H}^1(\Omega))\big)^*$ and $\mathrm{L}^2(\mathcal{Y};\mathrm{H}^1(\Omega)^*)$ as well as $\mathrm{L}^2(\mathcal{Y};\mathrm{H}^1(\Omega))$ and $\mathrm{H}^1(\Omega;\mathrm{L}^2(\mathcal{Y}))$ holds due to the underlying tensor product structure of the two-scale space $\mathrm{L}^2(\mathcal{Y};\mathrm{H}^1(\Omega))$. Overall testing (3.13) with Φ_ε gives

$$\left|\int_{\Omega\times\mathcal{Y}} \mathbb{A}\nabla_y U\cdot\nabla_y(\mathcal{T}_\varepsilon\varphi) + \mathbb{B}U\,\mathcal{T}_\varepsilon\varphi - \mathbb{F}\,\mathcal{T}_\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}y\right| \leq \varepsilon^{1/2}C\|\varphi\|_\varepsilon.$$

Step 2: Approximation errors. Using $\mathcal{F}_\varepsilon = \mathcal{T}_\varepsilon^*$ and $\nabla_y(\mathcal{T}_\varepsilon\varphi) = \mathcal{T}_\varepsilon(\varepsilon\nabla\varphi)$ gives

$$\left|\int_\Omega \mathcal{F}_\varepsilon(\mathbb{A}\nabla_y U)\cdot\varepsilon\nabla\varphi + \mathcal{F}_\varepsilon(\mathbb{B}U)\varphi - F_\varepsilon\varphi\,\mathrm{d}x\right| \leq \varepsilon^{1/2}C\|\varphi\|_\varepsilon. \tag{3.14}$$

Thanks to the Lipschitz continuity of $\mathbb{A}$ and $\mathbb{B}$ with respect to $x\in\Omega$ and the boundedness (2.3) of $\mathcal{F}_\varepsilon$, we obtain

$$\begin{aligned}
&\|\mathcal{F}_\varepsilon(\mathbb{A}\nabla_y U) - A_\varepsilon\,\mathcal{F}_\varepsilon(\nabla_y U)\|^2_{\mathrm{L}^2(\Omega)} \\
&= \int_{\Omega_\varepsilon^-}\left|\fint_{\mathcal{C}_\varepsilon(x)} \left(\mathbb{A}\left(z,\tfrac{x}{\varepsilon}\right) - \mathbb{A}\left(x,\tfrac{x}{\varepsilon}\right)\right)\nabla_y U\left(z,\tfrac{x}{\varepsilon}\right)\,\mathrm{d}z\right|^2\mathrm{d}x \\
&\leq \varepsilon^2\|\nabla_x\mathbb{A}\|_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})}\|\nabla_y U\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})}
\end{aligned}$$

and analogously $\| \mathcal{F}_\varepsilon(\mathbb{B}U) - B_\varepsilon \mathcal{F}_\varepsilon(U)\|^2_{\mathrm{L}^2(\Omega)} \leq \varepsilon^2 \|\nabla_x \mathbb{B}\|_{\mathrm{L}^\infty(\Omega\times\mathcal{Y})}\|U\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})}$. Inserting these two estimates into (3.14) yields

$$\left|\int_\Omega A_\varepsilon \mathcal{F}_\varepsilon(\nabla_y U)\cdot\varepsilon\nabla\varphi + B_\varepsilon \mathcal{F}_\varepsilon U\varphi - F_\varepsilon\varphi \,\mathrm{d}x\right| \leq \varepsilon^{1/2}C\|\varphi\|_\varepsilon.$$

Step 3: Folding mismatch. Thanks to Theorem 3.4 we can replace the folded functions $\mathcal{F}_\varepsilon U$ and $\mathcal{F}_\varepsilon(\nabla_y U)$ with the gradient folding $\mathcal{G}_\varepsilon U$ and $\varepsilon\nabla(\mathcal{G}_\varepsilon U)$, respectively, such that

$$\left|\int_\Omega A_\varepsilon\varepsilon\nabla(\mathcal{G}_\varepsilon U)\cdot\varepsilon\nabla\varphi + B_\varepsilon(\mathcal{G}_\varepsilon U)\varphi - F_\varepsilon\varphi\,\mathrm{d}x\right| \leq \varepsilon^{1/2}C\|\varphi\|_\varepsilon. \tag{3.15}$$

The choice $\varphi_\varepsilon = u_\varepsilon - \mathcal{G}_\varepsilon U \in \mathrm{H}^1(\Omega)$ is indeed admissible in (3.15) and φ_ε is also a test function for the original equation (2.4), i.e.

$$\int_\Omega A_\varepsilon\varepsilon\nabla u_\varepsilon\cdot\varepsilon\nabla\varphi_\varepsilon + B_\varepsilon u_\varepsilon\varphi_\varepsilon - F_\varepsilon\varphi_\varepsilon\,\mathrm{d}x = 0. \tag{3.16}$$

Subtracting (3.15) from (3.16) and exploiting the positive definiteness (2.6) gives

$$\alpha\|u_\varepsilon - \mathcal{G}_\varepsilon U\|^2_\varepsilon \leq \varepsilon^{1/2}C\|u_\varepsilon - \mathcal{G}_\varepsilon U\|_\varepsilon. \tag{3.17}$$

Finally, applying the triangle inequality, the boundedness (2.3) of $\mathcal{T}_\varepsilon$, estimate (3.7) for the folding mismatch, and the approximation error yields

$$\begin{aligned}
&\|\mathcal{T}_\varepsilon u_\varepsilon - U\|_{\mathrm{L}^2(\Omega;\mathrm{H}^1(Y))}\\
&\leq \|u_\varepsilon - \mathcal{G}_\varepsilon U\|_\varepsilon + \|\mathcal{G}_\varepsilon U - \mathcal{F}_\varepsilon U\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon\nabla(\mathcal{G}_\varepsilon U) - \mathcal{F}_\varepsilon(\nabla_y U)\|_{\mathrm{L}^2(\Omega)}\\
&\quad + \|\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon U - U\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})} + \|\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon(\nabla_y U) - \nabla_y U\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})} \leq \varepsilon^{1/2}C
\end{aligned}$$

which finishes the proof. □

4 Discussion of further error estimates

4.1 The non-degenerating case for $\gamma = 0$

We consider the non-degenerating elliptic equation

$$-\operatorname{div}(A_\varepsilon\nabla u_\varepsilon) + B_\varepsilon u = F_\varepsilon \quad \text{in } \Omega \tag{4.1}$$

with homogeneous Neumann boundary conditions and given data as in (2.5)–(2.7). It is well-known that $u_\varepsilon \rightharpoonup u$ weakly in $\mathrm{H}^1(\Omega)$ and u solves the effective equation

$$-\operatorname{div}(A_{\mathrm{eff}}\nabla u) + B_{\mathrm{eff}}u = F_{\mathrm{eff}} \quad \text{in } \Omega \tag{4.2}$$

with the same boundary conditions. The effective matrix A_{eff} is given via the standard unit-cell problem (for arbitrary vectors $\xi \in \mathbb{R}^d$)

$$\xi\cdot A_{\mathrm{eff}}(x)\xi = \min_{\Phi\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})}\int_{\mathcal{Y}}(\xi+\nabla_y\Phi)\cdot\mathbb{A}(x,y)(\xi+\nabla_y\Phi)\,\mathrm{d}y, \tag{4.3}$$

where $\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}) = \{\Phi \in \mathrm{H}^1(\mathcal{Y}) \mid \int_{\mathcal{Y}} \Phi \,\mathrm{d}y = 0\}$. The effective data B_{eff} and F_{eff} are the usual averages

$$B_{\mathrm{eff}}(x) = \int_{\mathcal{Y}} \mathbb{B}(x,y)\,\mathrm{d}y \quad \text{and} \quad F_{\mathrm{eff}}(x) = \int_{\mathcal{Y}} \mathbb{F}(x,y)\,\mathrm{d}y. \tag{4.4}$$

The regularity of $\mathbb{A}$ in (2.7) implies $A_{\mathrm{eff}} \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. If the domain Ω is additionally convex or of class $\mathcal{C}^2$, we obtain higher regularity of u, namely $u \in \mathrm{H}^2(\Omega)$, see e.g. [11, 14]. This higher regularity of the limit solution u is needed to derive quantitative estimates and it is not nearly as trivial as in the degenerating case $\gamma = 1$. The corrector U is the unique minimizer in (4.3) corresponding to $\xi = \nabla u(x)$ and it satisfies $U \in \mathrm{H}^1(\Omega;\mathrm{H}^1(\mathcal{Y}))$ thanks to the higher regularity of $\mathbb{A}$ and u with respect to $x \in \Omega$. Hence, we can state our main result for $\gamma = 0$.

Theorem 4.1. *Let the assumptions* (2.5)–(2.7) *hold and let u_ε and u solves the elliptic equation* (4.1) *and* (4.2)*, respectively. If u belongs to* $\mathrm{H}^2(\Omega)$*, then there exists a constant $C > 0$ such that*

$$\|u_\varepsilon - u\|_{\mathrm{L}^2(\Omega)} + \|\,\mathcal{T}_\varepsilon(\nabla u_\varepsilon) - [\nabla u + \nabla_y U]\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})} \le \varepsilon^{1/2}C.$$

Proof. The proof is analog to the one of Theorem 2.1 for $\gamma = 1$ and one shows that $u + \varepsilon\,\mathcal{G}_\varepsilon U$ is an approximate solution of the original equation (4.1). To control the periodicity defect in Step 1 we apply [9, Thm. 2.3], i.e. for every $\varphi \in \mathrm{H}^1(\Omega)$, there exists a function $\Phi_\varepsilon \in \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}))$ and a constant $C > 0$ such that $\|\Phi_\varepsilon\|_{\mathrm{H}^1(Y;\mathrm{L}^2(\Omega))} \le C\|\varphi\|_{\mathrm{H}^1(\Omega)}$ and $\|\,\mathcal{T}_\varepsilon(\nabla\varphi) - [\nabla\varphi + \nabla_y\Phi_\varepsilon]\|_{\mathrm{L}^2(\mathcal{Y};\mathrm{H}^1(\Omega)^*)} \le (\varepsilon + \varepsilon^{1/2})C\|\varphi\|_{\mathrm{H}^1(\Omega)}$. □

Remark 4.2 (Boundary conditions)**.** *For different boundary conditions the error estimates are functioning as before, however, one has to modify the approximating sequence so that the new boundary conditions are satisfied. In the case of homogeneous Dirichlet boundary conditions, a suitable candidate is $u + \varepsilon\,\mathcal{G}_\varepsilon(\rho_\varepsilon U)$, where ρ_ε denotes a cut-off function to guarantee zero conditions at the boundary $\partial\Omega$.*

4.2 Comparison with [8]

The corrector U admits the decomposition (see e.g. [15, Sec. 4])

$$U(x,y) = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(x) z_i(x,y), \tag{4.5}$$

where $z_i \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ is the unique minimizer corresponding to the standard basis vector e_i in $\mathbb{R}^d$. If A_ε is exactly periodic as in [8], i.e. $A_\varepsilon(x) = \mathbb{A}(x/\varepsilon)$, then $z_i \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$ is it as well. In this case we have the following error estimates for solutions u_ε and u of (4.1) and (4.2), respectively, with $\mathbb{B} \equiv 0$ and either homogeneous Neumann or Dirichlet boundary conditions.

Theorem 4.3 ([8, Prop. 4.3])**.** *If $u \in \mathrm{H}^2(\Omega)$, then there exists $C > 0$ such that*

$$\|u_\varepsilon - u\|_{\mathrm{L}^2(\Omega)} + \big\|\nabla u_\varepsilon - [\nabla u + \textstyle\sum_{i=1}^{d} \mathcal{Q}_\varepsilon\big(\frac{\partial u}{\partial x_i}\big) z_i(\frac{\cdot}{\varepsilon})]\big\|_{\mathrm{L}^2(\Omega)} \le \varepsilon^{1/2}C.$$

In any case – exactly periodic or not – the question is how to “fold” $U(x, y)$? We remark that $x \mapsto U(x, x/\varepsilon) \in \mathrm{L}^2(\Omega)$ is in general not satisfied for $U \in \mathrm{L}^\infty(\Omega; \mathrm{L}^2(\Omega))$, whereas for products $x \mapsto w(x)z(x/\varepsilon) \in \mathrm{L}^2(\Omega)$ holds true for $w \in \mathrm{L}^\infty(\Omega)$ and $z \in \mathrm{L}^2(\mathcal{Y})$ according to [15, Thm.4]. Hence, the construction $\sum_{i=1}^d \mathcal{Q}_\varepsilon \left(\frac{\partial u}{\partial x_i}\right)(x) z_i(x/\varepsilon)$ is admissible in $\mathrm{H}^1(\Omega)$ for exactly periodic coefficients. However, in the not-exactly periodic situation, the choice $\mathcal{Q}_\varepsilon \left(\frac{\partial u}{\partial x_i}\right)(x) z_i(x, x/\varepsilon)$ is not admissible. This technicality is circumvented in our approach by using the gradient folding operator and choosing the recovery sequence $u + \varepsilon\, \mathcal{G}_\varepsilon U$ respective $u + \varepsilon\, \mathcal{G}_\varepsilon(\rho_\varepsilon U)$. Nevertheless, for more regular data such as $\mathbb{A} \in \mathrm{C}^1(\overline{\Omega}; \mathrm{L}^\infty(\mathcal{Y}))$ and hence $z_i \in C^1(\overline{\Omega}; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$, the naive folding $z_i(x, x/\varepsilon)$ is again well-defined in $\mathrm{H}^1(\Omega)$ and the proof of [8, Prop. 4.3] seems to be valid, too.

The approach of [8] applies as well to systems of reaction-diffusion equations, cf. [6]. And our approach is also generalizable to semilinear parabolic equations with globally Lipschitz continuous right-hand sides, see Subsection 4.3.

4.3 Error estimates for parabolic equations

Our approach can be generalized to semilinear parabolic equations. Following e.g. [4, 6, 19, 27] one aims to apply Gronwall's Lemma and therefore estimates

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon - \mathcal{F}_\varepsilon U\|^2_{\mathrm{L}^2(\Omega)} &= \int_\Omega (\dot{u}_\varepsilon - \mathcal{F}_\varepsilon \dot{U})(u_\varepsilon - \mathcal{F}_\varepsilon U)\,\mathrm{d}x \\
&= \int_\Omega \underbrace{(\dot{u}_\varepsilon - \mathcal{F}_\varepsilon \dot{U})\varphi_\varepsilon\,\mathrm{d}x}_{\text{Term 1}} + \int_\Omega \underbrace{(\dot{u}_\varepsilon - \mathcal{F}_\varepsilon \dot{U})(\mathcal{G}_\varepsilon U - \mathcal{F}_\varepsilon U)\,\mathrm{d}x}_{\text{Term 2}}.
\end{aligned}
$$

Choosing the test function $\varphi_\varepsilon = u_\varepsilon - \mathcal{G}_\varepsilon U$ and inserting the reformulated equations, Term 1 can be treated as in the elliptic case. To control Term 2, in particular to obtain the convergence rate $\varepsilon^{1/2}$, we need the uniform boundedness $\sup_{\varepsilon>0} \|\dot{u}_\varepsilon\|_{\mathrm{L}^2(\Omega)} < \infty$ and estimate (3.8) of order $O(\varepsilon)$ for the folding mismatch, i.e. $|\text{Term 2}| \leq \varepsilon C$. This can be achieved by assuming either Ω is polyhedral or by extending all solutions, given data, and operators from Ω to Ω_ε^+ such that the duality $\mathcal{F}_\varepsilon^+ = (\mathcal{T}_\varepsilon^+)^*$ is applicable.

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